Ganita

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Section 1 - Notations

Chapter 1 - Vedic Chart

वेदः/संहिताः	ब्राह्मणम्	आरण्यकं	उपनिषद्	शिक्षा	श्रौतसूत्रम्	गृह्यसूत्रम्	धर्मसूत्रम्	शुल्वसूत्रम्
१.ऋग्वेदः शाकल, आश्वलायन बाष्कल शांखायन माण्डूकायन	ऐतरेय शांखायन/कौषीतकि	ऐतरेय कौषीतकि बाष्कल	ऐतरेय कौषीतकि बाष्कल	पाणिनीय ऋक्प्रातिशाख्य ऋक्पार्षद	आश्वल शांखायन/कौषीतकि	आश्वल शांखायन/कौषीतकि शाम्बव्य	वशिष्ठ	नास्ति

२.यजुर्वेदः कृष्णयजुर्वेदः शुक्लयजुर्वेदः (१) माध्यन्दिन (२)वाजसनेयि काण्व कठ कपिष्ठल तैत्तिरीय मैत्रायणी	कठ कपिष्ठल तैत्तिरीय मैत्रायणी शतपथ	बृहद् तैत्तिरीय मैत्रायणी	बृहद् तैत्तिरीय मैत्रायणी ईशोपनिषद् श्वेताश्वर कठ	वाजसनेयि तैत्तिरीय याज्ञवल्क्य माण्डव्य वाशिष्ठी भारद्वाज अवसाननि	कात्यायन बौधायन आपस्तम्ब भारद्वाज सत्याषाढ वैखानस कठ मैत्रायणी	कात्यायन बौधायन आपस्तम्ब भारद्वाज सत्याषाढ वैखानस कठ वाधूल	वशिष्ठ बौधायन आपस्तम्ब वैखानस हिरण्यकेशी हारीत विष्णु शंख	कात्यायन बौधायन आपस्तम्ब मैत्रायणी मानव वाराह वाधूल
३.सामवेद कौथुमीय राणायनीय जैमिनीय	प्रौढ वंश षडविंश सामविधान देवताध्याय जैमिनीय	तवलकार/जैमिनीय छान्दोग्य	छान्दोग्य केनोपनिषद्	नारदीय ऋक्तन्त्र पुष्यसूत्र	लाट्यायन द्राह्ययायण मशकसूत्र जैमिनीय खादिर	जैमिनीय गौतम खादिर गोभिल	गौतम	नास्ति

४.अथर्ववेदः पिप्पलाद शौनक मौद स्तौद जाजल जलद ब्रह्मवेद चारणवैद्य देवदर्श	गोपथ	नास्ति	प्रश्न मुण्ड माण्डूक्य	माण्डूकी अथर्ववेदप्रातिशाख्य शौनकीयाप्रातिशाख्य कौत्सव्या	वैतान	कौशिक	नास्ति	नास्ति

Chapter 2 - Numeral Notation

Inhabitants of the land of the Sindhu

- The discovery at the Mohenjo-Daro reveals that as early as 3000 B.C. the inhabitants of the land of the Sindhu, the Hindus built brick houses, planned cities, used metals such as gold, silver, copper and bronze and lived a highly organised life. The Vedas, (3000 B.C probably) hymns of praise and poems of worship, show a high state of civilisation. The Brāhmaṇa (2000 B.C) which follows Vedas is partly ritualistic and partly philosophical. In these works are to be found a well-developed system of metaphysical, social and religious philosophy and the jobs of most of the science and arts which have helped to make up the modern civilisation. Here, we find the beginnings of the science of mathematics. This Brāhmaṇa period was followed by more than 2000 years of continuous progress and brilliant achievements. There were a class of people, the Brāhmaṇas, who took the vow of property and devoted themselves from one generation to another, to the cultivation of the sciences and arts, religion and philosophy. The great epic the Rāmāyana was composed by Vālmīki, the father of Sanskrit poetry about 1000 B.C. Sanskrit grammar - Pāṇini 700 BC. Sciences of medicine and surgery Suśruta wrote about 600 BC. A century later, Mahāvīra and Buddha taught their unique system of religious and moral philosophy and the doctrine of Nirvāṇa. With the spread of these religions evolve, the Jaina and the Buddhist literatures. Some of the earlier Purāṇs, and dharma śastras where did it in about this time. 400 B.C. to 400 A.D. was a great period of activity and progress.

- **Physician** - Umāswāti and Charaka
- **Grammarian and philosopher** - Pāṇini and Patañjali
- **Politician** - Kauṭilya
- **Chemist** - Nāgārjuna
- **Poets** - Aśhvaghoṣa , Bhaśa and Kālidā

Chapter 3 - The decimal place value system

- ❖ The third and most important of the Hindu numeral notation is the decimal place value.In this system there are ten Symbols only, those called अंक(mark) for numbers 1-9 and Zero call शून्य(empty). Professor Halsted - 'the importance of the creation of the zero mark can never be exaggerated.

Forms-

All the Hindu scripts are derived from a common source that is Brāhmī script.

Nāgari forms -

The most important as well as the most widely used of the different Symbols are those belonging to the Nāgari script . The present form of these

१ २ ३ ४ ५ ६ ७ ८ ९ ०

Place of invention of the new system

The new system is the arrangement of the aṇka (digits). The arrangement in the old system was that the biggest numbers were written to the left of the smaller ones. The same arrangement continues in the new system with place value where the digits to the left due to their place or position have bigger values. The changes from old to new are found only in India. The earliest epigraphic instance of the new system was found in 594 AD.

Inventor-

It is not known who was the inventor of the new system, whether it was invented by some Scholars or conference of sages or gradual development of some form of abacus. The system became quite popular all over Northern India.

Time of invention -

The grant plates were documented. They were written by professional writers. The existence of such writers is mentioned in the southern Buddhist canons and in the epics. They have been called *lekhaka, lipikara* and later on divira,Karaṇa, kāyastha etc. Kalhaṇa - king of Kashmir employed for drafting legal documents title Paṭṭopādhyāya (the teacher of little deeds). The existence of manuals that are lekhapañcāśikā, lekhaprakaśa legal and state documents. According to health, the Greek alphabetic notation was invented in the 7th century B.C. but it came into general use only in the second century A.D. It took 800 years to get popular. In Arabia the new notation was introduced in the 8th century A.D. but it came into common use about 500 or 600 years later. In legal documents and in recording to historical dates the Arabs, even now use their old alphabetic notation. Epigraphic evidences shows that the new system was quite common in India in the 8th century and the old system in the middle of the 10th century. The exact date of the invention would be nearer the first century BC. Persistence of the old system

The names of the decimal place values given by the ancient mathematicians -

Chapter 4 - Table of the Place Value Notation

शंकरवर्मन (सद्रत्नमाला)

एकं दश शतं चाथ सहस्रमयुतं क्रमात्।

नियुतं प्रयुतं कोटिरर्बुदं वृन्दमप्यथ॥५॥

खर्वो निखर्वश्च महापद्मः शङ्कुश्चवारिधिः।

अन्त्यं मध्यं परार्धं च संख्या दशगुणोत्तराः॥६॥

भास्कराचार्यः (लीलावती)

एकदशशतसहस्रायुतलक्षप्रयुतकोटयः क्रमशः।

अर्बुदमब्जं खर्वनिखर्वमहापद्मशंकवस्तस्मात्॥२॥

जलधिश्चांत्यं मध्यं परार्धमिति दशगुणोत्तराः संज्ञाः।

संख्यायाः स्थानानां व्यवहारार्थं कृता पूर्वैः॥३॥

श्रीधराचार्यः (पाटीगणित)

एकं दश शतमस्मात्सहस्रमयुतं ततः परं लक्षम्।

प्रयुतं कोटिमथार्बुदमब्ज खर्व निखर्व च॥७॥

तस्मान् महासरोजं शङ्कु सरितां पतिं ततस्त्वन्त्यम्।

मध्यं परर्द्धमाहुर्यथोत्तरं दशगुणं तज्जाः॥८॥

गणितसरसंग्रह (महावीराचार्यः)

एकं तु प्रथमस्थानं द्वितीयं दशसंज्ञिकम् ।

तृतीयं शतमित्याहुः चतुर्थं तु सहस्रकम् ॥६३॥

पञ्चमं दशसाहस्रं षष्ठं स्याल्लक्षमेव च।

सप्तमं दशलक्षं तु अष्टमं कोटिरुच्यते ॥६४॥

नवमं दशकोट्यस्तु दशमं शतकोटयः ।

अर्बुदं रुद्रसंयुक्तं न्यर्बुदं द्वादशं भवेत् ॥६५॥

खर्वं त्रयोदशस्थानं महाखर्वं चतुर्दशम्।

पद्मं पञ्चदशं चैव महापद्मं तु षोडशम् ॥६६॥

क्षोणी सप्तदशं चैव महक्षोणी दशाष्टकम्।

शङ्खं नवदशं स्थानं महाशङ्खं तु विंशकम् ॥६७॥

क्षित्यैकविंशतिस्थानं महाक्षित्या द्विविंशकम्।

त्रिविंशकमथ क्षोभं महाक्षोभं चतुर्नयम्॥६८॥

स्थान	शंकरवर्मन	भास्कराचार्यः	श्रीधराचार्यः	महावीराचार्यः
1	एकं	एक	एकं	एक
10	दशं	दश	दशं	दश
10^2	शत	शत	शत	शत
10^3	सहस्रं	सहस्र	सहस्रं	सहस्र
10^4	अयुतं	अयुत	अयुतं	दश सहस्र

10^5	नियुतं	लक्ष	लक्षं	लक्षं
10^6	प्रयुतं	प्रयुत	प्रयुतं	दश लक्ष
10^7	कोटि	कोटि	कोटि	कोटि
10^8	अर्बुदं	अर्बुदं	अर्बुदं	दश कोटि
10^9	वृन्दं	अब्जं	अब्जं	शत कोटि
10^{10}	खर्व	खर्व	खर्व	अर्बुदं
10^{11}	निखर्व	निखर्व	निखर्व	न्यर्बुद
10^{12}	महापद्म	महापद्म	महासरोजं	खर्व
10^{13}	शङ्कु	शङ्कु	शङ्कु	महा खर्व
10^{14}	वारिधि	जलधि	सरितांपति	पद्म
10^{15}	अन्त्यं	अन्त्यं	अन्त्यं	महा पद्म
10^{16}	मध्यं	मध्यं	मध्यं	क्षोणी
10^{17}	परार्धम्	परार्धम्	परार्धम्	महाक्षोणी
10^{18}				शङ्ख
10^{19}				महा शङ्ख
10^{20}				क्षिति

10^{21}				महा क्षिति
10^{22}				क्षोभ
10^{23}				महा क्षोभ

Chapter 5- Word numerals

Explanation of the system -

- A system of place value notation was developed in India in the early centuries of the Christian era. In the system, the numerals are expressed by the name of the things, beings or concepts, which are in accordance with the teaching of the śāstras. Thus the number one may be denoted by anything e.g. moon, earth, number two maybe denoted by any pair e.g. the eyes, the hands. The system is used in works on astronomy, mathematics and metrics as well as in the dates of inscription. The number 1230, may be expressed in many ways-

  1. Kha-guṇa-kara-ādi
  2. Kha-loka-karṇa-candra
  3. Ākāśa-kāla-netra-dharā

The words denoting the numbers from one to nine and zero, with the use of the principle of place value , give us a very convenient method of expressing numbers by words Chronograms

Word numerals without place value -

In the Veda, the use of names of things to do not number is not found. We do find instances of numbers denoting things. In Ṛgveda, the number twelve denote ‘a year’. In the Atharvaveda, the number seven denote a group of seven things (the seven seas). There are instances of fractions having been denoted by word symbols e.g.

Kalā=1/16

Kuṣṭha =1/12

Śapha=¼

The earliest instances of a word being used to denote a whole numbers are found about 2000 BC in Śatapatha Brāhmaṇa and Taittiriya Brāhmaṇa. Chāndogya Upaniṣad also contains several instances. In the Vedāṅga Jyotiṣa(1200 BC) words for numerals have been used at several places. The Śrauta sutras of Kātyāna and Lātyāyana have the words Gāyatri for 24 and Jagati for 48. In Aitareya Brāhmaṇa the word Virāt has been used to denote '10' at one place and 30 at another place. The use of the word Symbols without place value is found in Piṅgala Chandaḥ sutra (200 BC). The principle of place value seems to have been applied to the word numerals between 200 BC and 300 A.D.

The word numeral with place value in its current form is found in Agni Purāṇa. The word system is used in the commentary of Bhaṭṭotpala on the Bṛhat-saṃhita has given a quotation from the original Puliśa Siddhānt(400 BC). The number expressed in the quotation is –

Kha(0) kha(0) aṣṭa(8) muni(7) rāma(3) aśvi(2) netra(2) aṣṭa(8) śara(5) rātripāḥ(1)

=1,582,237,800

Chapter 6 - Application of Mathematics

- Sutra by MAHAVIRACHARYA in GAṆITSĀRASAÑGHRA about the different fields where Mathematics is Used : (Verses - 9-16)

लौकिके वैदिके वापि तथा सामायिके ऽपि यः । व्यापारस्तत्र सर्वत्र संख्यानमुपयुज्यते ॥
कामतन्त्रेऽर्थशास्त्रे च गान्धर्वे नाटकेऽपि वा । सूपशास्त्रे तथा वैद्ये वास्तुविद्यादिवस्तुषु ॥
छन्दोऽलङ्ङ्कारकाव्येषु तर्कव्याकरणादिषु । कलागुणेषु सर्वेषु प्रस्तुतं गणितं परम् ॥
सूर्यादिग्रहचारेषु ग्रहणे ग्रहसंयुतौ । त्रिप्रश्वे चन्द्रवृत्तौ च सर्वत्राङ्गीकृतं हि तत्॥
द्वीपसागरशैलानां संख्याव्यासपरिक्षिपः । भवनव्यन्तरज्योतिर्लोककल्पाधिवासिनाम्॥
नारकाणां च सर्वेषां श्रेणीबन्धेन्द्रकोत्कराः । प्रकीर्णकप्रमाणाद्या बुध्यन्ते गणितेन ते ॥
प्राणिनां तत्र संस्थानमायुरष्टगुणादयः । यात्राद्याः संहिताद्याश्च सर्वे ते गणिताश्रयाः ॥
किं त्रैलोक्ये सचराचरे। यत्किंचिद्विस्तु तत्सर्वं गणितेन विना न हि ॥

9. In all those transactions which relate to worldly, Vedie or (other) similarly religious affairs, calculation is of use.

