Name (In BLOCK letters):

Roll/Application Number:

Category (Tick the appropriate one) : GEN/OBC-NCL/EWS/SC/ST/PwD

**Instructions**

1. This question booklet consists of 20 questions, divided into four sections, with 5 questions in each section.

2. Each question may have more than one correct options.

3. Each question carries 3 marks.

4. An examinee will be awarded 3 marks for a totally correct answer. For the questions containing multiple correct options, 1 mark will be given for partially correct answers, provided no wrong option has been chosen in addition. In all other cases, no marks will be awarded.

5. This question-cum-answer booklet must be returned to the invigilator before leaving the examination hall.

6. Please enter your answers ONLY on this page in the space given below.

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Total marks obtained:
Notations and conventions

Throughout this question paper, the following notations and conventions will be adopted:

1. \( \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C} \) denote the set of all integers, rationals, real numbers and complex numbers respectively.

2. For \( p \in [1, \infty) \), \( L^p(\mathbb{R}) \) denotes the set of all measurable functions \( f : \mathbb{R} \to \mathbb{C} \) with the property that \( \int_{\mathbb{R}} |f(t)|^p \, dt < \infty \).

3. \( L^\infty(\mathbb{R}) \) stands for the set of all bounded measurable functions from \( \mathbb{R} \) to \( \mathbb{C} \).

4. \( \mathbb{D}^2 := \{ z \in \mathbb{C} : |z| < 1 \} \) is the open unit disk in \( \mathbb{C} \).

5. \( \mathbb{H}^2 := \{ x + iy \in \mathbb{C} : y > 0 \} \) is the upper half plane in \( \mathbb{C} \).

6. \( \hat{\mathbb{C}} := \mathbb{C} \cup \{ \infty \} \) is the Riemann sphere.

7. For \( n \in \mathbb{N} \), we let \( S^n := \{ (x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} : x_1^2 + \cdots + x_{n+1}^2 = 1 \} \).

8. For \( n \in \mathbb{N} \) and a field \( \mathbb{K} \), \( M_n(\mathbb{K}) \) denotes the set of all \( n \times n \) matrices with entires from \( \mathbb{K} \). When \( \mathbb{K} = \mathbb{R} \) or \( \mathbb{C} \), we assume that \( M_n(\mathbb{K}) \) is identified with \( \mathbb{K}^{n^2} \) by the following map

\[
(a_{ij})_{i,j=1}^n \mapsto (a_{11}, \cdots, a_{1n}, \cdots, a_{n1}, \cdots, a_{nn}),
\]

and thus equipped with the natural metric topology. Consequently, the subgroups \( GL_n(\mathbb{K}), SL_n(\mathbb{K}) \) etc. inherit the subspace topology from \( M_n(\mathbb{K}) \).

9. Given any commutative ring \( R \) with identity and \( a \in R \), \( (a) \) denotes the principle ideal in \( R \) generated by the element \( a \).
Section A

1. Let \( g : \mathbb{R} \to \mathbb{R} \) be a bounded continuous function. Choose the correct statement(s) from the following:

   (a) The sequence \( \left\{ \int_{x_n}^{y_n} g(t) \, dt \right\}_{n=1}^{\infty} \) is Cauchy for any pair of Cauchy sequences \( \{x_n\}_{n=1}^{\infty} \) and \( \{y_n\}_{n=1}^{\infty} \) in \( \mathbb{R} \).

   (b) The function defined by

   \[ (x, y) \mapsto \int_{x}^{y} g(t) \, dt, \quad \text{for all } (x, y) \in \mathbb{R}^2, \]

   may not be differentiable at every point of \( \mathbb{R}^2 \).

   (c) The function defined in (1b) is continuous on \( \mathbb{R}^2 \) but not uniformly continuous on \( \mathbb{R}^2 \).

   (d) If \( g(x_0) \neq 0 \), then one can find open intervals \( I, J \) in \( \mathbb{R} \) containing \( x_0 \) such that the set

   \[ S := \left\{ (x, y) \in I \times J : \int_{x}^{y} g(t) \, dt = 0 \right\} \]

   is the graph of some continuously differentiable function \( \varphi : I \to J \), i.e.,

   \[ S = \{ (x, \varphi(x)) : x \in I \}. \]

2. Let \( f \) be a continuous real valued function on \( \mathbb{R} \) with compact support. Pick out the correct statement(s) from below:

   (a) \( f(\mathbb{R}) \) is measurable.

