	Indian Institute of Technology Kanpur Department of Mathematics and Statistics **WRITTEN TEST FOR PH.D. ADMISSIONS IN MATHEMATICS**				
Maximum Marks : 120	Date : December 11, 2017		Time : 90 Minutes		
Name of the Candidate					
Roll Number		Category (Tick One)	GEN	OBC	SC/ST/PwD

INSTRUCTIONS

(1) There are three sections; the first section has true/false questions, the second section is fill in the blanks and the third section has multiple choice questions.
- In the first section, every correct answer will be awarded 3 marks and a wrong answer will be awarded NEGATIVE 3 (-3) marks.
- In the second section, every correct answer will be awarded 3 marks and a wrong answer will be awarded 0 marks.
- The third section has one or two correct answers. In this section
  - each question has four choices.
  - if a wrong answer is selected in a question then that entire question will be awarded 0 marks.
  - the candidate gets full credit of 3 marks, only if he/she selects all the correct answers and no wrong answers; 1 mark will be awarded for an answer to a question if it is partially correct and a wrong answer is not selected.

(2) These question-cum-answer-sheets must be returned to the invigilator before leaving the examination hall.

(3) **Please enter your answers on this page in the space given below.**

True/False Questions

Question Number	Correct Option
1	
2	
3	
4	
5	

Fill In The Blanks Questions

Q. No.	Answer	Q. No.	Answer
1		6	
2		7	
3		8	
4		9	
5		10	

Multiple Choice Questions

Q. No.	Correct Option(s)	Q. No.	Correct Option(s)	Q. No.	Correct Option(s)	Q. No.	Correct Option(s)	Q. No.	Correct Option(s)
1		6		11		16		21	
2		7		12		17		22	
3		8		13		18		23	
4		9		14		19		24	
5		10		15		20		25	

Notations

I. We denote by $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$, the set of natural numbers, integers, rational numbers, real numbers and complex numbers, respectively.
II. We denote by S^n, the n-sphere, that is, $S^n = \{x \in \mathbb{R}^{n+1} : \sum_{i=1}^{n+1} x_i^2 = 1\}$.
III. We denote by S_n the permutation group on n symbols.
IV. The differential operator ∇^2 is given by

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

True/False [15 marks]

(1) If $f : \mathbb{R} \to \mathbb{R}$ is continuous with $\lim_{|x|\to\infty} |f(x)| = 0$, then f is uniformly continuous on $\mathbb{R}$.

(2) There is no continuous bijective map from the sphere S^2 to the circle S^1.

(3) The iterative method $x_{m+1} = g(x_m),\ m \geq 0$, with $g(x) = (x-2)^2 - 6$ for the solution of $x^2 - x - 2 = 0$ converges quadratically in a neighborhood of the root $x = 2$.

(4) Let $(x_e, y_e, z_e)^T$ be the solution of the linear system

$$\begin{aligned} 3x + y - 3z &= 1 \\ x + y - 2z &= 2 \\ 3x + 2y - z &= -3. \end{aligned}$$

If $(x_n, y_n, z_n)^T$ denotes the n-th Gauss-Seidel iteration and $\boldsymbol{e}_n = (x_n, y_n, z_n)^T - (x_e, y_e, z_e)^T, n = 0, 1, 2, \ldots$, denotes the error vector, then, $\|\boldsymbol{e}_n\|_2 \to 0$ as $n \to \infty$ for any non-zero vector $\boldsymbol{e}_0$, where $\|\boldsymbol{r}\|_2 = \left(\sum_{i=1}^3 r_i^2\right)^{1/2}$ for any $\boldsymbol{r} \in \mathbb{R}^3$.

(5) Let $\Omega = \{(x,y) \in \mathbb{R}^2 \mid x^2 + (y - 1/2)^2 < 1\}$ with its boundary $\partial\Omega$ and let $u(x,y)$ be the solution of the following boundary value problem

$$\begin{aligned} \nabla^2 u(x,y) &= 0, \quad (x,y) \in \Omega, \\ u(x,y) &= x^2 + y^2, \quad (x,y) \in \partial\Omega. \end{aligned}$$

Then, $\inf \{u(x,y) : (x,y) \in \Omega \cup \partial\Omega\} = 1/2$.

Fill in the blanks [30 marks]

(1) The value of the contour integral oriented counterclockwise,

$$\oint_{|z|=1} \frac{e^z}{z^{10}} dz$$

is ___________.

(2) Number of subgroups of S_3 is ___________.