10. In the science of love, in the science of wealth, in music and in the drama, in the art of cooking, and similarly in medicine and in things like the knowledge of architecture

11. In prosody, in poetics and poetry, in logic and grammar and such other things, and in relation to all that constitutes the peculiar value of (all) the (various) arts: the science of computation is held in high esteem.

12. In relation to the movements of the sun and other heavenly bodies, in connection with eclipses and the conjunctions of planets, and in connection with the triprasna and the course of the moon-indeed in all these (connections) it is utilised

13-14. The number, the diameter and the perimeter of islands, oceans and mountains; the extensive dimensions of the rows of habitations and halls belonging to the inhabitants of the(earthly) world, of the interspace (between the worlds), of the world of light, and of the world of the gods; (as also the dimen- sions of those belonging) to the dwellers in hell: and (other) miscellaneous measurements of all sorts-all these are made out by means of computation.

15. The configuration of living beings therein, the length of their lives, their eight attributes and other similar things, their progress and other such things, their staying together and such other things-all these are dependent upon computation (for their due measurement and comprehension).

16. What is the good of saying much in vain ? Whatever 'there is in all the three worlds, which are possessed of moving and non-moving beings-all that indeed cannot exist apart from measurement.

Section 2 - Arithmetics

Introduction

- One of the most prominent portions which is discussed in ancient India is arithmetic. The word *pāṭīgaṇita* was used for arithmetic. The word *pāṭīgaṇita* is consisting of two words namely, *pāṭī* means board and *gaṇita* means science of calculation. So *pāṭīgaṇita* means the science of calculation. It is believed that this term originated from Northern India. The oldest Sanskrit term for the board is *Falak* or not *pāṭī*. The word *pāṭī* seems to have entered into Sanskrit literature about the beginning of the 7th century A.D.one of the most prominent portions which is discussed in ancient India is arithmetic. The word *pāṭīgaṇita* was used for arithmetic. The word *pāṭīgaṇita* is consisting of two words namely, *pāṭī* means board and *gaṇita* means science of calculation. So *pāṭīgaṇita* means the science of calculation. It is believed that this term originated from Northern India. The oldest Sanskrit term for the board is *Falak* or not *pāṭī*. The word *pāṭī* seems to have entered into Sanskrit literature about the beginning of the 7th century A.D.

In arithmetic the eight fundamental operations i.e. addition, subtraction, multiplication, division, square, square root, cube and cube root are highly practised. In ancient literature the applications of all these areas are explained in great details. Without the applications of all these eight operations the mathematicians would not be able to practise arithmetic in any context.

Chapter 1 - Addition (सङ्ककलित)

Terminology
Hindu name for addition is संकलित
सम्+कलित =made together

संख्यावतां बहूनामेकीकरणं तदएव सङ्कलित । १ ।(पाटीगणित)
महासिद्धातं, आर्यभट्ट II

"Making it into one of several numbers is addition."
There are two processes of addition and subtraction given by our ancient mathematicians, that are I) direct process and ii) inverse process.

Method -
In the direct process of addition the numbers to be added are written down one below the other then put a line in the bottom. Then start doing the sum from the unit place(right side).
In the inverse process, the numbers standing in the last place (left side) are added together first then the next place and so on.
Now let us see the verses and methods given by the ancient mathematicians-

Verse

कार्यः क्रमादुत्क्रमतोऽथवाऽङ्कयोगोयथास्थानकमन्तरं वा ॥१३॥

-लीलावती

यथास्थानं व्युत्क्रमेण क्रमेण यदि वाङ्कयोः।

मेलनं युतिमत्राहुर्वियुतिं च वियोजनम्॥८॥

-सद्रत्नमाला

Glossary

१.क्रमात्- in a natural direction
२.उत्क्रमात्- reverse order or backwards
३.योग - add
४.यथास्थान- proper place
५.मेलनं- matching
६.वियोजन्- subtract

Meaning -

Addition or subtraction of numbers should be done in direct or reverse order according to the ranks.

Finding the sum of two quantities by adding the numbers in them either in direct order or in reverse order is called addition. Similarly finding the difference between them is subtraction.

Direct process-

	ल	अ	स	श	द	ए
	 +		4 6 9	7 3 0	1 8 9	5 0 5
Carry over digits			*1*	*1*	*1*	
	=	**2**	**0**	**1**	**9**	**0**

Inverse process-

- Started from the left

	को	ल	अ	स	श	द	ए
			2 **6**	**5** **3**	**9** **5**	**1** **2**	**2** **3**

	+		5	4	3	8	3
	(2+6+6)	1	4				
	(5+3+4)		1	2			
	(9+5+3)			1	7		
	(1+2+8)				1	1	
	(2+3+3)						8
	=	1	5	3	8	1	8

Exercise -

1) **2365+6321**
2) **5639+7653**
3) **4381+6409**
4) **3759+6081**

Chapter 2 - Subtraction (व्यवकलित)

Terminology
The Hindu name for subtraction is व्यवकलित
वि+अव+कलित = made apart

यदपास्तं सर्वधनात् तद्व्यवकलितं तु शेषकं शेषम् ।2।
महासिद्धांतं, आर्यभट्ट II

"The taking out (of some number) from the *sarvadhana* (total) is subtraction; what remains is called hsa (remainder)."

Method
The process is similar to the addition process.

- **Direct process of subtraction**

	अ	स	श	द	ए
 -	**2**	**4** **2**	**5** **7**	**7** **8**	**1** **2**
			-1	**-1**	
Remaining digits		**4-1=3**	**5-1=4** **Borrow 1=14**	**7 -1=6** **Borrow 1=16**	
=	**2**	**1**	**7**	**8**	**9**

- **Inverse process of subtraction**

Started from the left

	अ	स	श	द	ए
 -	2	3 3	5 1	7(17) 9	0(10) 2
=	2	0	4	8	8
		-	1	1	

=	2	0	3	7	8

Exercise -

1) 3421-2653
2) 5409-3285
3) 2876-1427
4) 6548-5321

Chapter 3 - Multiplication(प्रत्युत्पन्न)

Terminology
The Hindu name for multiplication is गुणन।
This term is the oldest one because it occurs in Vedic period.
Some interesting names for multiplication were given by the ancient mathematicians, that are- हनन, वध, क्षय which means 'killing' or 'destroying'. These names came into use after the invention of the new method of multiplication with the decimal place-value numerals; in the new method the multiplicand rubbed or destroyed by the new product.

गुण्यान्त्यस्थानोपरि गुणकाद्यं स्थापयेत् ततो गुणयेत् ।
गुणकस्थानैरखिलैर्गुण्यस्थानानि सर्वाणि ।।३।।

Method
There are multiple methods of multiplication that were given by the ancient mathematicians. We will discuss all the methods in detail.

ब्रह्मस्फुटसिद्धान्त

गुणकारखण्डतुल्यगुण्यो गोमूत्रिकाकृतोगुणितः।
सहितः प्रत्युत्पन्नो गुणकारकभेदतुल्यो वा॥५५॥
गुण्यो राशिर्गुणकार राशिनेष्टाधिकोनकेनगुणः।
गुण्येष्टवधोनयुतो गुणकेऽभ्यधिकोनके कार्यः॥५६॥ {ब्रह्मस्फुटसिद्धान्त (ब्रह्मगुप्त)}

Glossary

१. गुणक - multiplier
२.गुण्य - multiplicand
३.खण्ड - part
४. गोमूत्रिका - cow's urine, zigzagging
५.प्रत्युत्पन्न - multiplication

६.राशि - amount
७.इष्ट - sacrifice,wish
८.भेद- splitting,breaking

Meaning

In *Gomūtrikā,* multiplier will be split up into two or more parts or split up by their place value and then multiplied with the multiplicand (**गुण्य).** The multiplicand will be shifted according to the places of the multiplier , and the result of the multiplications added according to the places; this gives the result of the multiplication.

The multiplicand is multiplied by the multiplier as many times as there are component parts in the multiplier **,** this process is called *Bheda*.

Assume a number and then add or subtract from the multiplier. After that multiply the multiplicand with the added or subtracted number and multiply the multiplicand with the assumed number separately. Then the results of the multiplications would be subtracted or added accordingly.

Thus *Brahmagupta* has mentioned three methods:
(1.1)*Gomūtrikā (गोमूत्रिका)*
(1.2) *Bheda (भेद)*
(1.3) *Iṣṭa* (इष्ट)

The book entitled "In the History of Hindu Mathematics" the author has mentioned four types of processes namely - (1) *Gomūtrikā* (2) *Bheda* (3) *Iṣṭa* and (4) *Khaṇḍa* . But according to the commentary of *Pṛthudakaswāmi* there are three processes namely - (1)*Gomūtrikā* (2) *Bheda* (3) *Iṣṭa*. The commentator makes it clear that the *Khaṇḍa* and *Gomūtrikā* come together.
'गुणकारस्यैकदशादिस्थानीयखण्डतुल्यो गुण्यः स्थाप्यस्ततस्तैः खण्डैर्गोमूत्रिकाकृतो गुणितः सहितः प्रत्युत्पन्नोगुणनफलं भवति।' (commentary of *Pṛthudakaswāmi)*

1.1. Gomūtrikā (गोमूत्रिका)

The first process is *Gomūtrikā* . It is similar to the course of cow's urine, going alternately to the one side and to another side means moving in a zigzag way. The following illustration is based on the commentary of *Pṛthudakaswāmi*. Two methods come under this process.

(i) In the first method the digits of the multiplier would be separated by their places, then severally multiplied with the multiplicand and multiplicand would be shifted accordingly.
To explain it more let us see the example -

1863x145
1863 is multiplicand 145 is the multiplier .
This process looks like this-

1863 x 1
  1863 x 4
    1863 x 5

	अ	स	श	द	ए		

	8	6	3	x	1		
=	8	6	3				
	1	8	6	3	x	4	
=	7	4	5	2			
		1	8	6	3	x	5
=		9	3	1	5		
	2	1					
=	7	0	1	3	5		

(ii) The second method of *Gomūtrikā* is to split up the multiplier in parts.
Let us understand this process with an example -
325x112
112 is the multiplier , we will separate them in two parts one is 11 and second is 2.

325x2

स.	श.	द.	ए.
x	3	2	5 2
		1	
	6	5	0

325x11

	अ.	स.	श.	द.	ए.
			3	2	5

x			1	1	
		2	5		
=	3	5	7	5	

	अ.	स.	श.	द.	ए.
+	3	5	6 7	5 5	0
		1	1		
=	3	6	4	0	0

The final result is 325x112=36400

1.2 *Bheda* (भेद)

Second process is *bheda*. This process consists of two m(i) The multiplier is broken up into two or more parts whose sum is equal to it. The multiplicand is then multiplied severally by these and the results added.

Example
674x24
24 is the multiplier. Let us break this multiplier into two parts. So, 20 and 4 these are the two parts. At first the multiplicand will be multiplied by 20 and then by 4. After that we will add the results.
674x20 -

	अ.	स.	श.	द.	ए.
X Carry over digits		1	6	7 2	4 0
=	1	3	4	8	0

674x4 -

		6	7	4

x				4
carry		2	1	
=	2	6	9	6

	अ.	स.	श.	द.	ए.
 +	1	3 2	4 6	8 9	0 6
carry		1	1		
=	1	6	1	7	6

- The result of the multiplication 674x24=16176

1.3 *Iṣṭa (इष्ट)*

The third and last process is *Iṣṭa* . *Iṣṭa*means *icchā* (wish) but here the meaning is assumption. We have to assume a number according to the multiplier then add or subtract from the multiplier . Then multiply the multiplicand with the increased or diminished number and with the assumption as well. Then add or subtract the products respectively.

This process is of two types - 1) add an assumed number from the multiplier or 2) subtract an assumed number from the multiplier .

Let us see the first process with an example-

456x23

Assume number- 7

Here the multiplier is 23 and the assumed number is 7. Assumed number will be added with the multiplier.

23+7=30

In the first step, multiplicand will be multiplied by 30.

	X		4	5 3	6 0
		1	1		
=	1	3	6	8	0

Now we will multiply the multiplicand with the assumed number.

	स.	श.	द.	ए.
X	4	5	6 7
Carry over digits	3	4	
= 3	1	9	2

Now subtract the two products (13680, 3192)-

	अ.	स.	श.	द.	ए.
-	1	3 3	6 1	8 9	0 2
=	1	0	4	8	8

Now we will see an example of the second process–

456 x 25
Assume number - 8
Here the multiplier is 108 and the assumed number is 8. Assumed number will be subtracted from the multiplier.
25 - 5 = 20

In the first step we will multiply the multiplicand with the diminished number, then with the assumed number.

स.	श.	द.	ए.
x	4	5 2	6 0
1	1		
9	1	2	0

	स	श	द	ए

	X	4	5	6 5
Carry over digits		2	3	
=	2	2	8	0

Now we will add the two products (9120 and 2280).

		सहस्र	शतक	दशक	एकक
+		9 2	1 2	2 8	0 0
			1		
=	1	1	4	0	0

Exercise

1) 342x45
2) 764x78
3) 456x765
4) 321x143
5) 4087x5432
6) 7302x421

त्रिशतिका/पाटीगणितसार, श्रीधराचार्य

विन्यस्याधो गुण्यं कवाटसन्धिक्रमेण गुणराशेः ।

गुणयेद्विलोमगत्याऽनुलोममार्गेण वा क्रमशः ॥१८।।

उत्साय्योर्त्साय्य ततः कवाटसन्धिर्भवेदिदं करणम् ।

तस्मिंस्तिष्ठति यस्मात्प्रत्युत्पन्नस्ततस्तत्स्थः ॥१९।।

रूपस्थानविभागाद् द्विधा भवेत्खण्डसंज्ञकं करणम् ।

प्रत्युत्पन्नविधाने करणान्येतानि चत्वारि॥२०।।

Glossary

१.विनस्य - to place in a special way

२.कवाटसन्धि- door injunction

३.क्रमेण- in a natural direction

४.विलोम- reverse order

५.तत्स्थ- cross multiplication

६.रुपविभाग- parts multiplication

७.स्थानविभाग- separation of places

Meaning

Place the multiplicand below the multiplier as in the junction of two doors. Then multiply in the inverse or direct order and move the multiplier each time. This process is known as *Kavātasandhi.*

When the multiplier and multiplicand remain in the same place, then the process is known as *Tatstha*. The process of multiplication called khaṇḍa is of two types

1.रूपविभाग it depends on the multiplie, which is split up into two or more parts whose sum or product is equal to it.

2.स्थानविभाग in this process the digits standing in the different notational places (*sthāna*) of the multiplier are taken separately.

i) कवाटसन्धि (*Kavātasandhi*)

ii) तस्थ (*Tatstha)*

iii) खण्ड (*Khanda*)

1.Kavātasandhi (कवाटसन्धि)

It can be done in two ways i.e. inverse or direct.

