# Technical Report on Uplink Sum-Rate and Power Scaling Laws for Multi-User Massive MIMO-FBMC Systems

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This technical report derives the closed-form signal to interference ratio (SIR) for the uplink of a single-cell multi-user massive MIMO-FBMC system in the presence of a carrier frequency offset (CFO) and perfect channel state information (CSI) with the zero-forcing (ZF) receiver employed at the base station (BS). Numerical results are presented to compare the SIR of FBMC and OFDM-based massive MIMO systems in the presence of CFO.

Let $\epsilon_u$ represent the normalized CFO for the $u$th user in the cell. Similar to (3) of the manuscript, the signal received at the $n$th BS antenna in the presence of CFO, with the noise neglected, can be obtained as

$$y^n_{\text{cfo}}[l] = \sum_{u=1}^{U} \Big(s^u[l] * g^{n,u}[l]\Big) e^{j2\pi\epsilon_u l/M}, \quad \text{for} \;\; 1 \leq n \leq N. \tag{1}$$

Employing (1) and (2) from the manuscript, the expression for $y^n_{\text{cfo}}[l]$ can be expanded as

$$y^n_{\text{cfo}}[l] = \sum_{u=1}^{U} \sum_{i=0}^{L_h-1} g^{n,u}[i] \sum_{m=0}^{M-1} \sum_{k\in\mathbb{Z}} d^u_{m,k} p\big[l-i-kM/2\big] e^{j2\pi m(l-i)/M} e^{j\phi_{m,k}} e^{j2\pi\epsilon_u l/M}. \tag{2}$$

As shown in (4) in the manuscript, we have $p\big[l-i-kM/2\big] \approx p\big[l-kM/2\big]$ for $i \in [1 \;\, L_h]$. Thus, $y^n_{\text{cfo}}[l]$ above can be simplified to

$$y^n_{\text{cfo}}[l] = \sum_{u=1}^{U} G^{n,u}_m \sum_{m=0}^{M-1} \sum_{k\in\mathbb{Z}} d^u_{m,k} p\big[l-kM/2\big] e^{j2\pi ml/M} e^{j\phi_{m,k}} e^{j2\pi\epsilon_u l/M}, \tag{3}$$

where the channel's frequency response (CFR) $G^{n,u}_{\bar{m}}$ at the $m$th subcarrier is determined as $G^{n,u}_m = \sum_{l=0}^{L-1} g^{n,u}[l] e^{-j2\pi ml/M}$. As described in the paragraph below (3) of the manuscript, the demodulated signal $\tilde{y}^n_{\bar{m},\bar{k}}$ on the $n$th BS antenna at subcarrier $\bar{m}$ and symbol time $\bar{k}$ is obtained as $\tilde{y}^n_{\bar{m},\bar{k}} = \sum_{l=-\infty}^{+\infty} y^n_{\text{cfo}}[l]\chi^*_{\bar{m},\bar{k}}[l]$. Substituting $\chi_{\bar{m},\bar{k}}[l]$ from (2) of the manuscript and $y^n_{\text{cfo}}[l]$ from (3), $\tilde{y}^n_{\bar{m},\bar{k}}$ can be expanded as

$$\tilde{y}^n_{\bar{m},\bar{k}} = \sum_{u=1}^{U} G^{n,u}_m \sum_{m=0}^{M-1} \sum_{k\in\mathbb{Z}} d^u_{m,k} e^{j(\phi_{m,k}-\phi_{\bar{m},\bar{k}})} \lambda^{m,k}_{\bar{m},\bar{k}}(\epsilon_u), \tag{4}$$

where

$$\lambda^{m,k}_{\bar{m},\bar{k}}(\epsilon_u) = \sum_{l=-\infty}^{\infty} p\big[l - kM/2\big] p\big[l - \bar{k}M/2\big] e^{j2\pi(m-\bar{m})l/M} e^{j2\pi\epsilon_u l/M}. \tag{5}$$