   (b) The Lebesgue measure of \( f(\mathbb{R}) \) can be 0 even when \( f \) is nonconstant.

   (c) The boundary of \( f^{-1}(\mathbb{R} \setminus \alpha) \) has positive measure for at most countably many \( \alpha \in \mathbb{R} \).

   (d) For every \( p \in [1, \infty] \), there exists a continuous function \( g : \mathbb{R} \to \mathbb{R} \) which vanishes identically on \( \mathbb{R} \setminus f(\mathbb{R}) \) but \( g \notin L^p(\mathbb{R}) \).

3. Let \( \gamma : [0, 1] \to \mathbb{C} \) be continuously differentiable and \( \gamma^* \) denote its range. Assume \( \gamma(0) = \gamma(1) \). Define \( \eta_\gamma : \mathbb{C} \to \mathbb{C} \) by the following

   \[ \eta_\gamma(z) = \begin{cases} 
   \int_{\gamma} \frac{dw}{w - z} & \text{if } z \in \mathbb{C} \setminus \gamma^* \\
   1 & \text{if } z \in \gamma^* 
   \end{cases} \]

   Find the true statement(s) from below:

   (a) The restriction of \( \eta_\gamma \) on \( \mathbb{C} \setminus \gamma^* \) is locally constant.

   (b) \( \eta_\gamma \) vanishes at infinity, i.e. \( \forall \varepsilon > 0 \), there exists a compact subset \( K \) such that \( |\eta_\gamma(z)| < \varepsilon \) holds for any \( z \notin K \).

   (c) \( \eta_\gamma \) does not vanish identically on the complement of any compact subset of \( \mathbb{C} \).

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4. Let $f$ be an entire function. We define $\varphi : (0, \infty) \longrightarrow [0, \infty)$ by

$$\varphi(t) := \sup_{|z|=t} |f(z)|, \text{ for all } t > 0.$$ 

Which of the following statement(s) is/are always true?

(a) $\varphi$ is bounded.
(b) $\varphi$ has a zero, i.e. $\exists t_0 \in (0, \infty)$ such that $\varphi(t_0) = 0$.
(c) $\varphi(t) \xrightarrow{t \to \infty} 0$.
(d) $\varphi$ is continuous except at countably many points in every closed and bounded interval $[a, b] \subseteq (0, \infty)$.

5. Let $\mathcal{H}$ be a Hilbert space over $\mathbb{C}$. Consider a bounded linear operator $T : \mathcal{H} \longrightarrow \mathcal{H}$ such that $||Tv|| \leq ||v||$, for every $v \in \mathcal{H}$. Denote the adjoint of $T$ by $T^*$, which is defined by the following property:

$$\langle Tv, w \rangle = \langle v, T^*w \rangle, \text{ for all } v, w \in \mathcal{H}.$$ 

Pick the FALSE statement(s) from the following:

(a) $T^*v = v \Longrightarrow Tv = v$, where $v \in \mathcal{H}$.
(b) The converse of (5a) holds true.
(c) If $T$ is an isometry (i.e. $||Tv|| = ||v||$, for all $v \in \mathcal{H}$) then $T^*T = I$, where $I$ is the identity operator on $\mathcal{H}$.
(d) None of the above is true.
Section B

6. Consider the group $\mathbb{Z}/2019\mathbb{Z}$ with addition modulo 2019. Which of the following groups admit(s) an homomorphism onto $\mathbb{Z}/2019\mathbb{Z}$?

(a) $\mathbb{Z}/26247\mathbb{Z}$ with respect to addition modulo 26247.
(b) $\mathbb{Q}$ with respect to usual addition.
(c) $\{z \in \mathbb{C} : \exists n \in \mathbb{Z} \text{ such that } z^n = 1\}$ with respect to usual multiplication of complex numbers.
(d) $\{z \in \mathbb{C} : |z| = 1\}$ with respect to usual multiplication of complex numbers.

