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NUMERICAL SIMULATION of the INTERFACES of a JET STREAM

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Abstract
This paper aims to study the evolution of interfaces of a jet stream flowing in a stationary medium. The medium and the jet stream have been considered inviscid, surface tension effects have been neglected and there is a density jump across the interface between the stream and the media, within the jet stream and outside in the surrounding media density remains uniform. Due to discontinuity in velocities across the interface there exists vorticity at these interfaces and hence they have been modelled as a vortex sheet, everywhere else the system is assumed irrotational. The motion of the interfaces has been studied through the evolution of these vortex sheets which have been assumed to be comprised of a finite number of point vortices. The results indicate that the interface develops more roll ups for smaller Atwood numbers.

Keywords: *Vortex sheet; Point vortex; Kinematic conditions; Singularity*

I. INTRODUCTION

A jet of fluid flowing in a stationary media develops an instability at the interface due to the fact that it is a parallel shear flow. In such a system there is a sudden jump in velocity across the interface which leads to development of vorticity at the interfaces. For studying the time evolution of the interfaces we can model them as vortex sheets. A vortex sheet can be defined as a surface of infinitesimally small thickness at which all the vorticity is concentrated. In our analysis we will assume that the flow is irrotational everywhere in the flow. We also assume the fluids to be inviscid since there is a sudden jump in velocity across the interfaces of the jet and a finite viscosity of the fluid will mean that there is an infinite shear stress at the interface. So, we assume inviscid fluids because the existence of a vortex sheet is not feasible in viscous fluids. Also, we have not considered surface tension effects in this study.

Now since we are studying a 2D system the sheet reduces to a curve which we model as an array of point vortices. In a system of point vortices the motion of each of the vortices is due to the vectorial sum of the velocity induced at its location by the remaining vortices. Sohn et al. [1] studied the evolution of interface in Kelvin-Helmholtz instability, they did the modelling of the interface by assuming it to be an array of point vortices. Sohn [2] also applied the concept of modelling a vortex sheet as an array of point vortices in his study of the Rayleigh-Taylor instability.

The numerical computation for the interface becomes complicated due to the non-linear structures and steep vortex strength around the vortex cores. The computation of velocity of point vortices may develop singularity as shown by Moore [3]. For stable computations a desingularization parameter (Krasny [6]) is used in the velocity calculations. Another issue with numerical computation is the clustering of the vortices around the vortex cores and diverging at the outside region which leads to poor resolution of the interface.

A Study of fluid flowing in the form of a jet is important to many applications such as liquid issued from an orifice in diesel engines or from a fan spray nozzle. The breaking mechanism in these cases could be studied by inclusion of surface tension effects in a later study. Our aim in this paper is to perform accurate numerical simulation of the interfaces of a jet stream. The accuracy of the vortex method used by us lies in the fact that the evolution of the interface is studied in a Lagrangian manner rather than computation over a 2D grid.

II. MATHEMATICAL FORMULATION

A. Vortex method

We have two fluids let's say 1 and 2 which are on either sides of the two interfaces of the jet stream. Since we have assumed the fluids to be inviscid they will be governed by the Euler equations:

$$a_i = -\frac{1}{\rho_i}\nabla P_i \qquad [1]$$

Here i=1,2 describes the two fluids.

There is a discontinuity in velocity across both the interfaces which leads to development of vorticity at the interfaces. The interfaces can be modelled as vortex sheets and their strength known as vortex sheet strength can be defined as:

$$\gamma = \left(\vec{u}_1 - \vec{u}_2\right)\cdot \vec{s} \qquad [2]$$

Where $\vec{u}_1$ and $\vec{u}_2$ are the velocities above and below the interfaces and $\vec{s}$ is the unit tangent vector of the interfaces.

Now the interfaces have been for the purpose of numerical computation assumed to be comprised of a finite number of point vortices. The motion of the interface has been studied through the motion of these point vortices. The Lagrangian velocity of the interface is taken to be a weighted average of the velocities of the fluid above and below it.

$$\vec{q} = \vec{U} + \frac{1}{2}\gamma\alpha\vec{s} \qquad [3]$$

Here $\vec{U} = \dfrac{\vec{u}_1 + \vec{u}_2}{2}$ is the average of the velocities above and below the interface and is known as the principal vortex sheet velocity. The parameter α is the weighting parameter and if we take it to be 1 then the vortices follow the upper fluid and for the value -1 the vortices follow the lower fluid. In our computations we taken its value as 0 so, the interface will move with the principal velocity.

The motion of the interface can be described by the equation:

$$\frac{d\vec{X}}{dt} = \vec{q} \qquad [4]$$

Here the interface has been parameterized in terms of the arc length s.