Substituting $l = q + \frac{(k+\bar{k})M}{4}$ in the expression for $\lambda^{m,n}_{\bar{m},\bar{k}}(\epsilon_u)$ in (5), one obtains

$$\begin{aligned}
\lambda^{m,k}_{\bar{m},\bar{k}}(\epsilon_u) &= e^{j\pi(m-\bar{m})(k+\bar{k})/2} e^{j\pi\epsilon_u(k+\bar{k})/2} \sum_{q=-\infty}^{\infty} p\Big[q + \frac{(\bar{k}-k)M}{4}\Big] p\Big[q - \frac{(\bar{k}-k)M}{4}\Big] e^{j2\pi\big((m-\bar{m})+\epsilon_u\big)q/M} \\
&= e^{j\pi(m-\bar{m})(k+\bar{k})/2} e^{j\pi\epsilon_u(k+\bar{k})/2} A_p\left(\frac{(\bar{k}-k)M}{2}, (m-\bar{m}) + \epsilon_u\right),
\end{aligned} \tag{6}$$

where $A_p\left(M, \nu_0\right)$, termed as the discrete ambiguity function, is defined as

$$A_p\left(M, \nu_0\right) = \sum_{q=-\infty}^{\infty} p\Big[q + \frac{M}{2}\Big] p^*\Big[q - \frac{M}{2}\Big] e^{j2\pi\nu_0 q/M}.$$

The received signal $\tilde{y}^n_{\bar{m},\bar{k}}$ in (4) can be separated into the desired and interference component as

$$\tilde{y}^n_{\bar{m},\bar{k}} = \sum_{u=1}^{U} G^{n,u}_{\bar{m}} \left( d^u_{\bar{m},\bar{k}} \lambda^{\bar{m},\bar{k}}_{\bar{m},\bar{k}}(\epsilon_u) + J^u_{\bar{m},\bar{n}}(\epsilon_u) \right), \tag{7}$$

where the interference component

$$J^u_{\bar{m},\bar{k}}(\epsilon_u) = \sum_{(m,n)\in\Omega_{\bar{m},\bar{k}}} d^u_{m,k} e^{j(\phi_{m,k}-\phi_{\bar{m},\bar{k}})} \lambda^{m,k}_{\bar{m},\bar{k}}(\epsilon_u). \tag{8}$$

The above expression also exploited the fact that the dominant component of the interference arises from the neighborhood $\Omega_{\bar{m},\bar{k}}$ of the desired symbol at the index $(\bar{m}, \bar{k})$, and the CFR $G^{n,u}_{\bar{m}}$ can be assumed to be constant over this neighborhood. However, note that due to the presence of a CFO, the interference is not a purely imaginary quantity. This is in contrast to (6) of the manuscript, where the interference is a purely imaginary quantity in the absence of synchronization errors. The quantity $\lambda^{\bar{m},\bar{k}}_{\bar{m},\bar{k}}(\epsilon_u)$ in the signal component in (7) can be evaluated using (6) as

$$\lambda^{\bar{m},\bar{k}}_{\bar{m},\bar{k}}(\epsilon_u) = e^{j\pi\epsilon_u\bar{k}} A_p(0, \epsilon_u).$$

Substituting $\lambda_{\bar{m},\bar{k}}^{m,k}(\epsilon_u)$ from (6) together with the relationships $m - \bar{m} = m_0$, $k - \bar{k} = k_0$, $\phi_{m,k} = \frac{\pi}{2}(m+k) - \pi mk$, the expression for the interference in (8) can be recast as

$$J_{\bar{m},\bar{k}}^{u}(\epsilon_u) = \sum_{(m_0^{(0)},k_0^{(0)})\in\Omega_{\bar{m},\bar{k}}} d_{\bar{m}+m_0,\bar{k}+k_0}^{u} e^{j\pi(m_0+k_0-m_0k_0)/2} e^{-j\pi\bar{m}k_0} e^{j\pi\epsilon_u(k_0+2\bar{k})/2}$$
$$A_p\left(\frac{-k_0M}{2}, m_0+\epsilon_u\right). \tag{9}$$