7. Let $R$ be a commutative ring with identity. Let $a \in R$ be such that $a^{2019} = 0$ and $u$ be a unit in $R$. Then the cardinality of the quotient ring $R/(u + a)$ is

(a) 1
(b) same as that of $R$.
(c) 2019.
(d) 2018.

8. Consider the subfields $\mathbb{Q}(\sqrt{5})$ and $\mathbb{Q}(\sqrt{7})$ of $\mathbb{C}$. Which of the following statements is/are true?

(a) They are isomorphic as abelian groups.
(b) They are isomorphic as vector spaces.
(c) They are isomorphic as rings.
(d) They are isomorphic as fields.

9. Let $V$ be the subspace of the vector space of all $5 \times 5$ real symmetric matrices with the property that characteristics polynomial of each element in $V$ is of the form $x^5 + ax^3 + bx^2 + cx + d$. Then the dimension of $V$ is:

(a) 15.
(b) 14.
(c) 10.
(d) 12.

10. Suppose that $A$ is a $5 \times 5$ real matrix all of whose entries are 1. Find the correct one(s) from the statements given below.

(a) A is not diagonalizable over $\mathbb{R}$.
(b) A is idempotent.
(c) A is nilpotent.
(d) The minimal polynomial and the characteristics polynomial of $A$ are not same.
Section C

11. Let \( f, g : X \rightarrow Y \) be continuous maps where \( X \) is an arbitrary topological space and \( Y \) is a Hausdorff space. Find the true statement(s) from the following:

(a) The subset \( \{ x \in X : f(x) = g(x) \} \subseteq X \) is closed in \( X \).
(b) Even if \( f \neq g \), there can exist a dense subset \( D \subseteq X \) such that \( f(x) = g(x) \) for all \( x \in D \).
(c) If \( f : X \rightarrow Y \) is injective then \( X \) is also a Hausdorff topological space.
(d) None of the above statements is true.

12. The function given by \( z \mapsto \frac{az + b}{cz + d} \), where \( a, b, c, d \in \mathbb{R} \) such that \( ad - bc > 0 \), is a

(a) holomorphic map from \( \mathbb{D}^2 \) onto itself with an holomorphic inverse.
(b) holomorphic map from \( \mathbb{H}^2 \) onto itself with an holomorphic inverse.
(c) holomorphic onto function on \( \mathbb{C} \) with an holomorphic inverse.
(d) holomorphic map from \( \hat{\mathbb{C}} \) onto itself with an holomorphic inverse.

13. Which of the following statements is/are true?

(a) If \( X \subseteq \mathbb{R}^2 \) is path connected, then \( X \) is also path connected.
(b) Let \( X \subseteq \mathbb{R} \). Then \( X \) is connected if and only if \( X \) is path connected.
(c) For \( n \in \mathbb{N} \), let \( N \) and \( S \) be respectively the points \((0, \ldots, 0, 1)\) and \((0, \ldots, 0, -1)\) in \( \mathbb{R}^{n+1} \). Then \( S^n \setminus \{N, S\} \) is path connected.
(d) The set \( X = \{(x, y) \in \mathbb{R}^2 : xy = \pm 1, x > 0\} \) is path connected.

14. Consider the function \( f : (0, \infty) \rightarrow \mathbb{R} \) defined by \( f(x) = \sin \left( \frac{1}{x} \right) \), for all \( x \in (0, \infty) \). Pick the correct statement(s) from the following:

(a) \( f \) has countably many fixed points.
(b) For any \( a > 0 \), the restriction map \( f|_{(a, \infty)} : (a, \infty) \rightarrow \mathbb{R} \) has infinitely many fixed points.
(c) The restriction map \( f|_{(0, 1]} : (0, 1] \rightarrow \mathbb{R} \) has finitely many fixed points.
(d) The restriction map \( f|_{[1, \infty)} : [1, \infty) \rightarrow \mathbb{R} \) has no fixed point.