The principal velocity is dependent upon the vortex sheet strength of the interfaces and can be determined from the Birkhoff-Rott equation.

$$u - iv = \sum_{j=1}^{2} \frac{i}{2\lambda}\int_0^{\lambda} \tilde{\gamma}_j \cot\left(\frac{\pi\left(z - \tilde{z}_j\right)}{\lambda}\right) d\tilde{s}_j \qquad [5]$$

Here j=1,2 refer to the two interfaces. $z = x + iy$ is the position of the point at which the velocity is calculated in complex plane and $\tilde{z} = \tilde{x} + i\tilde{y}$ is the position of the interfaces in complex plane. u and v are the horizontal and vertical velocities respectively.

So, the velocities of the interfaces are dependent on the vortex sheet strength and hence we need an evolution equation of the vortex sheet strength in order to study the evolution of the interfaces. From the Euler equations we can derive that

$$\left(\vec{a}_1 - \vec{a}_2\right)\cdot\vec{s} = A\left(\vec{a}_1 + \vec{a}_2\right)\cdot\vec{s} \qquad [6]$$

Here $A = \dfrac{\rho_2 - \rho_1}{\rho_2 + \rho_1}$ is the Atwood number, the subscripts 1, 2 denote fluids above and below the interfaces , $\vec{a}_1$ and $\vec{a}_2$ are the accelerations. Now these accelerations can be determined in terms of the vortex strength by using the kinematic conditions for accelerations of the fluid above and below the interface.

$$\vec{a}_i = \frac{d\vec{u}_i}{dt} - \frac{1}{2}\gamma\left(\alpha \pm 1\right)\frac{\partial\vec{u}_i}{\partial s} \qquad [7]$$

Here + and - sign is for i = 2 and 1 i.e. below and above the interface respectively. Using eqns. [6] and [7] we can derive the equation for evolution of vortex strength as:

$$\frac{d\gamma}{dt} = 2A\frac{d\vec{U}}{dt}\cdot\vec{s} + \left(\frac{\alpha + A}{4}\right)\frac{\partial\gamma^2}{\partial s} - \left(1 + \alpha A\right)\gamma\frac{\partial\vec{U}}{\partial s}\cdot\vec{s} \qquad [8]$$

a. Numerical procedure

In the numerical simulation of the interfaces we assume that the interfaces are comprised of finite no. of point vortices located at $\left\{\vec{X}\right\}_{i=0}^{N}$ with vortex strength $\left\{\gamma\right\}_{i=0}^{N}$. The equation [5] with desingularization parameter in the discrete form can be written as:

$$u_{il} = \frac{1}{2}\sum_{k=1}^{2}\sum_{j\neq i} K_{jk} \frac{\sinh\left(2\pi\left(y_{il} - y_{jk}\right)\right)}{\cosh\left(2\pi\left(y_{il} - y_{jk}\right)\right) - \cos\left(2\pi\left(x_{il} - x_{jk}\right)\right) + \delta^2} \qquad [9]$$

$$v_{il} = -\frac{1}{2}\sum_{k=1}^{2}\sum_{j\neq i} K_{jk} \frac{\sin\left(2\pi\left(y_{il} - y_{jk}\right)\right)}{\cosh\left(2\pi\left(y_{il} - y_{jk}\right)\right) - \cos\left(2\pi\left(x_{il} - x_{jk}\right)\right) + \delta^2} \qquad [10]$$

here k and l denotes the interfaces and i, j denote the point vortices. Also here K_{jk} represents the local circulation strength.

Suppose we know the system at some time level n then we can solve for vortex strength at time level n+1 using eqn. [8], this equation also contains time derivative of velocity so, initially this term is estimated at previous time level n-1. After getting vortex strength at n+1 time level we update the interface and get its shape at n+1 level and then calculate the velocities at n+1 time level. Now we estimate the acceleration term in eqn. [8] with central difference $\dfrac{d\vec{U}}{dt} = \dfrac{\vec{U}_{n+1} - \vec{U}_{n-1}}{2\Delta t}$ and iterate this step till the convergence of acceleration.