In the above expression, the notation $(m_0^{(0)}, k_0^{(0)})$ represents all the $(m_0, k_0)$ points in the neighborbood excluding $(m_0, k_0) = (0, 0)$. For convenience, (7) can be written in a vectorial form as

$$\tilde{\mathbf{y}}_{\bar{m},\bar{k}}^{\text{cfo}} = \mathbf{G}_{\bar{m}} \tilde{\mathbf{d}}_{\bar{m}.\bar{k}}^{\text{cfo}}, \tag{10}$$

where $\tilde{\mathbf{y}}_{\bar{m},\bar{k}}^{\text{cfo}} = [\tilde{y}_{\bar{m},\bar{k}}^{1}, \tilde{y}_{\bar{m},\bar{k}}^{2}, \ldots, \tilde{y}_{\bar{m},\bar{k}}^{N}]^T$ is the $N \times 1$ received vector at the BS in the presence of CFO, $\mathbf{G}_{\bar{m}} = \mathbf{H}_{\bar{m}}\mathbf{D}^{1/2}$ is the $N \times U$ CFR matrix. The $U \times 1$ vector $\tilde{\mathbf{d}}_{\bar{m}.\bar{k}}^{\text{cfo}}$ comprises the data and interference caused by the CFO. Assuming perfect CSI knowledge at the BS, the signal after ZF processing at the BS is given as $\tilde{\mathbf{d}}_{\bar{m}.\bar{k}}^{\text{cfo}} = \mathbf{G}_{\bar{m}}^{\dagger} \tilde{\mathbf{y}}_{\bar{m},\bar{k}}^{\text{cfo}}$, where $\mathbf{G}_{\bar{m}}^{\dagger}$ is the pseudo-inverse of the CFR matrix $\mathbf{G}_{\bar{m}}$. The $u$th element of the vector $\tilde{\mathbf{d}}_{\bar{m}.\bar{k}}^{\text{cfo}}$, which comprises both the signal and interference components for the $u$th user, is given as

$$\tilde{d}_{\bar{m}.\bar{k}}^{u,\text{cfo}} = d_{\bar{m},\bar{k}}^{u} e^{j\pi\epsilon_u\bar{k}} A_p(0, \epsilon_u) + J_{\bar{m},\bar{k}}^{u}(\epsilon_u). \tag{11}$$

Exploiting the property that the transmitted OQAM symbols are real, the estimate of the symbol $d_{\bar{m},\bar{k}}^{u}$ for the $u$th user in the presence of CFO can be obtained as

$$\hat{d}_{\bar{m},\bar{k}}^{u} = \Re\left\{\frac{\tilde{d}_{\bar{m}.\bar{k}}^{u,\text{cfo}}}{e^{j\pi\epsilon_u\bar{k}} A_p(0, \epsilon_u)}\right\} = d_{\bar{m},\bar{k}}^{u} + \tilde{J}_{\bar{m},\bar{k}}^{u}(\epsilon_u), \tag{12}$$

where the real interference term $\tilde{J}_{\bar{m},\bar{k}}^{u}(\epsilon_u)$ is given as

$$\tilde{J}_{\bar{m},\bar{k}}^{u}(\epsilon_u) = \Re\left\{\frac{J_{\bar{m},\bar{k}}^{u}(\epsilon_u)}{e^{j\pi\epsilon_u\bar{k}} A_p(0, \epsilon_u)}\right\}. \tag{13}$$