Let us start with the direct process.

- We will take two numbers multiplicand and multiplier - 123x21
- In this process the multiplier would be placed up and multiplied in the below.

X	1	2	2 3	1
=	1	2	6	3

Shift the multiplier into the left.

X	1	2 2	1 6	3
		2x2=4	(2x1)+6=8	
=	1	4	8	3

shift to the left

	2	1		
X	1	4	8	3
=	2	5(4+1)	8	3

Inverse way-

123x21

Shift to the right

	2	1		
X		1	2	3
=	2	1	2	3

		2	1	

X	2	1	2	3
=	2	5	2	3

	सहस्र	शतक	दशक	एकक
X	2	5	2 2	1 3
			(3x2)+2=8	
=	2	5	8	3

2. Tatstha (तस्थ) -

This is called cross multiplication.

814(multiplicand)
526(multiplier)

	लक्षम्	अयुत	सहस्र	शतक	दशक	एकक
4x6					2	4
(6x1)+(2x4)				1	4	
(6x8)+(5x4)+(1x2)			7	0		
(2x8)+(5x1)		2	1			
8x5	4	0				
=	4	2	8	1	6	4

3.Rupa Vibhāga (रूपविभाग) - Brahmagupta named this process as *Bheda.*

4.Sthāna Vibhāga (स्थानविभाग) - the multiplicand would be divided by their place.

3254x462

400+60+2

	सहस्र	शतक	दशक	एकक
X	3	2	5	4 2
Carry over digit		1		
=	6	5	0	8

	लक्षम्	अयुत	सहस्र	शतक	दशक	एकक
X			3	2	5 6	4 0
Carry over digit				3	2	
=	1	9	5	2	4	0

	नियुत	लक्षम्	अयुत	सहस्र	शतक	दशक	एकक
X				3	2 4	5 0	4 0
Carry over digit		1	2	1			
=	1	3	0	1	6	0	0

	नियुत	लक्षम्	अयुत	सहस्र	शतक	दशक	एकक
+	1	1 3	9 0	6 5 1	5 2 6	0 4 0	8 0 0
Carry over digits		1	1	1			
=	1	5	0	3	3	4	8

Exercise -

1) 3098x532
2) 7640x4320
3) 5438x3076
4) 3219x2876

- ❖ गणितसारसंग्रह, महावीराचार्य

गुणयेद्गुणेन गुण्यं कवाटसन्धिक्रमेण संस्थाप्य।

राश्यर्धखण्डतत्स्थैरनुलोमविलोममार्गाभ्याम्॥१।।

Meaning

Placed the multiplicand and the multiplier one below the other in the manner of hinges of a door, the multiplicand should be multiplied by the multiplier, in accordance with either of the two methods namely normal or reverse.

Split up the multiplicand or the multiplier into two or more parts. Then multiply the multiplier by a factor of the multiplicand or multiply the multiplicand by a factor of the multiplier**.**

When the multiplier remains in the same position, then the process is known as *Tatstha*.

१.कवाटसन्धि (*Kavātasandhi*)

२.तत्स्थ (*Tatstha)*

३.खण्ड (*Khanda)*

We have discussed all the processes in the previous part.

- ❖ लीलावती, भास्कराचार्या

गुण्यान्त्यमकं गुणकेन हन्यादुत्सारितेनैवमुपान्तिमादीन् ।

गुण्यस्त्वधोऽधो गुणखण्डयुतस्तैः खण्डकैः संगुणितो युतो वा ॥ 15 ॥

भक्तो गुण: शुध्यति येन तेन लब्ध्या च गुण्यो गुणितं फलं वा ।

द्विधा भवेद्रूपविभाग एवं स्थानैः पृथग्वा गुणितः समेतः ॥ 16 ॥

इष्टोनयुकेन गुणेन निघ्नोऽभीष्टघ्नगुण्यान्वितवर्जितो वा ॥ 17 ||

Meaning -

Multiply the last digit of the multiplicand by the multiplier, and move forward. In this way, multiply the multiplicands from the penultimate one onwards. This process is called *rupaguṇa*.

By splitting up or breaking up the multiplicand or multiplier into two or more suitable parts whose sum will be equal to it should be multiplied by the multiplicand or multiplier**.**

By dividing the multiplier by any number which will be an aliquot part of it, multiply the multiplicand by that number and then by the quotient that result is the final product.

The division of the number is of two types- (i) on the basis of places or (ii) on the basis of addition. Thus, the multiplicand should be multiplied by the two or more parts of the multiplier or the multiplicand. At last, addition will take place and we can get the final result.

Assume a number and then add or subtract from the multiplier. After that multiply the multiplicand with the increased or diminished number and with the assumed number separately. Then the results of the multiplications would be subtracted or added accordingly.

1. खंडगुण (**khaṇḍa Guna**)
2. रूपगुण (**Rupa Guna)**
3. विभागगुण (**Vibhāgaguṇa)**
4. स्थानगुण (**Sthāna Guna)**
5. इष्टगुण (**Iṣṭa Guna)**

Let us see all the methods with examples for better understanding-

1.khaṇḍa Guna (खंडगुण) - 135x12

We will divide 135 into two parts 13 and 5.

	शतक	दशक	एकक
		1 1	3 2
Carry		3	

=	1	5	6

X	1	5 2
=	6	0

	सहस्र	शतक	दशक	एकक
 +	1	5	6 6	 0
Carry		1		
=	1	6	2	0

2.Rupa Guna (रूपगुण) -

135x12

100+35

- 100x12

	सहस्र	शतक	दशक	एकक
		1	0	0

x			1	2
=	1	2	0	0

- 35x12

	शतक	दशक	एकक
 x		3 1	5 2
Carry		6	
=	4	2	0

	सहस्र	शतक	दशक	एकक
 +	1	2 4	0 2	0 0
=	1	6	2	0

3.Vibhāgaguṇa (विभागगुण)

135x12

135=27x5

- 27x12

	शतक	दशक	एकक
x		2 1	7 2
Carry over digits		8	
=	3	2	4

- 324x5

	सहस्र	शतक	दशक	एकक
x		3	2	4 5
Carry over digits		1	2	
=	1	6	2	0

4.Sthāna Guna (स्थानगुण)

135x12

100+30+5

	सहस्र	शतक	दशक	एकक
 x		1	0 1	0 2
=	1	2	0	0

	शतक	दशक	एकक
 x		3 1	0 2
=	3	6	0

	दशक	एकक
 x	 1	5 2
=	6	0

	सहस्र	शतक	दशक	एकक

+	1	2 3	0 6 6	0 0 0
Carry over digits		1		
=	1	6	2	0

5. Iṣṭa Guna (इष्टगुण) -

135x12

Assume number is - 4

12-4=8

	सहस्र	शतक	दशक	एकक
x		1	3	5 8
Carry over digits		2	4	
=	1	0	8	0

	शतक	दशक	एकक
	1	3	5

x			4
Carry over digits	1	2	
=	5	4	0

	सहस्र	शतक	दशक	एकक
x	1	0 5	8 4	0 0
Carry over digits		1		
=	1	6	2	0

Exercise

1) 8021x345
2) 5631x765
3) 2086x432
4) 5193x2186
5) 3280x2081

- सद्रत्नमाला, शंकरवर्मन

गुण्यान्त्योपान्त्यादीन सर्वान् गुण्येद् गुणेन पृथगङ्कान्।

गुण्यान् गुणखण्डसमान् खण्डैस्तैर्वाथ तद्युतिर्गुणनम्॥९॥

Meaning

Multiply separately, the last, last but one etc. digits of the multiplicand by the multiplier. The sum of these (in accordance with the place values in the multiplicand) is the product. Or multiply, severally, the multiplicand by any number of terms into which the multiplier is split. The sum of these is the product.

१.खंड

२.स्थानगुण

These processes are mentioned with detail in the previous part.

Chapter 4 - Division

Two Methods discussed below from Sadratanamala and Lilavati

यद्घ्नो हारो हार्यसमस्तत्फलं हरणे भवेत् । हार्याद् धृतिः स्वानधिकहारकेण तथोत्क्रमात् ।।१० ।।

किं नाम हरणम् ? (In First stanza question is asked)
Second stanza with procedure
c b = a
c is Result
b = divisor
a = dividend
हारः × हरणफलम् = हार्यः

यद्घ्नः = येन् हन्ति इति
हन्ति = गुणयति
अनधिकम् = स्वानधिकम् ।
स्वानधिक हारक = स्वानधिक: हारक : (हारः)
उत्क्रमात हार्यात्

Keep dividing till you get a reminder smaller than the divisor.
Example : 256 / 8 = 32

भाज्यादधरः शुध्यति यद्गुणः स्यात् अन्त्यात् फलं तत् खलु भागहारे
समेन केनाप्यवत्य हारभाज्यौ भवेद् वा सति सम्भवे

अन्वय : यद्गुणः हर: अन्त्यात् भाज्यात शुध्यति तत् खलु भागहारे फुलम स्यत्। सम्भवे तु सति केनापि वा समेन हारमाज्यों अपवर्य (फल) भवेत् ।

भाज्यम् = No. of Dividend = Dividend
हर , हार = Divisor
शुध = to divide fully अपवर्त्य = divided by common factor

Chapter 5 - Square

Sanskrit term for square is *varga or kṛti*. The origin of the term is from, graphical representation of a square. The commentator Parameśvara : The Four sided figure whose sides are equal and both of whose diagonals are also equal is called *samacaturaśra.*

Method first occurred in the Brāhma-sputa-siddhānta, it was known to Āryabhaṭa 1 as he has given the square method. Srīdhara , Mahāvīra , Bhāskara II and others have given methods to square. What is the need to include these many methods to solve problems in books ?
In Modern school methods we can see only 2 - 3 methods to solve the given problem, student rote learn and use in exams, but if we need to increase skills like *Thinking Capacity* , *Creative Problem Solving* , *Decision Making , etc.* in students they should be given different methods than 2 - 3. No need to limit students on given selected formulas if answers can be concluded using other methods as well. This was the same technique used by Mahāvira in Ganita-Sārā-Sangraha by introducing many different methods to solve Squares in the Sūtras. Important to note in terms of sanskrit , a short sūtra (अल्पाक्षरम् असन्दिग्धं सारवत् विश्वतोमुखम् । अस्तोभम् अनवद्यं च सूत्रः सूत्रविदो विदुः ॥) can explain more than one formula, see below :

Mahāvira in Ganita-Sāra-Sangraha Stated:
तृतीये वर्गपरिकर्मणि करणसूत्रं (गणितसारसंग्रह)

द्विसमवधो घातो वा स्वेष्टोनयुतद्वयस्य सेष्टकृतिः। एकादिद्विचयेच्छागच्छयुतिर्वा भवेद्वर्गः ॥ २९ ॥
In this Rule 29 Mahāvira gave three methods
Expressing Algebraically, comes out :
$a \times a = a^2$ (The Multiplication of two equal quantities) ;
$(a + x) (a - x) + x^2 = a^2$ (Multiplication of the quantities obtained by the Subtraction and Addition of any assumed quantity , together with addition of the square of assumed quantity.);
$1+ 3 + 5 + 7 +....$ to a terms = a^2(The sum of a series in arithmetical progression of which 1 is the first term 2 is common difference and the number of terms wherein is that of which the square is required gives rise to the required square.)

द्विस्थानप्रभृतीनां राशीनां सर्ववर्गसंयोगः । तेषां क्रमघातेन द्विगुणेन विमिश्रितो वर्गः ॥ ३० ॥
In Rule 30 स्थान *here is place in notation. After interpretation the rule works out to be*
The square of numbers consisting of two or more places is equal to the sum of the squares of all the numbers in all the places combined with twice the product of those numbers taken two at a time in order.
$(1234)^2 = (1000^2 + 200^2 + 30^2 + 4^2) + 2(1000 \times 200) + 2(1000 \times 30) + 2(1000 \times 4) + 2 (200\times30) + 2 \times (200 \times 4) + 2 \times(30 \times 4) = 1522756$

$(1 + 2 + 3 + 4)^2 = (1^2 + 2^2 + 3^2 + 4^2) + 2(1\times 2 + 1 \times 3 + 1 \times 4 + 2 \times3 + 2 \times 4 + 3 \times 4)$
$= 100$

कृत्वान्त्यकृति हन्याच्छेषपदैर्द्विगुणमन्त्यमुत्सार्य । शेषानुत्सार्यैवं करणीयो विधिरयं वर्गे ॥ ३१ ॥
Get the square of the last figure(right to left) and then multiply this last figure , after it is doubled and pushed on to the right by one notational place, by the figures found in the remaining places. Each of the remaining figures in the number is to be pushed on by one place and then dealt with similarly.
Understanding rule with example:

Square of 131:
$1^2 = 1$

2 × 1 × 3 = 6

2 × 1 × 1 =	2	
3^2=	9	
2 × 3 × 1 =	6	
1^2=	1	(1)
	1 7 1 6 1	

In Sadratnamālā most interesting concept of Kaṭapayādi in introduced to remember square of numbers.

नञावचश्च शून्यानि संख्या कटपयादयः ।

मिश्रे तूपान्त्यहल् संख्या न च चिन्त्यो हलस्वरः ॥३॥ (कटपयादिसङ्ख्यानियमः)
(३.३)

१	२	३	४	५	६	७	८	९	०
क	ख	ग	घ	ङ	च	छ	ज	झ	ञ
ट	ठ	ड	ढ	ण	त	थ	द	ध	न
प	फ	ब	भ	म					
य	र	ल	व	श	ष	स	ह		पदस्य आदौ स्वराः

वर्गपरिकर्म
तुल्योभयहतिर्वर्ग एकतः क्रमशः पदैः।
का वा धेनुस्तटे शुभ्रा तुङ्गो धावेद् वृषो यदि ॥११॥

In above sūtra Katapayādi Paddhatī is used :

द्वयोः समानयोः सङ्ख्यायोः गुणनम् वर्गः इति। एकतः क्रमशः पदैः निर्दिष्टः वर्गः =

का	वा	धे नुः	त टे	शु रा	तु गः	धा वे	वृ षः	य दि
१	४	९०	६१	५२	६३	९४	४६	१८
१	४	९	१६	२५	३६	४९	६४	८१

स्थाप्योऽन्त्यवर्गः शेषोऽपि द्विघ्नान्त्यघ्नो निजोपरि।
उपान्त्यादिम् अथोत्सार्य भूयोऽप्येवं क्रिया कृतिः॥ १२ ॥

Process left to right. The digit on the extreme left is called अन्त्य,that on its right is उपन्त्य next to last. अन्त्य is squared and placed and remaining multiplied from left to right by twice the last digit , placed as a part of the square already placed starting from next place. Procedure continues until digits are finished.