15. Consider the following functions:

\[
f : GL_n(\mathbb{C}) \rightarrow \mathbb{C} \setminus \{0\}, \quad f(A) := \det A, \quad \text{for all } A \in GL_n(\mathbb{C});
\]

and

\[
g : \mathbb{R} \rightarrow M_2(\mathbb{R}), \quad g(x) := \begin{pmatrix} \cos x & -\sin x \\ \sin x & \cos x \end{pmatrix}, \quad \text{for all } x \in \mathbb{R}.
\]

Choose the correct one(s) from the statements given below:

(a) Let \( \mathcal{K} \) denote the Cantor set and \( GL_n(\mathcal{K}) \) denote the set of all \( n \times n \) invertible matrices having entries from \( \mathcal{K} \). Then \( f(GL_n(\mathcal{K})) \) is closed.
(b) Let $\mathcal{K}$ be as above. Then $g(\mathcal{K})$ is closed.

(c) $GL_n(\mathbb{C})$ has infinitely many closed subgroups containing $SL_n(\mathbb{C})$.

(d) All the above three statements are true.
16. If the function $K : [0,1] \times [0,1] \rightarrow \mathbb{R}$ is such that

$$u(t) = c_1 + c_2 t + \int_0^t K(t,s) f(s) \, ds$$

is the general solution to the ODE

$$\frac{d^2}{dt^2} u(t) = f(t), \quad 0 < t < 1,$$

where $f$ is continuous on $[0,1]$, then $K(t,s) =$

(a) $s - t$.
(b) $t - s$.
(c) $t(s - t)$.
(d) $s(t - s)$.

17. The differential equation

$$y = x \frac{dy}{dx} - \left( \frac{dy}{dx} \right)^2$$

has more than one solutions passing through the point

(a) $(0,1)$.
(b) $(1,1)$.
(c) $(2,1)$.
(d) $(2,-1)$.

18. Let $u \in \mathcal{C}^2(\mathbb{R} \times [0,\infty))$ solves the initial value problem for the wave equation in one dimension:

$$
\begin{cases}
  u_{tt}(x,t) - u_{xx}(x,t) = 0 & \text{for all } (x,t) \in \mathbb{R} \times [0,\infty) \\
  u(x,0) = f(x) & \text{for all } x \in \mathbb{R} \\
  u_t(x,0) = g(x) & \text{for all } x \in \mathbb{R},
\end{cases}
$$

where $f$ and $g$ are infinitely differentiable functions with compact supports. For $t > 0$, define

$$K(t) := \frac{1}{2} \int_{-\infty}^{\infty} u_t^2(x,t) \, dx \quad \text{and} \quad P(t) := \frac{1}{2} \int_{-\infty}^{\infty} u_x^2(x,t) \, dx.$$ 

Choose the correct one(s) from the following statements.

(a) The function $K(t) + P(t)$ is a constant function of time.
(b) The function $K(t) + P(t)$ can be a non constant function of time.
(c) The function $K(t) + P(t)$ is always continuous.
(d) The function $K(t) + P(t)$ is a polynomial of degree 3.

19. Let $u : \mathbb{R}^2 \to \mathbb{R}$ be a $C^1$ function (i.e., both partial derivatives are continuous). Consider the following problem:

\[
\begin{aligned}
  u_t(x,t) + u_x(x,t) &= 0 & \text{for all } (x,t) \in \mathbb{R}^2 \\
  u(x,x) &= 1 & \text{for all } x \in \mathbb{R}.
\end{aligned}
\]

Which of the following statements is/are correct?

(a) The above problem has unique solution.
(b) The above problem has infinitely many solutions.
(c) There exists a solution $u$ of the above problem such that $u(1,0) = 5$.
(d) The above problem has at most finitely many solutions.

20. Consider the following function

\[
\phi : \mathbb{R} \setminus \{0\} \to \mathbb{R}, \quad \phi(r) := \frac{1}{2r} \int_{1-r}^{1+r} u(s) \, ds, \quad \text{for all } r \in \mathbb{R} \setminus \{0\};
\]

where $u'' = 0$ on $\mathbb{R}$ with $u(1) \neq 0$. Then

(a) $\phi'(r) = 0$ for all $r \in \mathbb{R} \setminus \{0\}$.
(b) $\phi(1) = u(1)$.
(c) $\lim_{r \to 0} \phi(r) = u(0)$.
(d) $\phi$ is an odd function.
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