Using (9), the interference term $\tilde{J}_{\bar{m},\bar{k}}^{u}(\epsilon_u)$ can be evaluated as

$$\tilde{J}_{\bar{m},\bar{k}}^{u}(\epsilon_u) = \Re\left\{\frac{J_{\bar{m},\bar{k}'}^{u}(\epsilon_u)}{e^{j\pi\epsilon_u\bar{k}} A_p(0, \epsilon_u)}\right\} = \left(\frac{1}{A_p(0, \epsilon_u)}\right) \sum_{(m_0^{(0)},k_0^{(0)})\in\Omega_{\bar{m},\bar{k}}} d_{\bar{m}+m_0,\bar{k}+k_0}^{u}$$
$$\cos\left(\frac{\pi}{2}(m_0+n_0-m_0n_0+\epsilon_u n_0) - \pi\bar{m}n_0\right) A_p\left(\frac{-k_0M}{2}, m_0+\epsilon_u\right), \tag{14}$$

where we have also exploited the fact that the ambiguity function $A_p(m, \nu_0)$ is real [1]. Since the OQAM symbols $d^u_{m,k}$ are zero mean i.i.d. with variance $P_d$, the variance of the interference $\tilde{J}^u_{\bar{m},\bar{k}}(\epsilon_u)$ above can be obtained as

$$\mathbb{E}\big[|\tilde{J}^u_{\bar{m},\bar{k}}(\epsilon_u)|^2\big] = \left(\frac{P_d}{A_p^2(0,\epsilon_u)}\right) \sum_{(m_0^{(0)},k_0^{(0)})\in\Omega_{\bar{m},\bar{k}}} \cos^2\left(\frac{\pi}{2}(m_0 + n_0 - m_0 n_0 + \epsilon_u n_0)\right) A_p^2\left(\frac{-k_0 M}{2}, m_0 + \epsilon_u\right). \tag{15}$$

Following the rules described in (10) of the manuscript, the QAM symbol after OQAM to OQAM conversion is obtained from (12) as

$$\hat{c}^u_{\bar{m},\bar{k}} = c^u_{\bar{m},\bar{k}} + \tilde{v}^{u,\text{cfo}}_{\bar{m},\bar{k}}, \tag{16}$$

where $c^u_{\bar{m},\bar{k}} = d^u_{\bar{m},2\bar{k}} + jd^u_{\bar{m},2\bar{k}+1}$ and $\tilde{v}^{u,\text{cfo}}_{\bar{m},\bar{k}} = \tilde{J}^u_{\bar{m},2\bar{k}} + j\tilde{J}^u_{\bar{m},2\bar{k}+1}$ when $\bar{m}$ is even, and $\bar{m}$, $c^u_{\bar{m},\bar{k}} = d^u_{\bar{m},2\bar{k}+1} + jd^u_{\bar{m},2\bar{k}}$ and $\tilde{v}^{u,\text{cfo}}_{\bar{m},\bar{k}} = \tilde{J}^u_{\bar{m},2\bar{k}+1} + j\tilde{J}^u_{\bar{m},2\bar{k}}$ otherwise. Since the interference terms $\tilde{J}^u_{\bar{m},2\bar{k}}$ and $\tilde{J}^u_{\bar{m},2\bar{k}+1}$ are zero-mean independent variable with equal variances, the term $\tilde{v}^{u,\text{cfo}}_{\bar{m},\bar{k}}$ after OQAM to QAM conversion has the variance of $\mathbb{E}[|\tilde{v}^{u,\text{cfo}}_{\bar{m},\bar{k}}|^2] = 2\mathbb{E}[|\tilde{J}^u_{\bar{m},\bar{k}}|^2]$. Thus, from (16), the SIR at the index $(\bar{m}, \bar{k})$ of the $u$th user in the presence CFO of can be expressed as

$$\text{SIR}^u_{\bar{m},\bar{k}} = \frac{\mathbb{E}[|c^u_{\bar{m},\bar{k}}|^2]}{\mathbb{E}[|\tilde{v}^{u,\text{cfo}}_{\bar{m},\bar{k}}|^2]} \tag{17}$$

$$= \frac{2P_d A_p^2(0,\epsilon_u)}{\sum_{(m_0^{(0)},k_0^{(0)})\in\Omega_{\bar{m},\bar{k}}} 2P_d \cos^2\left(\frac{\pi}{2}(m_0 + n_0 - m_0 n_0 + \epsilon_u n_0)\right) A_p^2\left(\frac{-k_0 M}{2}, m_0 + \epsilon_u\right)}.$$