956 [9 अन्त्य ; 5 उपन्त्य]

9 5 6

Placed 9^2 8 1
Add $2 \times 9 \times 5$ 9 0
Add $2 \times 9 \times 6$ 1 0 8

---

9 1 0 8
Add 5^2 2 5
Add $2 \times 5 \times 6$ 6 0

---

9 1 3 9 0
Add 6^2 3 6

---

9 1 3 9 3 6

खण्डद्वयहतिर्द्विघ्नी खण्डद्विकृतियुत् कृतिः।
यद्वाभीष्टोनादढ्यवधोऽभीष्ट वर्गयुता कृतिः ॥ १३ ॥

In above sūtra 2 technique has been given
$a^2 = b^2 + c^2 + 2bc$ (1)
$a^2 = (a + k)(a - k) + k^2$

Chapter 6 - Cube(घन)

In Sūtras of Ganita-Sārā-Sangraha and Sadratnamalā there are given different methods of solving questions related to Cube.

Mahāvira in Ganita-Sāra-Sangraha Stated:
पञ्चमे घनपरिकर्म करणसूत्रं (गणितसारसंग्रह)

त्रिसमाहतिर्घनः स्यादिष्टोनयुतान्यराशिघातो वा। अल्पगुनणितेष्टकृत्या कलितो वृन्देन चेष्टस्य ॥ ४३॥
In Rule 43 Expressing Algebraically, comes out :

$a \times a \times a = a^3$ (The product of three equal quantities);
$a(a + b)(a - b) + b^2(a - b) + b^3 = a^3$ (The product obtained by the multiplication of given quantity by that given quantity as diminished by a chosen quantity and then again by that given quantity as increased by the same chosen quantity , when combined with the square of the chosen quantity as multiplied by the least of the above)

इष्टादिद्विगुणेष्टप्रचयेष्टपदान्वयोऽथ वेष्टकृतिः । व्येकेष्टहतैकादिद्विचयेष्टपदैक्ययुक्ता वा ॥ ४४॥

एकादिचयेष्टपदे पूर्वं राशि परेण संगुणयेत् । गुणितसमासस्त्रिगुणश्चरमेण युतो घनो भवति ॥ ४५॥

अन्त्यान्यस्थानकृतिः परस्परस्थानसंगुणा त्रिहता । पुनरेवं तद्योगः सर्वपदघनान्वितो वृन्दम् ॥ ४६ ॥

अन्त्यस्य घनः कृतिरपि सा त्रिहतोत्सार्य शेषगुणिता वा । शेषकृतिस्त्र्यन्त्यहता स्थाप्योत्सार्यैवमत्र विधिः ॥ ४६ ॥

Chapter 7 - Square root (वर्गमूलम्)

Terminology
Hindu terms for square root is 'कृतेः पदम्' or 'वर्गमूलम्'.
पदम् - It means a foot of the leg, foot, a square on a chessboard, cause, side, place, part or portion.
मूलम् - It means root of a plant, lower part of the plant, basis, foundation, cause or origin.
करणी is found to have been used in the Sulba works and prakrita literature. But in later times, this term is reserved for a surd, i.e. the square root which cannot be evaluated.

So figuratively it implies the base or the lowest part or the cause of something. Here it is the base or lowest part of a वर्ग or कृति i.e. the square.
Out of these the term मूलम् is the oldest. It appears in अनुयोगद्वारसूत्र (c 100)

Definition
पदं कृतिर्यत् तत् । - ब्रहमस्फूटसिद्धान्त 18.35

पदं = square root
कृति = square

Meaning
The square root of a square is that of which it is square.
Methods
Two totally different procedures to find the square root can be found in the ancient Indian texts.

I) By Arbitrary number (इष्टाङ्केन वर्गमूलक्रिया)
This procedure can be learned by the students of the standard 4th as it involves the methods : Multiplication, division & average. Interestingly, this exact method is applied in algorithmic systems by computers.

Verse
इष्टाप्तेष्टैक्यार्धमिष्टमवशिष्टं कृतेः पदम् । - सद्रत्नमाला १.१९

Steps to follow
i) Choose any number as इष्टाङ्क
ii) Divide the square number by इष्टाङ्क. Here quotient will be obtained.
iii) Take average of इष्टाङ्क & quotient. This average becomes इष्टाङ्क.
iv) Repeat the whole process till the divisor equals the quotient which is the square root.

Examples
1. √625 = ?
इष्टाङ्क = 20
625 ÷ 20 = 31
Average = 20+31 = 51÷2 = 25
Now इष्टाङ्क = 25
625 ÷ 25 = 25
Here quotient = divisor = 25
Hence √625 = 25

II) Basic Method

This method is convenient for the students of standards 6th to 8th. Remembering two digit perfect squares is required for this method.

This method is common in most of the Ancient indian sanskrit texts such as सद्रत्नमाला, पाटीगणितम्, आर्यभटीयम्, गणितसारसङ्ग्रह & लीलावती.

Verses
शुद्धवर्गस्य मूलेन द्विघ्नेन अवर्गततः हतम् ।
तदादिमूलं तद्वर्गः शोध्यो वर्गात्पुनस्तथा ॥
- सद्रत्नमाला १.१४.

भागं हरेतवर्गात्नित्यं द्विगुणेन वर्गमूलेन ।
वर्गाद्वर्गे शुद्धे लब्धे स्थानान्तरे मूलम् ॥
- आर्यभटीयम् २.४

Steps to follow
1. Start labelling the digits from units placed as वर्ग & अवर्ग alternately.
   eg. In the number 2916,
   6 = वर्ग,
   1= अवर्ग ,
   9 = वर्ग,
   2 = अवर्ग

2. Now start from the 1st वर्गस्थान from the left. Subtract maximum perfect square from the वर्गस्थान number. Here प्रथममूल is obtained as the root of that subtracted square.

अवर्ग	वर्ग	अवर्ग	वर्ग
2	9	1	6
-			

2 5

---

0 4
So, प्रथममूल = 5.

3. Now drag the number down from अवर्गस्थान and divide it by (प्रथममूल × 2). Here, द्वितीयमूल is obtained as a quotient of the division.

अवर्ग वर्ग अवर्ग वर्ग
2 9 1 6
-
2 5

---

10) 4 1 (4 → द्वितीयमूल
-
4 0

---

0 1

4. Now, subtract the square of the द्वितीयमूल. repeat the same procedure till all the digits get used. Having remainder as 0 at the end of the procedure is the tally point of this method which ensures that the whole method has been done correctly.

अवर्ग वर्ग अवर्ग वर्ग
2 9 1 6
10) 4 1 (4 → द्वितीयमूल
-
4 0

---

0 1 6
-
1 6

---

0 0 → Tally point

5. Now arrange the प्रथममूल, द्वितीयमूल, etc in the order from left to right to get the final square root.
In the solved example,
प्रथममूल - 5, द्वितीयमूल - 4
Final square root = 54

Example
√459684

अ	व	अ	व	अ	व	
4	5	9	6	8	4	
-3	6					→ 6 = 1st root

12)	9	9 (8 → 2nd root				÷ (6×2)
-	9	6				
	0	3	6			
-	6	4				$- 8^2$
		-2	8 → Thus tells the 2nd root is wrong.			
12)	9	9 (7 → taking the previous number, 7 = 2nd root				
-	8	4				
	1	5	6			
	-	4	9			-7^2
134)	1	0	7	8 (8→3rd root÷ (67×2)		
	-1	0	7	2		
			0	6	4	
				-6	4	-8^2
				0	0→ Tally point	

Final Square root = 678

-III) पङ्क्ति Method

This method is the additional method to the previously explained most common method. पङ्क्ति method tallies the answer one more time. This method is followed by पाटीगणितम्, गणितसारसङ्ग्रह & लीलावती. However पाटीगणितम्, गणितसारसङ्ग्रह focuses mainly on पङ्क्ति part and लीलावती follows simultaneously both the methods.

Verses

अन्त्यौजादपहृतं कृतिमूलेन द्विगुणेन वर्गमूलेन ।
लब्धकृतिस्त्याज्यौजे द्विगुणदलं वर्गमूलफलम् ॥
- गणितसारसङ्ग्रह २.३६

विषमात्पदतस्त्यक्त्वा वर्गस्थानच्युतेन मूलेन ।
द्विगुणेन भजेच्छेषं लब्धं विनिवेशयेत्पङ्क्तौ ॥२५॥
तद्वर्गं संशोध्य द्विगुणं कुर्वीत पूर्ववल्लब्धम् ।
उत्सार्य ततो विभजेच्छेषं द्विगुणीकृतं दलम् ॥२६॥
- पाटीगणितम्

त्यक्त्वान्त्याद्विषमात्कृतिं द्विगुणयेन्मूलं समे तध्दृते
त्यक्त्वा लब्धकृतिं तदाद्यविषमाल्लब्धं द्विनिघ्नं न्यसेत् ।
पङ्क्त्यां पङ्क्तिहृते समेऽन्यविषमात्यक्त्वाप्तवर्गं फलं
पङ्क्त्यां तद्विगुणं न्यसेदिति मुहुः पङ्क्तेर्दलं स्यात्पदम् ॥
- लीलावती सूत्र १०

1. Follow the previously mentioned common method as it is.
2. Make a table of two columns out of which one column is मूल & other is पङ्क्ति.
3. Each time place the मूल in the मूल column and twice of the मूल in पङ्क्ति column. Add the numbers in पङ्क्ति column by placing the digit one place to the right. (स्थान-अपकर्षण)
4. The number got by addition is the number with which the obtained number in the second step is to be divided.

Example

√151321 = 3 8 9	मूल	पङ्क्तिः
अ व अ व अ व		
1 5 1 3 2 1		
- 9	3	3 × 2 = 6
		+
6) 6 1 (8	8	8 × 2 = 1 6 (स्थान अपकर्षण)
- 4 8		7 6
1 3 3		+
- 6 4		
76) 6 9 2 (9	9	9 × 2 = 0 1 8
- 6 8 4		**7 7 8 ÷ 2 = 3 8 9**
8 1		
- 8 1		
0 0		

Analysis of the Method

1. In this way, this method tallies the answer one more time.
2. It also gives a ready number to divide which is required in the next step. Eg. in the above example we get a number 76 which is in the next step a divisor.
3. By placing the next number at the one place to the right position, it eases the calculation.
4. As we doubling the root & placing it in a पङ्क्ति column. It gives a number which is naturally twice of the root.

❖ **Importance of the Ancient Indian Methods**

1. Methods are precise, easy and accurate.
2. Methods include the fixed 2-steps, which when followed give accurate answers.
3. It has a tally point by which accuracy of the answer can be checked.
4. Method itself tells if the answer goes wrong by giving a negative number as the output.

Methods in the SSC & NCERT Syllabus

Nowadays different methods for finding a square root are taught in the schools. These methods are explained below.

I) SSC Method
√2916

2	2916
2	1458
3	729
3	243
3	81
3	27
3	9
3	3
	1

1. Keep dividing by the smallest possible prime number. Till the last number obtained as 1.
2. In the divisors column, take 1 common number out of two and multiply them to obtain the root.

Eg. In this example, Root = 2 × 3 × 3 × 3 = 54

Limitations

1. It involves guessing the divisor.So calculations may go wrong.
2. As the divisor is always a prime number, guessing the numbers above 100 or 1000 becomes difficult.
3. As the number increases it becomes more lengthy and consumes time & is difficult to calculate. So, this method is only restricted to four-digit numbers.

II) NCERT Method
√2916

	Example	**Method**
	54	Answer
	29 16	1. Place the digits in the group of two
- 5^2	- 25	2. Subtract Biggest perfect square. Here, its root is the first digit of the required square root.
5 × 2 = 10	04 16	3. Multiply its root by 2
- 10**4** × **4**	- 4 16 0 00	4. Find the number which when placed in the right of the previous number and multiplied by the same will get subtracted from the square number. In this example, the number is 4.

Limitations

1. This method also involves guessing the number in the 4th step.
2. These steps are lengthy and time consuming.
3. There are high chances that the final answer will go wrong.

- **Comparing the Ancient method & Formula**

{Formula = [(a + b)² = a² + 2ab + b²] }

Example - 14^2

Ancient Method	Formula
1= अन्त्य , 4= आदि	$(10+4)^2$
अन्त्य^2 = 1 2×1 × 4 = 0 8 आदि^{2} = 0 1 6	10^2 = 100 +2×10×4 = 80 + 4^2 = 16
14^2 = 1 9 6	**14^2 = 1 9 6**

- Thus the Ancient method avoids zeroes & hence complications in the calculation.
- 'स्थान अपकर्षण' manner i.e. leaving one place and placing to the right helps to reach to a accurate answer.
- उपान्त्य^2 should not be places in 'स्थान अपकर्षण' manner.

Chapter 8 - Cube root (घनमूलम्)

The first description of the operation Cube root is found in the आर्यभटिय. The method is mentioned below as a basic method.

I) By Arbitrary number (इष्टाङ्केन वर्गमूलक्रिया)
This procedure can be learned by the students of the standard 4th as it involves the methods : Multiplication, division & average.
Verse
घनमूलं द्विराप्तेष्टयोगार्धमविशेषितम् । - सद्रत्नमाला १.१९
Steps to follow
1. Choose any number as इष्टाङ्क
2. Divide the cube number by इष्टाङ्क. Here quotient$_1$ (q_1) will obtain.
3. Divide the q_1 by इष्टाङ्क. Here quotient$_2$ (q_2) will obtain.
4. Take average of इष्टाङ्क & quotient$_2$. This average becomes इष्टाङ्क.
5. Repeat the whole process till divisor equals to the quotient$_2$ which is the square root.

Example
$\sqrt[3]{512}$ = ?

इष्टाङ्क$_1$ = 10
$512 \div 10 = 51 \rightarrow q_1$
$51 \div 10 = 5 \rightarrow q_2$
$(5+10) \div 2 = 7 \rightarrow$ इष्टाङ्क$_2$

$512 \div 7 = 73 \rightarrow q_1$
$73 \div 7 = 10 \rightarrow q_2$
$(10+7) \div 2 = 8 \rightarrow$ इष्टाङ्क$_3$

$512 \div 8 = 64 \rightarrow q_1$
$64 \div 8 = 8 \rightarrow q_2$
Here,
इष्टाङ्क$_3$ = q_2 = cube root = 8

II) Basic Method

This method is convenient for the students of standards 6th to 8th. Remembering three digit perfect cubes is required for this method.