The initial shape of the interfaces have been taken as a sinusoidal curve of wavelength 1, $y = a_0 \sin 2\pi x$ and an initial distribution for the circulation strength of the point vortices has been derived from linear theory :

$$K_i = \left(\Delta U + 2\pi a_0\left(\Delta U * A\sin 2\pi x - \frac{2\left|\Delta U\right|\sqrt{r}}{1 + r}\cos 2\pi x\right)\right)\frac{1}{N} \qquad [11]$$

III. RESULTS AND DISCUSSION

The numerical computation of the evolution of interfaces has been done for an initial disturbance amplitude of 0.01 which is small enough for performing linear analysis initially. The domain can be selected according to the number of wavelengths that we want to simulate, in the results shown we have simulated two wavelengths. The surface tension has been ignored in our analysis although it can be studied later. The results presented here are for Atwood number A = 0.05 and we have considered two cases, case a) Initial Phase difference between the interfaces is 0 (Fig. 1) and case b) Initial Phase difference is π (Fig.2). If we start with initially smooth interfaces they stay smooth. It was observed that the formation of roll ups in the interfaces gets weaker with the increase of Atwood number as can be seen in Fig. 3 with At = 0.5.

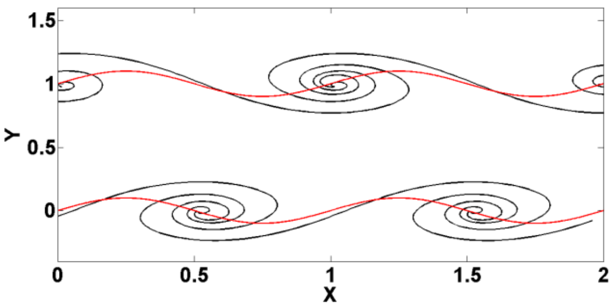

Figure 1 Configuration of the interfaces at t = 2.5 s (Phase difference = 0 and At = 0.05)

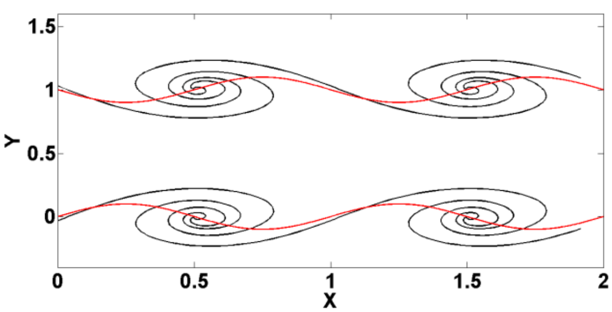

Figure 2 Configuration of the interfaces at t = 2.5 s (Initial Phase difference = π and At = 0.05)

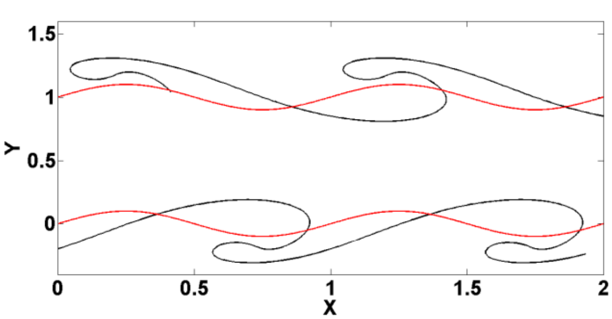

Figure 3 Configuration at t = 1.4 s (Initial phase difference = 0 and At=0.5)

In figures 1, 2 and 3 the amplitudes of the initial disturbance has been amplified to aid in visualising the phase difference. We can also see the variation of vortex strength at late times and see that in the region of vortex cores it grows steeper resulting in the need for additional measures like the desingularization parameter in velocity calculations.

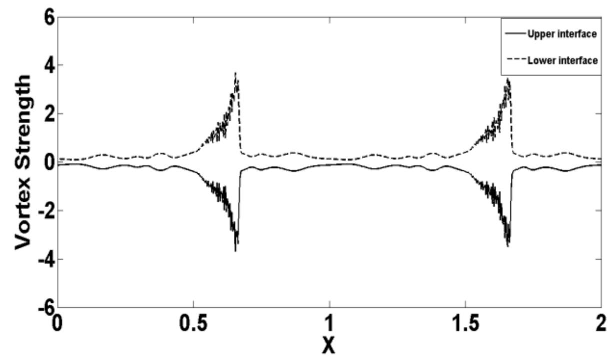

Figure 4 Variation of vortex strength with point vortex position (Initial Phase difference = π and at t = 2.5 s At = 0.05)

IV. CONCLUSIONS

We have studied the evolution of the interfaces of a jet stream in a stationary medium. The results presented in this paper can be validated with the work of Hashimoto et al. [4] and Minion et al. [5] on shear layers. The interface is unstable because of the generation of vorticity as a result of jump in velocity across the interface. As can be seen in the results a small initial sinusoidal disturbance grows and the interface rolls up at late times. The roll ups become weaker as we increase the Atwood number.

V. ACKNOWLEDGEMENT

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