(3) A 2×2 real matrix A has an eigenvalue 2 and its determinant is 6. Then the sum of entries of the principal diagonal of A is ___________.

(4) Let $A = \{x \in S_6 \mid x \text{ is a product of two disjoint 3-cycles}\}$. The number of elements in A is ________ .

(5) Let G denote the group $\{1, 5, 7, 11, 13, 17, 19, 23, 25, 29, 31, 35\}$ under multiplication modulo 36. Then the order of 5 in G is ________ .

(6) Number of 5-Sylow subgroups of the symmetric group S_5 is ________ .

(7) If $y = y(x)$ is the solution of the initial value problem

$$xy' = y + y^2, \quad y(-2) = -2,$$

then $y(-3) =$ ________ .

(8) If $y = y_p(x)$ is a particular solution of $y'' - y' = x^2$, then $y_p(x) =$ ________ .

(9) The curve $y = y(x)$ satisfies $y'' = y' - 1$ and touches the $x-$axis at the origin. Then $y(x) =$ ________ .

(10) Let $u(x,t)$ be the solution of the initial value problem

$$u_t = 2u_{xx}, \qquad 0 < x < 1;\ t > 0,$$
$$u(x,0) = 1 - x^2, \quad 0 \le x \le 1,$$
$$u(0,t) = 1 - 4t, u(1,t) = -4t, \quad t > 0.$$

If P and Q respectively denote the maximum and minimum values of $u(x,t)$ in the closed rectangle $R = \{0 \le x \le 1,\ 0 \le t \le 2\}$, then $P - Q =$ ________ .

Questions with one or two correct choices [75 marks]

(1) Let $\mathcal{P}_n$ denote the set of all polynomials of degree at most n. The quadrature rule

$$Q(f) = \frac{1}{18}\left(5f\left(\frac{1-\sqrt{3/5}}{2}\right) + 8f\left(\frac{1}{2}\right) + 5f\left(\frac{1+\sqrt{3/5}}{2}\right)\right)$$

for approximation of $I(f) = \int_0^1 f(x)dx$ is exact for all f in

a. $\mathcal{P}_4$ **b.** $\mathcal{P}_5$ **c.** $\mathcal{P}_6$ **d.** $\mathcal{P}_7$

(2) Consider the initial value problem

$$y'' + y = x\big|\sin(1/x)\big|, \qquad x \in (-1,1),$$

with $y(0) = y'(0) = 1$. Which of the following is true

a. It has infinitely many solutions.
b. It has a unique solution.
c. It has exactly two solutions.
d. It has no solution.

(3) Let $\Omega = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$ be a unit disk in $\mathbb{R}^2$ and $\partial\Omega$ be the boundary of Ω. Let $u(x,y)$ be the solution of the Dirichlet problem

$$\nabla^2 u = 0 \quad \text{for } (x,y) \in \Omega$$
$$u(x,y) = 1 + 2x^2 \quad \text{for } (x,y) \in \partial\Omega.$$

Then $u(3/4, 1/4)$ is equal to

a. 17/8 **b.** 1/2 **c.** 5/2 **d.** 3/2

(4) Let G be a group and H be a normal subgroup of G such that H is generated by an element a of order 6. Let $b \in G$. Then bab^{-1} is

a. a or a^2 **b.** a or a^3 **c.** a or a^4 **d.** a or a^5

(5) Let G be a group. Then which of the following statement(s) is/are true ?

a. If $G/Z(G)$ is cyclic then G need not be abelian, where $Z(G)$ is the centre of G.
b. If G has at least two elements then there always exists a nontrivial homomorphism from $\mathbb{Z}$ to G.
c. If $|G| = p^3$ for some prime p, then G is necessarily abelian.
d. If G is nonabelian, it may not have a nontrivial automorphism.

(6) Which of the following statement(s) is/are true ?

a. $\mathbb{R}$ and $\mathbb{C}$ are isomorphic as additive groups.
b. $\mathbb{R}$ and $\mathbb{C}$ are isomorphic as rings.
c. $\mathbb{R}$ and $\mathbb{C}$ are isomorphic as fields.
d. $\mathbb{R}$ and $\mathbb{C}$ are isomorphic as vector spaces over $\mathbb{Q}$.