This method is common in most of the Ancient indian sanskrit texts such as सद्रत्नमाला, पाटीगणितम्, आर्यभटीयम्, गणितसारसङ्ग्रह & लीलावती

Verses

1. घनमूलस्य वर्गेण त्रिघ्नेनाघनतोऽन्त्यतः ।
   लब्धस्य वर्गस्त्रयादिघ्नः शोध्याश्चाद्याद् घनाद् घनः ॥ - सद्रत्नमाला १.१८

2. अन्त्यघनादपहृतघनमूलकृतित्रिहृतिभाजिते भाज्ये ।
   प्राक्त्रिहताप्तस्य कृतिः शोध्यां शोध्ये घनेऽथ घनम् ॥ - गणितसारसंग्रह १.५३

3. अघनात् भजेत् द्वितीयात् त्रिगुणेन घनस्य मूलवर्गेण ।
   वर्गस्त्रिपूर्वगणितः शोध्यः प्रथमात् घनश्च घनात् ॥ - आर्यभटीयम् २.५

4. आद्यं घनस्थानमथाघने द्वे पुनस्तथाऽन्त्यात् घनतो विशोध्य ।
   घन पृथक्स्थं पदमस्य कृत्या त्रिघ्न्या तदाद्यं विभजेत् फलं तु ॥
   पङ्क्त्यां न्यसेत् तत्कृतिमन्त्यनिघ्नीं त्रिघ्नीं त्यजेत् तत्प्रथमात् फलस्य ।
   घनं तदाद्यात् घनमूलमेवं पङ्क्तिर्भवेदेवमतः पुनश्च ॥ - लीलावती १४, १५

5. छेदो घनाद् द्वितीयाद् घनमूलकृतिस्त्रिसंगुणाप्तकृतिः ।
   शोध्या त्रिपूर्वगुणिता प्रथमाद् घनतो घनो मूलम् ॥ - ब्रह्मस्फूटसिद्धान्त १२.७

6. घनपदमघनपदे द्वे घन(पद)तोऽपास्य घनपदोमूलम् ।
   संयोज्य तृतीयपदस्याषस्तदनष्टवर्गेण ॥२९॥
   एकस्थानोनतया शेषं त्रिगुणेन संभजेत्तस्मात् ।
   लब्धं निवेश्य पङ्क्त्यां तद्वर्गः त्रिगुणमन्त्यहतम् ॥ ३० ॥
   जहयादुपरिमराशेः प्राग्वद् घनमादिमस्य च स्वपदात् ।
   भुयस्तृतीयपदस्याथ इत्यादिकर्विधिर्मूलम् ॥ ३१ ॥ - पाटीगणितम्

Steps to follow

1. Label the digits from the unit's place as घन, अघन, अघन.
2. Subtract maximum perfect cubes. Here, root = प्रथमफल
3. Drag the next number and divide the number obtained by (3×प्रथमफल^2)
4. Subtract (3×प्रथममूल×द्वितीयफल^2)
5. Subtract (द्वितीयफल)^3
6. Repeat the whole process till the remainder obtained as 0. This is the tally point.

Example

$\sqrt[3]{19683}$

	अघन	घन	अघन	अघन	घन	
	1	9	6	8	3	
	-	8				→ 1st root = 2
12)	1	1	6 (9 → 2nd root			
-	1	0	8			

		0	8	8	
	-	4	8	6 →not possible, ∴ taking previous number	
12)	1	1	6 (8 → 2nd root		
	-	9	6		
	0	2	0	8	
	-	3	3	0 →not possible, ∴ taking previous number	
12)	1	1	6 (7 → 3rd root		
	-	8	4		
	0	3	2	8	
	-	2	9	4	
			3	4	3
		-	3	4	3
			0	0	0

Final Cube root = 27

In this way, within the 5 steps the cube root of the perfect cube is obtained.

Now let's compare this method with the formula we use today for finding the cube.

❖ **Comparing the Ancient Cubing Method with the today's Formula**

Ancient Method	**Formula : $(a+b)^3 = a^3+3ab^2+3ba^2+b^3$**
Eg. 27^3	Eg. 27^3
आदि = 7, अन्त्य = 2	$(20+7)^3$
अन्त्य^3 = 2^3 = 8 3 × आदि^{2} × अन्त्य = $3 \times 7^2 \times 2 = 294$ 3 × अन्त्य^2 × आदि = $3 \times 2^2 \times 7 = 84$ आदि^{3} = 7^3 = 0343	20^3 =8000 $3 \times 7^2 \times 20 = 2940$ $3 \times 20^2 \times 7 = 8400$ 7^3 = 343
27^3 = 19683	**27^3 = 19683**

- Thus the calculations are the same. But the ancient method avoids zeroes and eases the calculation by placing the numbers by the method called '**स्थान अपकर्षणेन लेखनम्**'.

❖ **Comparing the Ancient Method to find Cube and Cube root**

Cube	**Cube Root**

27^3 **:** आदि = 7, अन्त्य = 2	$\sqrt[3]{19683}$
अन्त्य^3 = 2^3 = 8 →(i) 3 × अन्त्य^2 × आदि = 3 × 2^2 × 7 = 84 →(ii) 3 × आदि^{2} × अन्त्य = 3 × 7^2 × 2 = 294 →(iii) आदि^{3} = 7^3 = 0343 →(iv) ——— **27^3 = 19683**	अ घ अ अ घ 1 9 6 8 3 - 8 →(i) (Subtracting 2^3) ——— 12) 1 1 6 (9 (Dividing by 3 × 2^2) -1 0 8 ——— 8 8 - 4 8 6 (Subtracting 3 × 2 × 9^2) ——— Not possible 12) 1 1 6 (8 (Taking previous number) - 9 6 ——— 2 0 8 - 3 3 0 (Subtracting 3 × 2 × 8^2) ——— Not possible 12) 1 1 6 (7 - 8 4 ——— 3 2 8 - 2 9 4→(iii)(Subtracting 3 × 2 × 7^2) ——— 3 4 3 - 3 4 3 →(iv) (Subtracting 7^3) ——— 0 0 0

- If you compare the steps involved in the ancient method of finding a cube and cube root, (i), (ii), (iii), (iv) numbers are exactly the same.
- The steps involved in finding a cube (multiplication) and cube-root (division) just help to find the exact number that needs to be added or subtracted.

Chapter 9 - Fractions

Early use

1. त्रिपादूर्ध्व उदैत्पुरुषः पादोऽस्येहाभवत्पुनः । ततो विष्वङ्व्यक्रामत्साशनानशने अभि ।।
   - ऋग्वेद 10.90.4

Here, the term त्रिपाद is used which means ¾ and पाद means ¼.
This references regarding the fraction ¾ as त्रिपाद in ऋग्वेद is probably the oldest record of a composite fraction known to us.

2. सोमविक्रयित्सोमं ते क्रीणानि महान्तं बह्वर्हं बहु शोभमानं, कलया ते क्रीणानि कुष्ट्या ते क्रीणानि, शफेन ते क्रीणानि, पदा ते क्रीणानि ।।

- मैत्रायणी संहिता 3.7.7

Here, i) कला = 1/16th
ii) कुष्ट = 1/12th
iii) शफ = 1/8th
iv) पाद = 1/4th

3. पुच्छे प्रादेशमुपधाय सर्वमाग्निं पञ्चमभागीयाभिः प्रच्छादयेत् | पञ्चदशभागीयाभिः सङ्ख्यां पूरयेत्||
   Here, पञ्चदशभाग = 1/15th
   Unit fraction are denoted by the use of cardinal number (पञ्चदश) with term 'भाग' or 'अंश'.
4. पञ्चदशभागोऽष्टाङ्गुलम् | - कात्यायन शुल्बसूत्र ५.८
   Here, पञ्चदशभाग = 1/15th

5. प्रक्रमेण वा ससप्तमभागेन प्रक्रमार्थः | - कात्यायन शुल्बसूत्र ६.४
   Here, सप्तमभाग = 1/7th

6. सत्र्यंशरूपद्वितयेन पञ्चत्र्यंशेन षष्ठे वद मे विभज्य |
   दर्भीयगर्भाग्रसुतीक्ष्णबुद्धिश्चेदस्ति ते भिन्नहृतौ समर्था ॥

   सत्र्यंशद्वितीय = 2 1/3
   षष्ठ = 1/6
   त्र्यंश = 1/3
   Here अंश = 1/3
   षष्ठ = षष्ठभाग = 1/6

   Sometimes, 'भाग' word is committed for the sake of metrical convenience. When fractions have unit numerators only the denominators are mentioned.

Terminology

Term	Meaning
1. भिन्न	broken European terms like fractio, fraction, roupt, rotto are derived from latin word fractus which means broken too. These are translations of Sanskrit term भिन्न.
2. भाग, अंश	parts/ portion
3. कला	It was earlier used for 1/16. Later used as fraction in शुल्बसूत्र.

- **Reduction to a common denominator (जातिचतुष्टयम्)**

The fractions having different denominators can't be added or subtracted. That's why to carry out addition and subtraction in these fractions, reducing them to a common denominator is necessary. The 4 ways to do so are given under the topic of जातिचतुष्टयम् in लीलावती by भास्कराचार्य II. The point to be note here is भास्कराचार्य II has mentioned all these processes within 1 or even in the half verse.

- भागजातिः (समच्छेदविधानम्)

अन्योन्यहराभिहतौ हरांशौ राश्योः समच्छेदविधानम् ।
मिथो हराभ्यामपवर्सिताभ्यां यद्वा हरांशौ सुधियात्र गुण्यौ ॥१॥
-लीलावती, भिन्नपरिकर्माष्टक

Glossary
समच्छेदविधानम् = Reduction to common denominator
हर = denominator
अश = numerator
हत = multiplied
अन्योन्यहराभिहतौ = cross multiplied

Explanation
There are two methods of this process.
i) The fractions are reduced to the common denominator by the cross multiplication.
E.g ⅔ + ¾ = (2×4 + 3×3) / (4×3) = (8 + 9) / 12 = 17/12

ii) Simplification of the fractions and multiplying them with a common number to get a common denominator.
Eg. 3/2 + ⅚ = 3×3/2×3 + ⅚ = 9/6 + ⅚ = (9+5)/6 = 14/6 = 7/3

- प्रभागजातिः

लवा लवघ्नाश्च हरा हरघ्ना भागप्रभागेषु सवर्णनं स्यात् ।
-लीलावती, भिन्नपरिकर्माष्टक ॥२॥

Glossary
लव = numerator
प्रभाग = fraction of the fraction
-घ्न = multiplied
सवर्णनं = simplified

Meaning
In the case of multiplication, numerator is multiplied by numerator and denominator is multiplied by denominator.

Example
¾ + ⅔ = 3×2/4×3 = 6/12 = 1/2

- भागानुबन्ध & भागापवाह

Verse

छेदघ्नरूपेषु लवा धनर्णमेकस्य भागाधिकोनश्चेत् ॥२॥
स्वांशाधिकोनः खलु यत्र तत्र भागानुबन्धे च लवापवाहे ।
तलस्यहारेण हरं निहन्यात् स्वांशाधिकोनेन तु तेन भागात् ॥ ३ ॥.
-लीलावती, भिन्नपरिकर्माष्टक

Meaning with an example
When a fraction of the number is added to the same number, the process is भागानुबन्ध.
eg. 4 + 4/3; Here 1/3rd part of a 4 is added.

The last line of the verse denotes the process of reduction to the common denominator.

When a fraction of a number is subtracted from a same number then the process is भागापवाह.
Eg. 3 - ¾; Here 1/4th part of 3 is subtracted or taken away from whole 3.

भिन्नपरिकर्माष्टकम् (Operations of Fractions)

अष्टक means octet. So here basic 8 operations of fractions are mentioned which are as follows:

1) Addition
2) Subtraction
3) Multiplication
4) Division
5) Square
6) Cube
7) Square root
8) Cube root

- भिन्नसङ्कलितव्यवकलिते

Verse

योगोऽन्तरं तुल्यहरांशकानां कल्प्यो हरो रूपमहारराशेः ।

-लीलावती, भिन्नपरिकर्माष्टक ॥४॥

Glossary

1. योग - addition
2. अन्तर - subtraction
3. तुल्य - same
4. हर - denominator
5. अंश - Numerator
6. रूप - Constant
7. अहारराशि - whole number

Meaning with an Example

Here, two rules are mentioned.

i)Addition & subtraction of only those fractions having the same denominator is possible.

Eg. a) 3/2 + 2/3 is not possible without reducing them to a common denominator.

b) ⅔ + 5/3 = (2+5)/3 = 7/3 is possible because both the fractions have the same denominator i.e. 3.

ii) If a whole number is added/subtracted to a fraction, its denominator shall be imagined as 1.

Eg. 3 = 3/1

- भिन्नगुणनम्

Verse

अंशाहतिच्छेदवधेन भक्ता लब्धं विभिन्ने गुणने फलं स्यात् ॥

-लीलावती, भिन्नपरिकर्माष्टक ॥४॥

अन्वय

विभिन्ने गुणने अंशाहतिः छेदवधेन भक्ता फलं स्यात् ।

Meaning

विभिन्ने गुणने - In the case of multiplication of fractions.

अंशाहतिः - multiplication of numerator

छेदवधः - Product of denominator

So,
अंशाहतिः
—--------- = फलम्
छेदवधः

I.e. N1/D1 × N2/D2 × N3/D3 = N1×N2×N3 / D1×D2×D3

Example
$\frac{2}{3} \times \frac{5}{6}$ = 2×5 / 3×6 = 10/18 = 5/9

- भिन्नभागहरणम्

Verse
छेदं लवं च परिवर्त्य हरस्य शेषः ।
कार्योऽथ भागहरणे गुणनाविधिश्च ॥
- लीलावती, भिन्नपरिकर्माष्टक ॥५॥

Meaning
For division of the fractions, the denominator & numerator of the divisor are exchanged and the fraction thus formed is multiplied.
eg . A/B ÷ C/D
Here, C/D = divisor; exchanging Numerator & denominator = D/C
A/B ÷ C/D = A/B × D/C = AD/BC

Example
3/4 ÷ 2/5 = 3/4 × 5/2 = 3×5 / 5×4 = 15/20 = 3/4

- भिन्नवर्गादयः

Verse
वर्गे कृती घनविधौ तु धनी विधेयौ ।
हरांशयोरथ पदे च पदं विधेयम् ।। - लीलावती, भिन्नपरिकर्माष्टक ॥५॥

Explanation
When square, cube, square root or cube root of a fraction is to be calculated then take the square, cube, square root or cube root of both numerator and denominator.

Examples
1)$(\frac{2}{3})^2 = 2^2/3^2$ = 4/9
2) $(\frac{2}{3})^3 = 2^3/3^3$ = 8/27
3)$\sqrt{(4/25)} = \sqrt{4}/\sqrt{25}$ = 2/5
4) $\sqrt[3]{(\frac{1}{8})} = \sqrt[3]{1}/\sqrt[3]{8} = (\frac{1}{8})$

Chapter 10 - Regula Falsi (इष्टकर्म)

The rule of false position is found in all the Hindu works. Bhaskar II gives prominence to this method and calls it as इष्टकर्म.

