Department of Mathematics & Statistics

Ph.D admission written test

Time: 1 Hour **Marks 60**

NAME: **December 5, 2015**

Instructions

1. Write your name in **BOLD** letters.
2. For a real number x, we denote by $[x]$ the largest integer less than or equal to x.
3. We denote by $\mathbb{Z}$, the set of integers, $\mathbb{Q}$, the set of rational numbers, $\mathbb{R} =$, the set of real numbers and $\mathbb{C}$ the set of complex numbers.
4. Marking Scheme
   - True or False. You will be awarded 2 marks for each correct answer -1 mark for each wrong answer.
   - Choose the correct answers. You will be awarded 2 marks for each correct answer -1 mark for each wrong answer.
   - In each part 0 mark for the questions not attempted.

1 True or False.

You need to just mark your answer as True or False.

1. Let $a > 1$ be a real number and $x_n := (1 + a^n)^{\frac{1}{n}}$. Then the sequence (x_n) converges.
2. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function and B be bounded subset of $\mathbb{R}$. Then $f(B)$ is a bounded subset of $\mathbb{R}$.
3. There exists a linear transformation $T : \mathbb{R}^4 \to \mathbb{R}^3$ such that
$$\ker(T) := \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : x_1 + x_2 + x_3 + x_4 = 0\}$$
and
$$\text{Image}(T) := \{(x, y, z) \in \mathbb{R}^3 : x + y + z = 0\}.$$
4. Every group of order 177 is cyclic.

5. Let R be a commutative ring with unity and let I be the set of all the non–units of R. Then I is a maximal ideal in R.

6. The polynomial $X^3 + 15X^2 + 36$ irreducible in $\mathbb{Q}[X]$.

7. There exists a non-constant analytic function $f : \mathbb{C} \to \{z = x + iy \in \mathbb{C} : y > 0\}$.

8. Let H be a real Hilbert space and $T : H \to H$ be a bounded linear map such that $\langle Tx, x \rangle = 0$ for all x in H. Then $Tx = 0$ for all $x \in H$.

9. The degree of the polynomial that interpolates a given function at $n + 1$ distinct points is exactly n.

10. Inverse Laplace trasform of $\pi/2 - \tan^{-1}(s/2)$ is $\sin(2t)/t$.

2 Choose the correct answer(s).

There may be more than one correct answer. You need to choose the correct atnswers and mark them.

1. A quadrature rule on the interval $[-1, 1]$ uses the quadrature points $x_0 = -\alpha$ and $x_1 = \alpha$, where $0 < \alpha \leq 1$:

$$\int_{-1}^{1} f(x)dx \approx \omega_0 f(-\alpha) + \omega_1 f(\alpha).$$

If this formula is exact for polynomials of as high a degree as possible, then which of the following options (is) are correct?
(A) $\omega_0^2 + \omega_1^2 = 1$ (B) $\omega_0^2 + \omega_1^2 = 2$ (C) $\alpha = 1/\sqrt{2}$ (D) $\alpha = 1/\sqrt{3}$.

2. If $y = x\sin(x) + x^2$ is a solution of the seventh order ordinary differential equation

$$a_7 y^{(7)} + a_6 y^{(6)} + a_5 y^{(5)} + a_4 y^{(4)} + a_3 y^{(3)} + a_2 y^{(2)} + a_1 y^{(1)} + a_0 y = 0,$$

where a_i, $i = 0, 1, \cdots, 7$ are constants. Then which of the following statements (is) are true?

(a) $a_7 + a_3 = a_5$.
(b) $a_6 - a_2 = a_3$.
(c) $\sum_{i=0}^{7} a_i = 4$.
(d) $\sum_{i=0}^{7} a_i = 3$.

(A) $a_7 + a_3 = a_5$ (B) $a_6 - a_2 = a_3$ (C) $\sum_{i=0}^{7} a_i = 4$ (D) $\sum_{i=0}^{7} a_i = 3$

3. Given that the differential equation

$$f(x,y)\frac{dy}{dx} + x^2 + y = 0$$

is exact and $f(0,y) = y^2$, then $f(1,2)$ is

(a) 5.