(7) Let R denote the ring $\{0, 1, 2, \ldots, 20\}$ under addition and multiplication modulo 21. Then the number of invertible elements in R is

a. 1 **b.** 4 **c.** 8 **d.** 12

(8) Let R and S be two commutative rings with unity and $f : R \to S$ be a ring homomorphism. Then which of the following statement(s) is/are true ?

a. Image of an ideal of R is always an ideal in S.
b. Image of an ideal of R is an ideal of S if f is injective.
c. Image of an ideal of R is an ideal of S if f is surjective.
d. Image of an ideal of R is an ideal of S if S is a field.

(9) Which of the following statement(s) is/are true ?

a. For every $n \in \mathbb{N}$ there exists a commutative ring with unity whose characteristic is n.
b. There exists a integral domain with unity whose characteristic is 57.
c. For every postive integers m and n, the characteristic of the ring $\frac{\mathbb{Z}}{m\mathbb{Z}} \times \frac{\mathbb{Z}}{n\mathbb{Z}}$ is mn.
d. For a prime number p, a commutative ring with unity of characteristic p contains a subring isomorphic to $\frac{\mathbb{Z}}{p\mathbb{Z}}$.

(10) Let A and P be 3×3 real matrices such that P is invertible and $P^{-1}AP$ is diagonal. Then the INCORRECT statement(s) is/are :

a. all eigenvalues of A must be real.
b. all eigenvalues of A must be distinct.
c. A has three linearly independent eigenvectors.
d. the subspace spanned by eigenvectors of A is $\mathbb{R}^3$.

(11) Let A be a 2×2 matrix of rank 1. Then A is

a. diagonalizable and non-singular.
b. diagonalizable and nilpotent.
c. neither diagonalizable nor nilpotent.
d. either diagonalizable or nilpotent.

(12) The characteristic polynomial of a matrix A is $x^2 - x - 1$. Then

a. A^{-1} does not exist.
b. A^{-1} exists but cannot be determined from the data.
c. $A^{-1} = A + I$.
d. $A^{-1} = A - I$.

(13) Let $f : \mathbb{R} \to \mathbb{R}$ be defined as $f(x) = \sin(x^2)$. Which of the following statement(s) is/are true?

a. f is uniformly continuous on $\mathbb{R}$.
b. f is NOT uniformly continuous on $\mathbb{R}$.
c. f is uniformly continuous on $(0, 1)$.
d. f is NOT uniformly continuous on $(0, 1)$.

(14) Consider the sequence $(a_n)_{n=1}^{\infty}$ defined by

$$a_n = \begin{cases} \frac{1}{n}, & \text{if } n = 2^k, k = 0, 1, 2, ... \\ 0, & \text{otherwise.} \end{cases}$$

Then

a. $\sum_{n=1}^{\infty} a_n < \infty$ but $\lim_{n\to\infty} (na_n)$ does not exist.
b. $\sum_{n=1}^{\infty} a_n$ is divergent but $\lim_{n\to\infty} (na_n) = 0$.
c. $\sum_{n=1}^{\infty} a_n < \infty$ and $(na_n)_{n=1}^{\infty}$ has a subsequence with limit 1.
d. $\sum_{n=1}^{\infty} a_n < \infty$ and $\lim_{n\to\infty} (na_n) = 0$.

(15) Consider $(\mathbb{R}^2, d)$ with the usual Euclidean metric d. Let $X = \{(x, \frac{1}{x}) \in \mathbb{R}^2 \mid x > 0\} \cup \{(0, y) \in \mathbb{R}^2 \mid y \geq 0\} \cup \{(x, 0) \in \mathbb{R}^2 \mid x \geq 0\}$. Then

a. X is open but not closed.
b. X is neither open nor closed.
c. X is closed but not open.
d. X is open and closed.

(16) Let $f, g : [0, 1] \to \mathbb{R}$ be functions defined as

$$f(t) = \begin{cases} \frac{\sin t}{t}, & t \neq 0 \\ 0, & t = 0, \end{cases} \qquad g(t) = \begin{cases} \frac{\sin t}{t^2}, & t \neq 0 \\ 0, & t = 0. \end{cases}$$

Then

a. both f and g are Riemann integrable on $[0, 1]$.
b. f is Riemann integrable but g is not Riemann integrable on $[0, 1]$.
c. g is Riemann integrable on $[0, 1]$ but f is not Riemann integrable on $[0, 1]$.
d. both f and g are NOT Riemann integrable on $[0, 1]$.