Verse
उद्देशकालापवदिष्टराशिः क्षुण्णो ह्रतोंऽशै रहितो युतो वा।।॥
इष्टाहतं दृष्टमनेन भक्तं राशिर्भवेत्प्रोक्तमितीष्टकर्म ॥ १० ॥ - लीलावती, इष्टकर्मप्रकार

Meaning
Any number, assumed, is treated as specified in the particular question, being multiplied and divided, increased or diminished by fractions (of itself); then the given quantity, being multiplied by the assumed number and divided by that (which has been found) yields the number sought. This is called the process of supposition.
∴ (दृष्टं × इष्टराशिः) / answer obtained by इष्टराशि = Original number

Example
पचघ्नः स्वत्रिभागोनो दशभक्तः समन्वितः ॥
राशित्र्यंशार्द्धपादैः स्यात्को राशिद्व्युनसप्ततिः ॥११॥ - लीलावती, इष्टकर्मप्रकार

If a number is multiplied by 5 then decreased by ⅓rd of the previous value. The answer thus obtained is then divided by 10; then added to the ⅓rd, ½, ¼th part of the assumed number; then the answer obtained is 68. Find the original number.

Solution:

Suppose the number is 6; इष्टराशिः = 6
6 × 5 = 30
30 - ⅓ (30) = 30 - 10 = 20
20 ÷ 10 = 2
2 + (6/3 + 6/2 + 6/4) = 2 + 13/2 = 17/2
Original number = (68 × 6) ÷ 17/2
= 68 × 6 × 2/17 =
= 48

Glossary
क्षुण्णः = multiplied
द्वि-ऊन-सप्ततिः = 68
राशि-त्र्यंश = 1/3rd of इष्टराशि
राशि-अर्ध = 1/2rd of इष्टराशि
राशि-पाद = ¼rd of इष्टराशि
स्वत्रिभाग = 1/3
ऊन - less
दृष्टं = Final answer in the example
इष्टराशिः = assumed number
समन्वितः = added

Chapter 11 - Method of Inversion

Verse
गुणकारा भागहरा भागहरास्ते भवन्ति गुणकाराः ।
यः क्षेपः सोऽपचयोऽपचयः क्षेपश्च विपरीते ॥ - आर्यभटीय. गणिताध्याय.२८

Glossary
विपरीत - inversion method
गुणकार - multiplier
भागहर - divisor
क्षेप - addition
अपचय - subtraction

Meaning
In method of Inversion,
Multipliers are replaced by divisors.
Additives are replaced by Subtractive.

Example
A number, if multiplied by 2; then increased by 1;then divided by 5;then multiplied by 3; then diminished by 2; then divided by 7; the result thus obtained is 1. Find the number.

Solution :

Starting from the last number 1, in the reverse order, inverting the operations, the result is
1 x 7 = 7; 7 + 2 = 9; 9 ÷ 3 = 3; 3 x 5 = 15; 15 - 1 = 14; 14 ÷ 2 = 7
Therefore, Number is 7.

Chapter 12 - Rule of three (त्रैराशिकम्)

करणसूत्र :
प्रमाणमिच्छा च समानजाति आद्यन्तयोः स्तः फलमन्यजातिः ।
मध्ये तदिच्छाहतमादिहृत्स्यादिच्छाफलं व्यस्तविधिर्विलोमे ॥१७॥ - लीलावती, त्रैराशिकप्रकार

Terms
प्रमाणं - आदिः
फलं - मध्यः
इच्छा - अन्तः

Glossary
फलं इच्छाहतं - फल x इच्छा
आदिहृत् - divided by आदि
समानजाति - same unit
विलोम - inverse proportion
प्रमाण - Quantity whose फल is given
फल - result of the प्रमाण
इच्छा - quantity for which फल isn't given
इच्छाफल - that which is asked to find

Formula
- In direct proportion,

इच्छाफल = (फल x इच्छा) / प्रमाण

- So in the case of inverse proportion,

(प्रमाण x फल)/ इच्छा = इच्छाफल

- प्रमाण and इच्छा have the same unit. फल has a different unit

Example

1. पङ्गुः प्रयाति कश्चिद्दिवसत्रितयेन योजनाष्टांशम् ।
   योजनशतं स यास्यति निगद्यतां केन कालेन ॥ - क्रियाक्रमकरी

I.e. When a paralysed person walks ⅛ th of योजन within 3 days. How many days will he need to walk a distance of 100 योजन ?

Here, ⅛ योजन = प्रमाण, 3 days = फल

इच्छा = 100 योजन

To find - इच्छाफल = No. of days he need to walk a distance of 100 योजन

There is a direct proportion between distance and number of days.

इच्छाफल= (फल × इच्छा) / प्रमाण

= (3 × 100) / ⅛

= 2400 days

- **व्यस्तत्रैराशिकम् (Inverse rule of three)**

Concept

इच्छा वृद्धो फले ह्रासो ह्रासे वृद्धिः फलस्य तु ।

व्यस्तं त्रैराशिकं तत्र ज्ञेयं गणितकोविदैः ॥१८॥ - लीलावती. व्यस्तत्रैराशिकप्रकार

If इच्छा increases then फल decreases and if इच्छा decreases then फल increases; this is called व्यस्तत्रैराशिकम्

Area of Applications

जीवानां वयसो मौल्ये तौल्ये वर्णस्य हेमनि ।

भागहारे च राशीनां व्यस्तं त्रैराशिकं भवेत् ॥ १॥ - लीलावती. व्यस्तत्रैराशिकप्रकार

व्यस्तत्रैराशिकम् can be applied in the following cases.

i) When the age of an animal its price decreases.

ii) In the case of the gold, if the carats are increased, the quantity of the gold in the same amount (price) decreases.

iii) When the divisor increases, the quotient decreases.

Example

सप्ताढकेन मानेन राशौ सस्यस्य मापिते ।

यदि मानशतं जातं तदा पञ्चाढकेन किम् ॥३॥ - लीलावती. व्यस्तत्रैराशिकप्रकार

If an agricultural yield is measured with the container of 7 आढक, it is 100 times of the container of 7 आढक. Then how many times of the container of 5 आढक will it take to measure same amount of the yield?

प्रमाण = 7 आढक

फल = 100 times

इच्छा = 5 आढक

इच्छाफल is to be found.

There is an inverse relation.

इच्छाफल = (प्रमाण × फल)/ इच्छा = 7 × 100 / 5 = 140 times

So, it will be measured in 140 times of the container of 5 आढक.

Chapter 13 - Rule of Five and more (पञ्चराशिकादिकम्)

करणसूत्र
पंचसप्तनवराशिकादिकेऽन्योन्यपक्षनयनं फलच्छिदाम् ॥
संविधाय बहुराशिजे वधे स्वल्पराशिवधभाजिते फलम् ॥१९॥ - लीलावती, पञ्चराशिकादिप्रकार

Explanation

1. This verse tells about the operation of पञ्चराशिकम्, सप्तराशिकम् and नवराशिकम्. When the quantities are in direct relation, इच्छाफल = (product of all quantities in इच्छा column × फल) / product of all quantities in प्रमाण column
2. When the quantities are in inverse relation. In this case, exchange the terms in प्रमाण and इच्छा column which are in inverse relation and carry out the same process explained above.
3. So the quantities in the numerator are more than the quantities in the denominator.

Example

1. पञ्चराशिकम् -

मासे शतस्य यदि पञ्च कलान्तरं स्याद्वर्षे गते भवति किं वद षोडशानाम् ॥
कालं तथा कथय मूलकलान्तराभ्यां मूलं धनं गणक कालफले विदित्वा ॥ १ ॥
- लीलावती, पञ्चराशिकादिप्रकार

If in 1 month interest is 5 units on an amount of 100 units; then after 1 year how much will be the interest on the amount of 16 units. How much time will it take to obtain the same interest? By knowing the period of 12 months and interest rate of 48/5, find out the principal amount.

1) To find: Interest on 16

प्रमाण	इच्छा	फल	इच्छाफल
100 units (Principal) 1 month (duration)	16 units (Principal) 12 months (duration)	5 units (interest)	Interest = ?

There is a direct relation between duration & interest and Principal-interest.
इच्छाफल = (फल × इच्छा) / प्रमाण = (5 × 16 × 12)/100× 1 = 48/5
So 48/5 units will be the the interest on amount of 16 units

2) To find: time required to obtain the interest of 48/5 units

प्रमाण	इच्छा	फल	इच्छाफल
100 units (Principal) 5 units (interest)	16 units (Principal) 48/5 units (interest)	1 month (duration)	Duration = ?

Duration and interest have a direct relation.

For getting the same interest, if the amount is increased then the period decreases. So Principal and duration have an inverse relation.

So interchanging Principal quantities in प्रमाण and इच्छा column.

प्रमाण	इच्छा	फल	इच्छाफल
16 units (Principal) 5 units (interest)	100 units (Principal) 48/5 units (interest)	1 month (duration)	Duration = ?

Here, इच्छाफल = इच्छा × फल / प्रमाण
= [1 × (48/5) × 100]/5 × 16
= 12 months
So, it will take 12 months to get 48/5 units interest.

3) To find: By knowing the period of 12 months and interest rate of 48/5, find out the principal amount.

प्रमाण	इच्छा	फल	इच्छाफल
1 month 5 units (interest)	12 months 48/5 units (interest)	100 units (principal)	Principal=?

Principal and interest have a direct relation.
For getting the same interest, if the amount is increased then the period decreases. So Principal and duration have an inverse relation.

So interchanging Principal quantities in प्रमाण and इच्छा column.

प्रमाण	इच्छा	फल	इच्छाफल
12 months 5 units (interest)	1 month 48/5 units (interest)	100 units (principal)	Principal=?

इच्छाफल = इच्छा × फल / प्रमाण
= (100 × 1 × 48/5) / (12 × 5)
= 16 units will be the principal.

2. नवराशिकम्
पिंडे येऽर्कमितांगुलाः किल चतुर्वगांगुला विस्तृतौ पट्टा दीर्घतया चतुर्दशकरास्त्रिशलभन्ते शतम् ॥
विस्तृतिपिण्डदैयमितयो येषां चतुर्वर्जिताः एता पट्टास्ते वद मे चतुर्दश सखे मूल्यं लभन्ते कियत् ॥४॥
- लीलावती, पञ्चराशिकादिप्रकार

30 Planks having 12 अंगुल height, 16 अंगुल breadth and 14 कर length are sold for a price of 100 units. When 14 such planks having measurements less than 4 are sold, how much money will one get?

प्रमाण	इच्छा	फल	इच्छाफल
30 12 अंगुल height 16 अंगुल breadth 14 कर length	14 8 अंगुल height 12 अंगुल breadth 10 length	100 (price)	?

There is a direct relation between measurements and price. Also between the no. of planks and the price.

इच्छाफल = (14 × 8 × 12 × 10 × 100) / (30 × 12 × 16 × 14)
= 50 / 3 units will one get on selling 14 planks having mentioned measurements.

Chapter 14 - Barter System (भाण्डप्रतिभाण्डकम्)

The meaning of the term भाण्डप्रतिभाण्डकम् is commodity for commodity.
Barter system is an extended application of पञ्चराशिकम्.

यत्र केनचित् कस्मैचित् किञ्चिद् भाण्डं दत्वा ततोऽन्यत् किञ्चित् प्रतिभाण्डमुपादीयते तदर्थं गणितं भाण्डप्रतिभाण्डकम् । तत्र च व्यस्तपञ्चराशिकस्य यो विधिरुक्तरूपः स एव तथैव योज्यः ।
- क्रियाक्रमकरी

Meaning - When an object is obtained in exchange of another object. The calculation related to this case are included in भाण्डप्रतिभाण्डकम्. In this method व्यस्तपञ्चराशिक is used.

Barter is a method of trading where goods are exchanged directly for one another without using money as an intermediary. It is an old method of exchange. People exchanged services and goods for other services and goods in return.
That's why problems based on the barter system are found in our ancient texts.

Example
द्रम्मेण लभ्यत इहाम्रशतकत्रयं चेत् ।
त्रिंशत्पणेन विपणौ वरहाडिमानि ।
आम्रैर्वदाशु दशभिः कति दाडिमानि ।
लभ्यानि तद्विनिमयेन भवन्ति मित्र ॥१॥ - लीलावती, भाण्डप्रतिभाण्डकविधि

प्रमाण	इच्छा	फल	इच्छाफल
300 आम्रफलानि 16 पणाः	30 दाडिमानि 1 पणः	10 आम्रफलानि	?

As it is application of व्यस्तपञ्चराशिक, inverse relation is necessary. Here with the same money, more दाडिमानि can be purchased as compared to the आम्रफलानि.

So interchanging the positions of fruits in the प्रमाण and इच्छा.

प्रमाण	इच्छा	फल	इच्छाफल
300 आम्रफलानि 1 पणः	30 दाडिमानि 16 पणाः	10 आम्रफलानि	?

इच्छाफल = (30 × 10 × 16) / 300 × 1
= 16

Chapter 15 - Concept and Operations of Zero

- **Why do we need Zero?**

अथैकादिस्थानस्थिताङ्कगणनाया यत्राङ्काभावस्तत्राङ्काभावद्योतनार्थं शून्यं निवेश्यते । इतरथा दशशतसहस्रादीनामभेदप्रसङ्गः । - बुद्धिविलासिनि, गणेशदैवज्ञ

When there is absence of any number at any decimal place (units,tens,etc) then 0 is placed to denote that absence. So, 0 shall also be considered as a number. Otherwise there will be no difference between hundred and thousand.
So, zero is a number. Therefore the operations with the same shall also be mentioned. Thus the whole background of zero is given.

This discussion is significant for knowing the importance and the origin of the zero as a number. Interestingly the zero appears in the works of many ancient scholars. Some of those are mentioned below.

- ब्रह्मस्फूटसिद्धान्त, ब्रह्मगुप्त

1. **Addition with Zero**

समैक्यं खम् । ऋणमैक्यं च धनमृणधनशून्ययोः शून्ययोः शून्यम् । - ब्रह्मस्फूटसिद्धान्त १८.३०

Explanation

- समैक्यं खम् - When a positive and negative number of the same magnitude are added then the answer is zero.
  Eg. (+3) + (-3) = 0

- ऋणमैक्यं च धनमृणधनशून्ययोः
  i) ऋणशून्ययोः ऐक्यं ऋणम् - When the negative number and zero are added then result is the same negative number.
  Eg. (-3) + 0 = (-3)

  ii) धनशून्ययोः ऐक्यं धनम् - When the positive number and zero are added then result is the same positive number.
  Eg. 4 + 0 = 4

- शून्ययोः शून्यम् - When a zero added to zero, the answer is zero.