(b) 4.

(c) 6.

(d) 0.

(A)5 (B)4(C)6(D)0

4. Let $\Omega = \{(x,y) : x^2 + (y-2)^2 < 4\}$ with its boundary $\partial\Omega$. Consider the boundary value problem

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad \text{in } \Omega,$$
$$u = x^2 - y^2 \qquad \text{on } \partial\Omega.$$

Then $\max\{u(x,y) : (x,y) \in \Omega \cup \partial\Omega\}$ is

(a) 0

(b) 1

(c) 2

(d) 3

(A) 0 (B) 1 (C) 2 (D) 3

5. Let $f : [0,1) \to \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$ be the map defined by $f(t) = (\cos t, \sin t)$. Which of the following statement(s) is (are) true?

(a) The map f is a one-one continuous map not on-to.

(b) The map f is a one-one and on-to continuous map.

(c) The map f is one-one and on-to continuous map and it is a homeomorphism.

(d) The map f is one-one and on-to continuous map and it is not a homeomorphism.

6. Let $f : [0,4] \to [1,3]$ be a differentiable function such that $f'(x) \neq 1$ for all $x \in [0,4]$. Then the function f has

(a) at most one fixed point.

(b) unique fixed point.

(c) no fixed point.

(d) more than one fixed point.

7. Which of the following statement(s) is (are) true?

(a) There exists a continuous one-one function from $[a, b]$ to (a, b) for any two real numbers $a < b$.

(b) There exists a continuous on-to function from (a, b) to $[a, b]$ for any two real numbers $a < b$.

(c) Every continuous function $f : [1, 10] \to (2, 8)$ has a fixed point.

(d) There exists a differentiable function $f : \mathbb{R} \to \mathbb{R}$ such that $f'(x) = [x]$ for x in $\mathbb{R}$.

8. We say two points (x_1, y_1) and (x_2, y_2) are equivalent iff $(x_2, y_2) = t(x_1, y_1)$ for some $t > 0$. If they are equivalent we denote by $(x_1, y_1) \sim (x_2, y_2)$. Let $Y := \frac{\mathbb{R}^2}{\sim} := \{[(x, y)] : (x, y) \in \mathbb{R}^2\}$ denote the quotient space under the quotient topology induced by the map $\pi : \mathbb{R}^2 \to Y$ defined by $\pi(x, y) := [(x, y)]$.

Which of the following statement(s) are true?

(a) The space Y is T_1 but not T_2.

(b) The space Y is neither T_1 nor T_2.

(c) The space Y is compact.

(d) The space Y is not compact.

9. Let $C([0, 1]) := \{f : [0, 1] \to \mathbb{R} : f \text{ is continuous}\}$ be the normed linear space with the norm $\| f \|_\infty := \sup\{|f(t)| : t \in [0, 1]\}$ and Y be the vector subspace of $C[0, 1]$ defined by $Y := \{f \in C[0, 1] : f \text{ is differentiable and } f' \text{ is continuous}\}$ with the norm $\| f \|_1 := \sup\{|f(t)| : t \in [0, 1]\} + \sup\{|f'(t)| : t \in [0, 1]\}$. Which of the following statements are true?

(a) The space $(Y, \| \|_\infty)$ is a Banach space.

(b) The space $(Y, \| \|_1)$ is a Banach space.

(c) The map $T : (Y, \| \|_1) \to (C[0, 1], \| \|_\infty)$ defined by $T(f) := f'$ is continuous.

(d) The map $I : (C[0, 1], \| \|_\infty) \to (Y, \| \|_1)$ defined by $I(f) := \int_0^x f(t)dt$ is continuous.

10. The number of connected component of $\mathbb{R}^2 \setminus \mathbb{Q} \times \mathbb{Q}$ is

(a) 1,

(b) 2,

(c) countably infinite,

(d) uncountable.