(17) Let $p(z)$ be a non-zero polynomial in complex variable z. Let

$$f(z) = p(z)e^{\frac{1}{z}} \quad \text{for} \quad z \in \mathbb{C} \setminus \{0\}.$$

Then
a. f has a removable singularity at $z = 0$.
b. f has a pole at $z = 0$ with residue equal to 0.
c. f has an essential singularity at $z = 0$.
d. f has a pole at $z = 0$ with residue equal to 1.

(18) For $z \in \mathbb{C}$, $\lim_{|z| \to \infty} |e^z|$

a. does not exist in $\mathbb{R}$.
b. is equal to 1.
c. is equal to 0.
d. is ∞.

(19) If $a = \lim_{n\to\infty}(1 + \frac{1}{n^2})^n$ and $b = \lim_{n\to\infty}(1 + \frac{1}{n})^{n^2}$, then

a. $a = 1, b = \infty$. **b.** $a = 0, b = 1$. **c.** $a = \infty, b = 1$. **d.** $a = 1, b = 0$.

(20) Let

$$\ell^2 = \{x = (x_n) : x_n \in \mathbb{R}, \sum_{n=1}^{\infty} |x_n|^2 < \infty\} \text{ with } ||x||_2 := \left(\sum_{n=1}^{\infty} |x_n|^2\right)^{\frac{1}{2}}.$$

If $A = \{x = (x_n) \in \ell^2 : x_n = 0$ for all but finitely many n's$\}$, then

a. A is open but not closed.
b. A is both open and closed.
c. A is closed but not open.
d. A is neither open nor closed.

(21) $(C[0,1], ||.||_\infty)$ denotes the set of all real-valued continuous functions on $[0,1]$ with $||f||_\infty := \sup\{|f(t)|, t \in [0,1]\}$. For each $x \in [0,1]$, define

$$Tf(x) = \int_0^x f(t)dt.$$

Then

a. T is injective but not surjective.
b. T is surjective but not injective.
c. T is bijective.
d. T is neither injective nor surjective.

(22) Let $f : (S, d_S) \to (T, d_T)$ be a bijective continuous function of metric spaces, and $f^{-1} : (T, d_T) \to (S, d_S)$ be the inverse function. Which of the following statement(s) is/are true?

a. f^{-1} is always a continuous function.
b. f^{-1} is continuous if (S, d_S) is compact.
c. f^{-1} is continuous if (T, d_T) is compact.
d. f^{-1} is continuous if (S, d_S) is connected.

(23) Consider $\ell^2 = \{x = (x_n) : x_n \in \mathbb{R}, \sum_{n=1}^{\infty} |x_n|^2 < \infty\}$. Let $T, S : \ell^2 \to \ell^2$ be defined as

$$T(x_1, x_2, x_3, \cdots) = (x_1, \frac{1}{2}x_2, \frac{1}{3}x_3, \cdots) \quad \text{and} \quad S(x_1, x_2, x_3, \cdots) = (0, x_1, x_2, x_3, \cdots)$$

for all $x = (x_n) \in \ell^2$. Which of the following statement(s) is/are true?

a. S and T have eigenvalues.
b. T does not have any eigenvalues but S has eigenvalues.
c. T has eigenvalues but S does not have any eigenvalues.
d. Neither T nor S have eigenvalues.

(24) $C[0,1]$ denotes the set of all real-valued continuous functions on $[0,1]$. Let $\{f_n\}$ be a sequence of functions in $C[0,1]$ such that $\lim_{n\to\infty} f_n(x) = f(x)$ for each $x \in [0,1]$. Then

a. f is continuous and $\int_0^{1-\frac{1}{n}} f_n \to \int_0^1 f$ as $n \to \infty$.
b. f is continuous but $\int_0^{1-\frac{1}{n}} f_n \not\to \int_0^1 f$ as $n \to \infty$.
c. If $f_n \to f$ uniformly, then f is continuous and $\int_0^{1-\frac{1}{n}} f_n \to \int_0^1 f$ as $n \to \infty$.
d. If $f_n \to f$ uniformly, then f is continuous but $\int_0^{1-\frac{1}{n}} f_n \not\to \int_0^1 f$ as $n \to \infty$.

(25) Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. Let A be a bounded set in $\mathbb{R}$ and B be a closed and bounded set in $\mathbb{R}$. Then

a. $f(A)$ is bounded and $f(B)$ is closed and bounded.
b. If $x_n \to x$ in A, then $f(x_n) \to f(x)$ in $f(A)$.
c. $f(A)$ is not bounded and $f(B)$ is closed.
d. $f(A)$ is not bounded and $f(B)$ is bounded.