Eg. 0 + 0 = 0

2. **Subtraction with zero -**
   शून्यविहीनमृणमृणं धनं धनं भवति शून्यमाकाशम् । - ब्रह्मस्फूटसिद्धान्त १८.३२

   Explanation
   - शून्यविहीनं ऋणं ऋणं धनं धनं -

   i) If a zero is subtracted from a positive number, the answer is the same positive number.
   Eg. 4 - 0 = 4

   ii) If a zero is subtracted from a negative number, the answer is the same negative number.
   (-3) - 0 = (-3)

   - शून्यं आकाशं : zero - zero = zero

3. **Multiplication with zero**
   शून्यर्णयोः खधनयोः खशून्ययोर्वा वधः शून्यम् । - ब्रह्मस्फूटसिद्धान्त १८.३३

   Explanation
- शून्यर्णयोः - zero × negative = zero
  Eg. 0 × (-3) = 0

- खधनयोः - zero × positive = zero
  Eg. 0 × 4 = 0

- खशून्ययोः वा - zero × zero = zero

4. **Division with Zero**
   खं खभक्तं खम्। - ब्रह्मस्फूटसिद्धान्त १८.३४
   Explanation
- खं खभक्तं खम् : zero ÷ zero = zero

5. **Concept of तच्छेद**

खोद्धृतमृणं धनं वा तच्छेदं खमृणधनविभक्तं वा ।
ऋणधनयोर्वर्गः स्वं - ब्रह्मस्फूटसिद्धान्त १८.३५

   Explanation
- खोद्धृतं ऋण धनं वा तच्छेदं
  Positive/negative ÷ zero = तच्छेद
  Eg. 4/0 = तच्छेद or (-4)/0 = तच्छेद

- खम् ऋणधन विभक्तं वा
  zero ÷ positive/negative : तच्छेद / zero
- ऋणधनयोः वर्गः स्वं : (positive/negative)2 = positive
  Eg. $(2)^2 = 4$; $(-2)^2 = 4$

6. **Square, Square root, Cube, Cube root of zero**
   खं खस्य पदं कृतिर्यत् तत् ॥ - ब्रह्मस्फूटसिद्धान्त १८.३५
   Explanation
   $\sqrt{0} = 0$
   $\sqrt[3]{0} = 0$

❖ लीलावती, भास्कराचार्य : शून्यकर्मप्रकार

योगे खं क्षेपसमं वर्गादौ खं खभाजितो राशिः ।
खहरः स्यात्खगुणः खं खगुणश्चिन्त्यश्च शेषविधौ ।।६।।
शून्ये गुणके जाते खं हारश्चेत् पुनस्तदा राशिः ।
अविकृत एव ज्ञेयस्तथैव खेनोनितश्व युतः ।।७।।

i) योगे खं क्षेपसमं : Any number if added to zero, result is that number only.
Eg. $0 + x = x$

ii) वर्गादौ खं =
Square, square root, cube and cube root of zero is zero only.
i.e. $0^2 = 0$
$0^3 = 0$
$\sqrt{0} = 0$
$\sqrt[3]{0} = 0$

iii) खभाजितो राशिः खहरः -
When a number is divided by zero it is called as खहरः

iv) खगुणः खं - If a number is multiplied by zero, the result is zero.

v) खगुणश्चिन्त्यश्च शेषविधौ - If a multiplier is zero, then don't be in a hurry to write answer as zero. Give it a thought for a while by observing further steps.
Then two such cases are mentioned.

a) If a number is multiplied by zero and then immediately divided by zero, two zeroes in the numerator and denominator cancel each other giving the same number as an answer. Eg. $p \times 0 / 0 = p$
b) If a zero is added or subtracted from a number, the number remains unchanged.
   Eg. $p + 0 = p$
   $p - 0 = p$

खं - 0
क्षेप - that which is added
खहरः - that quantity whose divisor is zero
खगुणः - that quantity whose multiplier is zero
अविकृत - unchanged

- ❖ **Nature of the खहर**

अस्मिन् विकारः खहरे न राशावपि प्रविष्टेष्वपि निःसृतेषु ।
बहुष्वपि स्याल्लयसृष्टिकालेऽनन्तेऽच्युते भूतगणेषु यद्वत् ॥२०॥ - बीजगणित, भास्कराचार्य

Meaning - At the time of creation, the beings are born from Brahman and at the time of destruction of the universe, the beings merge into the Brahman. But as the Brahman is infinite and all-pervasive, it doesn't make any difference to the Brahman.
Similarly, if any number is added to खहर or subtracted from खहर, it doesn't make any difference to the खहर.
Here the idea of infinity seems to be getting introduced. But it is not directly mentioned. This idea is likely to be originated from इशोपनिषद where the verse of similar meaning is mentioned.
The verse is as follows.
ॐ पूर्णमदः पूर्णमिदं पूर्णात् पूर्णमुदच्यते। पूर्णस्य पूर्णमादाय पूर्णमेवावशिष्यते॥

- ❖ **Nature of खगुण**

खगुणः खमिति । खेन गणो राशिः खं भवेत् । अत्र जीवन्मुक्तदृष्टान्तः । - बुद्धिविलासिनि, गणेशदैवज्ञ

The quantity multiplied by zero, becomes 0. Here, the example given is जीवन्मुक्त. जीवन्मुक्त is a person who has got a knowledge of the ultimate reality i.e. Brahman. So he has become one with the Brahman but still as his life is not yet over, so he is living. The person here knows Brahman, becomes brahman.
Similarly, quantity multiplied by zero, becomes 0.

- ❖ शून्यषड्विधम् । - बीजगणित, भास्कराचार्य

Here, six rules for zero are mentioned.
खयोगे वियोगे धनर्ण तथैव च्युतं शून्यतस्तद्विर्यासमेति ॥ १६ ॥
If zero is added to or subtracted from any number, that number remains as it is; its positivity or negativity remains the same. But if from zero something is removed, its sign changes.

वधादौ वियत् खस्य खं खेन घाते । बहारो भवेत् खेन भक्तश्च राशिः ॥ १८ ॥
If zero is multiplied by any number or divided by any number, the product is zero. If any number is divided by zero the result is a quantity with zero divisor.

Section 3 - Algebra

Chapter 1 - Introduction

Hindu name of algebra is bījagaṇita. bīja means element/analysis and gaṇita means The science of calculation so, bījagaṇita is "The science of calculation with elements". The Epithet date is as far back as the time of Prathudakaswami (860) who used it. Brahmagupta (628) calls algebra kuṭṭaka-gaṇita or kuṭṭaka. kuṭṭaka means "pulveriser"(dealing particularly with the subject of indeterminate equations).

Science of algebra was named in the beginning of the 7th century. Algebra is also called avyakta-gaṇita. (the science of calculation with unknowns). It is in contradistinction to the name - vyakta gaṇita.
Avyakta - unknown.
Vyakta - known

Vyakta ganita- arithmetic, geometry, mensuration

<u>Algebra definition</u>

Bhāskara II - Analysis (bija) is certainly the innate intellect assisted by the various symbols (varna), which, for the instruction of Duller intellects has been expounded by the ancient sages enlighten mathematicians as the sun irradiates the Lotus with the help of various symbols has now obtained the name of algebra. Algebraic analysis requires Intelligence and sagacity. Analysis is certainly clear intelligence.

Distinction from arithmetic

Both deal with symbols. In Arithmetic the values of symbols are vyakta (known). In Algebra the values are avyakta (unknown).

Bhaskara II - mathematicians have declared algebra to be computation attendant with demonstration else there would be no distinction. This dictum is evident in the treatment of the guṇa karma in Lilāvati and the madhyamāharaṇa in the bijganita. Method of demonstration has been stated to be always of two kinds

1) geometrical (kṣetragata)
2) symbolical (rāśigata)

Importance of algebra

Brahmagupta (628)

प्रायेण यतः प्रश्नाः कुदाकाराद्रतेन शक्यंते।

ज्ञातुं वक्ष्यामि ततः कुदाकारं सह प्रश्नैः ॥१॥

-ब्रह्मस्फूटसिद्धान्त, कुट्टकाध्याय(१८)

Bhāskara II (1150)

Bhaskara II has defined algebra thus:
उत्पादकं यत् प्रवदन्ति बुद्धेरधिष्ठितं सांख्या ।
व्यक्तस्य कृत्स्नस्य तदेकबीजम् व्यक्तमीशं गणितं च बन्दे।।१।।
भाज्यो हार क्षेपकश्चापत्र्य केनाप्यादौ सम्भवे कुट्टकार्थम्।
येनच्छिन्नौ भाज्यहारौ न तेन क्षेपश्चैतदिदुष्टमुद्दिष्टमेव।।१।।
- बीजगणितम्, भास्कराचार्य

Nārāyana (1350)

एकमनेकस्योक्तं(नित्यं) व्यक्तस्य गुणवतो जगतः ।

गणनाविधेश्च बीजं ब्रह्म च गणितं च तद् बन्दे ॥१॥

व्यक्तक्रियया ज्ञातुं प्रश्नानखिली भवन्ति नाल्पधियः।

बीजक्रियां च तस्माद् वच्मि व्यक्तां सुबोधां च॥६॥

-बीजगणितावतंस, नारायण पंडित

Origin of Hindu Algebra

-Śulba sutra (800-500 BC)

-Brāhmaṇa (2000 BC)

But it was mostly geometrical. The geometrical method of the transformation of a square into a rectangle having a given side in śulba is equivalent to the solution of a linear equation in one unknown.

Chapter 2 - Technical Terms

Right from the vedic period, a variety of the words for algebraic terms were coined. Few of these are mentioned below.

1. **Coefficient**

There is no systematic use of any special term for the coefficient. But some references are there.

Scholars	**Word used as Coefficient**
1. ब्रह्मगुप्त	संख्या (number) गुणक/गुणाकार (multiplier)
2. पृथुदकस्वामी, श्रीपति, भास्कर II	अङ्क (number) प्रकृति (multiplier)
3. Former works	रूप

2. Unknown Quantity

Terms used	Information
स्थानांग सूत्र	Before 300 BC
यावत् तावत्	Meaning - as many as, so much as, an arbitrary quantity
यदृच्छा वाञ्चा कामिक	Meaning - any desired quantity These words are used in the treatise called बख्शाली.
गुलिका	This term is used by आर्यभट I.The meaning is shot.
अव्यक्त	The meaning is unmanifested or not known. This term is more commonly used from the 7th Century.

Chapter-3 Symbols of Operations

There are no special symbols for fundamental operation in Bakshali Manuscript.

Abbreviations are used for this operations -

Yu (यु) - yuta युत(addition)

+ - kṣaya क्षय (subtraction)

Gu (गु) - guṇita गुणित (multiplication)

Bhā(भा) - bhājita भाजित (division)

Mū (मू) - mūla मूल (square root)

Zero(०) - vacant place

Symbols for powers and Roots -

The symbols for power and roots are sanskrit abbreviation-

Square - va(varga)

Cube - gha(ghana)

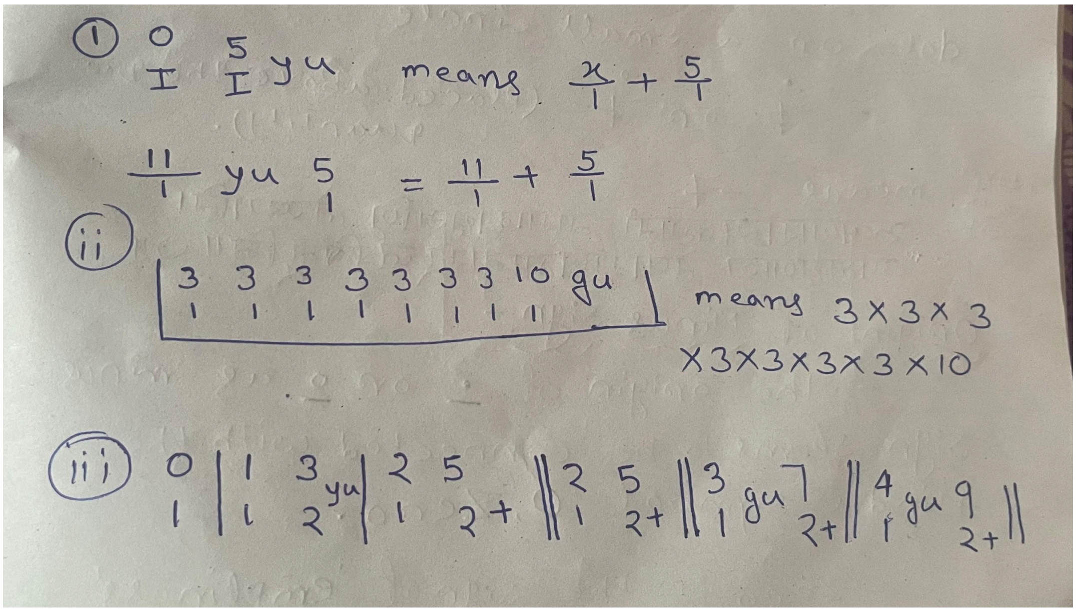
4th - va-va(varga-varga)

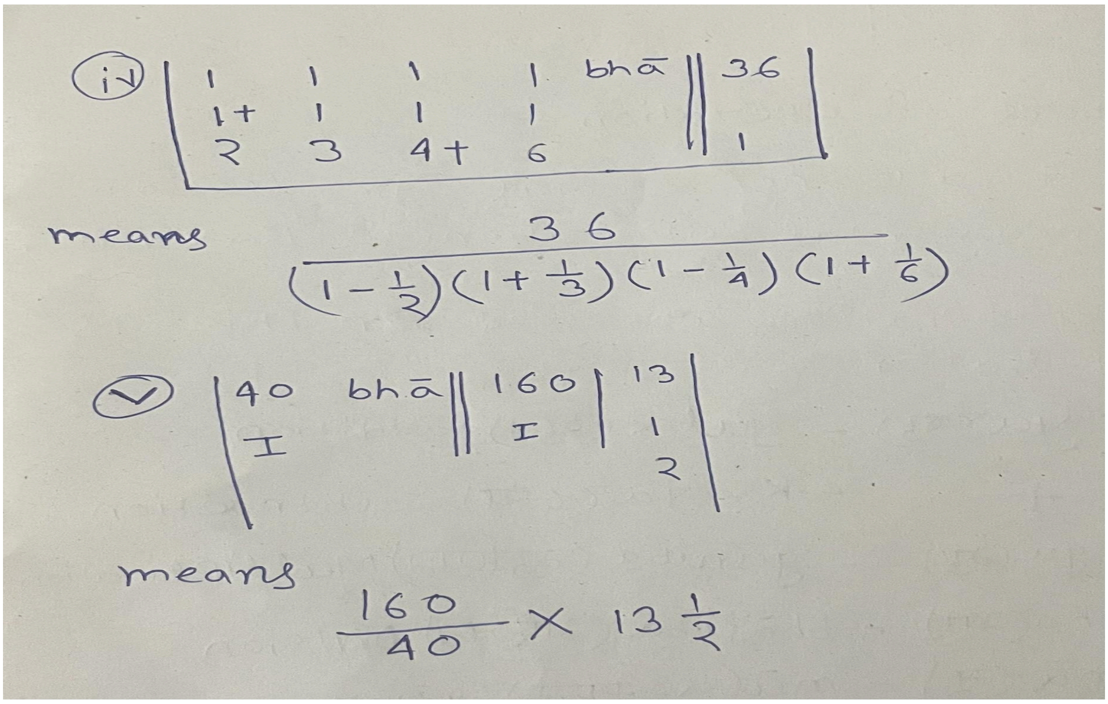
5th - va-gha-ghā(varga-ghana-ghāta)

6th - gha-va(ghana-varga)

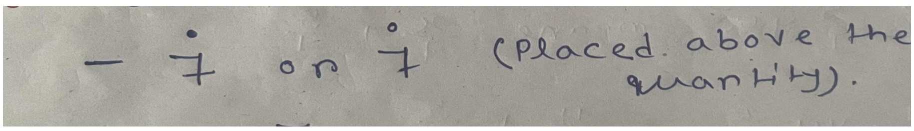
7th - va-va-gha-ghā(varga-varga-ghana-ghāta)

Instances from the Bakshali Manuscript -





In later Hindu mathematics the symbol for subtraction is a dot or a small circle.