It follows form [2], [3] that for the $u$th user, the variance of the interference caused by the CFO in OFDM systems at subcarrier $\bar{m}$ is given as

$$\frac{2P_d \sin^2(\pi(\bar{m} + \epsilon_u))}{M} \sum_{m=0,m\neq\bar{m}}^{M-1} \frac{1}{\sin^2\left(\frac{\pi(m+\epsilon_u)}{M}\right)}. \tag{18}$$

Thus, the SIR at subcarrier $\bar{m}$ of the $u$th user for the OFDM-based massive MIMO systems in the presence of CFO is given as

$$\overline{\text{SIR}}^u_{\bar{m}} = \frac{2P_d}{\frac{2P_d \sin^2(\pi(\bar{m} + \epsilon_u))}{M} \sum_{m=0,m\neq\bar{m}}^{M-1} \frac{1}{\sin^2\left(\frac{\pi(m+\epsilon_u)}{M}\right)}}.$$

Fig. 1 compares the SIR per user for both the FBMC and OFDM-based single-cell massive MIMO systems as a function of the normalized CFO (normalized to the subcarrier spacing).

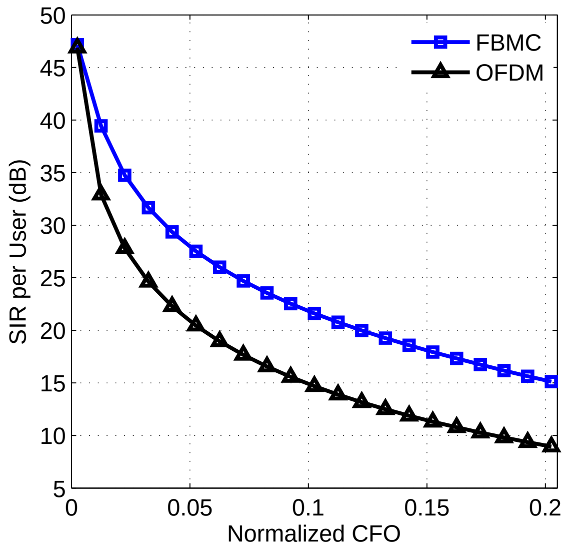


**Fig. 1:** SIR comparison of OFDM and FBMC-based single-cell massive MIMO systems with the ZF receiver processing at the BS in presence of CFO and perfect CSI. Number of BS antennas $N = 64$, number of users $U = 8$ and number of channel taps $L = 2$.

It can be observed that when the CFO is close to zero, both the systems have similar SIR performance. However, as the CFO increases, the FBMC waveform significantly outperforms the OFDM waveform in terms of SIR since the latter experiences significant ICI due to the poor frequency localization of the time domain rectangular pulse. On the other hand, FBMC systems experience significantly lower ICI due to the well-localized pulse shape (both in frequency as well as in time), which makes these systems robust against the CFO and well-suited for practical scenarios with synchronization impairments.

## References

[1] B. Farhang-Boroujeny, "OFDM versus filter bank multicarrier," *IEEE Signal Process. Mag.*, vol. 28, no. 3, pp. 92–112, 2011.

[2] H. S. Sourck, Y. Wu, J. W. M. Bergmans, S. Sadri, and B. Farhang-Boroujeny, "Sensitivity analysis of offset QAM multicarrier systems to residual carrier frequency and timing offsets," *Signal Processing*, vol. 91, no. 7, pp. 1604–1612, 2011.

[3] P. H. Moose, "A technique for orthogonal frequency division multiplexing frequency offset correction," *IEEE Trans. Communications*, vol. 42, no. 10, pp. 2908–2914, 1994.