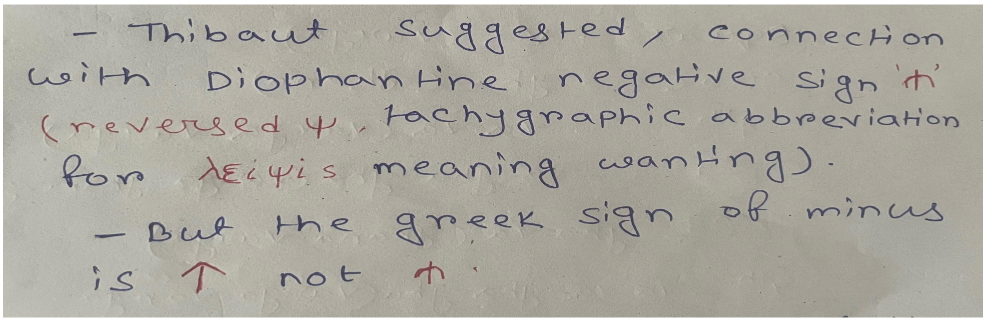
It means -7.

रुपीणामव्यक्तानां नामीद्यक्षराणि लेख्यानि ।

उपलक्षणाय तेषामृणगीनामूर्ध्वं बिन्दूनि ।।७।।
बीजगणितावतंस, नारायण पंडित

Origin of Minus sign -

❖ The origin of . Or o as minus sign seems to be connected with the Hindu symbol 0 (zero). Zero is the sign of emptiness in Bakshali treatise. The origin of Bakshali minus sign (+) has been the subject of much conjecture.



Hoernle presumed it as the abbreviation of ka (sanskrit word kanita) or nū (nyūna) which means diminished. In Brāhmī character it is denoted by a 'cross'. Datta rejects all the theory and believes it would be the abbreviation kṣa from kṣaya(decrease). The sign of kṣa whether in Brāhmī character or Bakshali differs from the simple cross.

Chapter - 4 Algebraic Operations

- **Addition/Subtraction**

Verse
अव्यक्तवर्ग-घनवर्गवर्गपञ्चगत षड्गतादीनां ।
तुल्यानां सङ्कलितव्यवकलिते पृथगतुल्यानाम् ॥
- ब्राह्मस्फूटसिद्धान्त १८.४१

Meaning
Of the unknowns, their squares, cubes, fourth powers, fifth powers, sixth powers, etc., addition and subtraction are (performed) of the like; of the unlike (they mean simply their) statement severally.

योगः अन्तरं तेषु समान जात्योर्विभिन्न जात्योश्च पृथक् स्थितिश्च ॥२२॥
- भास्कर II , बीजगणित

Addition and subtraction are performed of those of the same species Utili) amongst unknowns; of different species they mean their separate statement.

- **Multiplication**

Verse
सदृशद्विवधो वर्गस्त्र्यादिवधस्तद् गतोऽन्यजातिवधः ।

अन्योऽन्यवर्णघातो भावितकः पूर्ववच्छेषम् ॥
- ब्राह्मस्फूटसिद्धान्त १८.४२

Meaning

The product of two like unknowns is a square; the product of three or more like unknowns is a power of that designation. The multiplication of unknowns of unlike species is the same as the mutual product of symbols; it is called भावित (product or factum).

Verse

यावत्तावत्कालको नीलकोऽन्यो वर्णः पीतो लोहितश्चैतदाधाः ।
अव्यक्तानां कल्पिता मानसंज्ञास्वत्सङ्ख्यानं कर्तुमाचार्यवर्यैः ॥ २१॥
- बीजगणित, भास्कराचार्य

Meaning

Here, भास्कराचार्य II talk about the team given by पूर्वाचार्य (i.e. the scholars before him).
कालक, नीलक, पतिक, लोहिलक संज्ञाड ase for denoting like terms.
literal meaning of कालक - that which is black नीलक that which is blue
These terms are for denoting that operation can happen only in the like terms. Colour should not mix with one another. This is the visual representation of the rule that Algebraic operations can happen only in like terms.

Division and more

स्याद्रूपवर्गाभिहतौ तु वर्णौ द्वित्र्यादिकानाम् ।
वधे तु तद्वर्गघनादयः स्युस्तद्भावितं चासमजातिघाते ।
भागादिकं रूपवदेव शेषं व्यक्ते यदुक्तं गणिते तदत्र ॥२६॥ - बीजगणित, भास्कराचार्य

Thus he explains multiplication in Algebra & for division, he says it is the same as that of in व्यक्तगणित.
He includes वर्ग, वर्गमूल, घन, घनमूल, भागाकार in the term 'भागादि'

Chapter 5 - Laws of signs

- **सङ्कलनम् (Addition)**

Verse

धनयोर्धनं ऋणमृणयोः धनवर्णयोरन्तरं समैक्यं खम् ।
ऋणमेक्यं च धनमृणधनशून्ययोः शून्ययोः शून्यम् ॥
- ब्रह्मस्फूटसिद्धान्त १८.३०

Glossary

अन्तरं = difference
ऐक्यं = sum

Explanation

This verse explains the 6 rules.
1. धनयोः धनम् - When two positive numbers are added then the result is always positive.
Eg. 2 + 3 = +5

2. ऋणं ऋणयोः - When two negative numbers are added then the result is always negative.
Eg. (-2) + (-5) = (-7)

3. धन-ऋणयोः अन्तरं - When one positive and one negative number is added then the answer can positive or negative and the result is their difference.

4. समैक्यं खम् - When a positive and negative number of the same magnitude are added then the answer is zero.

5. ऋणमैक्यं च धनमृणधनशून्ययोः

i) ऋणशून्ययोः ऐक्यं ऋणम् - When the negative number and zero are added then result is the same negative number.

ii) धनशून्ययोः ऐक्यं धनम् - When the positive number and zero are added then result is the same positive number.

6. शून्ययोः शून्यम् - When a zero added to zero, the answer is zero.

- **व्यवकलनम् (subtraction)**

Verse

ऊनमाधिकाद्विशोध्यं धनं धनादृणमृणादधिकमूनात् ।
व्यस्तं तदन्तरं स्यादृणं धनं धनमृणं भवति ॥
शून्यविहीनमृणमृणं धनं धनं भवति शून्यमाकाशम् ।
शोध्यं यदा धनमृणादृणं धनाद्वा तदा क्षेप्यम् ॥
-ब्रह्मस्फूटसिद्धान्त १८.३१,३२

Glossary

विशोध्यं, विहीनं = subtracted
ऋणं = negative
धनं = positive
आकाशं, शून्यं = zero
क्षेप्यम् = sum

Explanation

1. ऊनम् अधिकाद् विशोध्यं धनम् - When the positive number is subtracted from the greater positive number the result is positive and their difference.
Eg. 9 - 5 = 4

2. ऋणं ऋणाद् - When the negative number is subtracted from a greater negative number the answer is their difference and it is negative.
Eg. (-9) - (-4) = (-5)

3. शून्यविहीनं ऋणं ऋणं धनं धनं -

i) If a zero is subtracted from a positive number, the answer is the same positive number.
Eg. 4 - 0 = 4

ii) If a zero is subtracted from a negative number, the answer is the same negative number.
Eg. (-5) - 0 = (-5)

4. शून्यं आकाशं : zero - zero = zero

5. धनाद् अधिकं ऊनाद् व्यस्तं तद् अन्तरं स्याद् ऋणं धनं धनं ऋणं भवति -
i) When a greater positive number is subtracted from the smaller one, the answer is their difference and it is negative.
Eg. 4 - 9 = (-5)

ii) When the greater negative number is subtracted from the smaller one, the answer is their difference and it is positive.
Eg. (-3) - (-9) = 6

6. शोध्यं यदा धनं ऋणाद् ऋणं धनाद् वा तदा भवति क्षेप्यम् -
i) If a positive number is subtracted from the negative number, answer is their sum and it is negative
Eg. (-2) - (5) = (-7)

ii) If a negative number is subtracted from a positive the answer is their sum and it is positive,
Eg. 5 - (-3) = 8

- **गुणनम् (Multiplication)**

Verse
ऋणमृणधनयोर्घातो धनमृणयोh धनवधोधनं भवति ।
शून्यर्णयोः खधनयोः खशून्ययोर्वा वधः शून्यम् । -ब्रह्मस्फूटसिद्धान्त १८.३३

Glossary
ऋणं = negative
धनं = positive
ख, शून्य = zero
वधः, घातः = Multiplication
Explanation
In the case of multiplication, the answer is always the same number that we get by multiplying. So the rules are about the sign of the result.

1. ऋणं ऋणधनयोः घातः - negative × positive = negative
2. धनं ऋणयोः - negative × negative = positive
3. धनवधो धनं - positive × positive = positive
4. शून्यर्णयोः - zero × negative = zero
5. खधनयोः - zero × positive = zero
6. खशून्ययोः वा - zero × zero = zero

- **हरणम् (Division)**

Verse
धनभक्त धनम् ऋणहृतमृणं भवति खं खभक्तं खम्।
भक्तमृणेन धनमृणं धनेन हृतम् ऋणमृणं भवति ॥ ३४ ॥
खोद्धृतमृणं धनं वा तच्छेदं खमृणधनविभक्तं वा ।
ऋणधनयोर्वर्गः स्वं खं खस्य पदं कृतिर्यत् तत् ॥ ३५ ॥ -ब्रह्मस्फूटसिद्धान्त

Glossary

- विभक्तं, भक्तं, हृतम् = divided
- कृति = square
- पदं = square root

Explanation

- धनभक्तं धनम् : positive ÷ positive = positive
  Eg.
- ऋणहतम् ऋनं धनम् भवति - negative ÷ negative = positive
  Eg.
- खं खभक्तं खम् : zero ÷ zero = zero
- भक्तम् ऋणेन धनं ऋण धनेन हतम् ऋण ऋणं भवति :
  positive ÷ negative = negative
  Eg.

  negative + positive = negative
  Eg.
- खोद्धृनं ऋण धनं वा तच्छेदं
  Positive/negative ÷ zero = तच्छेद
  Eg.
- खम् ऋणधन विभक्तं वा
  zero ÷ positive/negative : तच्छेद / zero
  Eg.
- ऋणधनयो: वर्गः स्वं : (positive/negative)2 = positive
  Eg.
- खं खस्य पद कृतिं यन् तत् : $(\text{zero})^2$ = zero

Section 4 - Geometry

Chapter 1 - History of Indian Geometry (Jyamiti)

Introduction

Before understanding any knowledge system understanding कः, किम्, कुतः, कुत्र, कदा, किमर्थम्, केन, कथम्, कीदृशम्, कस्मिन् all this questions has to be answered, What is Ancient Indian Geometry ? What is the relevance? Why should we learn about it? What is Sulbā? Why were they used? Who used it? When was it in use? How can we learn from it? These are general questions we come up with in mind when we learn about Indian geometry, and to answer these questions we will further deal with different concepts from Indian Geometry.

History is divided into :

Vedic Period **Post Vedic Period**

The question of date arises only in the case of works of the Vedic Age and those of the Mediaeval period. After conclusion by most Scholars we can take general estimate and place Vedāṅgas in period 1500 B.C. to 750 B.C. The period 400 BC to 400 AD seems to have been period of great activity and progress, during this period flourished the great Jaina metaphysics Umāsvati, Patañjali (Grammarian and Philosopher), Kautilya, Nāgārjuna, Craka, Aśvagjoṣa, Bhāsa and Kālidāsa. Siddhāntas, Sūrya, Pitāmaha Vaśiṣṭha, Parāśara and Others.

शिक्षा कल्पो व्याकरणं निरुक्तं छन्दसां चयः । ज्योतिषामयनमं चैव वेदांदगानि षडेव तु ।।

शिक्षा - *Phonetics ,* छंदस् - *Metronomy,* व्याकरण - *Grammar,* निरुक्त- *Etymology,* ज्योतिष- *Astronomy* कल्प - *Rules for the rituals and the ceremonials, Thus in Vedāṅgas we find the foundation of almost all the essential branches of human knowledge. The last 2 Vedāṅgas - Jyotiṣ and Kalpa contain the developments in Mathematics of the age.*

6 Vedāṅgas : The Vedāṅga Jyotiśa (c 1200 B.C) gives Gaṇita the highest place of honour as mentioned in the below shloka , among the sciences which form the Vedāṅga.

यथा शिखा मयूराणां नागानां मणयो यथा । तद्वद् वेदाङ्गशास्त्राणां गणितं मूर्धनि स्थितम्।।

"As the crests on the heads of peacocks , as the gems on the hoods of snakes , so is gaṇita at the top of the sciences known as Vedāṅga."

The subject treated in the Hindu Gaṇita of the early renaissance period consisted of the following :

परिकर्म व्यवहरो रज्जु रासी कलासवर्णा या । यावत्तावति वर्गो घनो ततः वर्गोवर्गो विकल्पो ता ।।
(Sthānāṅgasūtra Sūtra 747)

परिकर्म Fundamental operations , व्यवहर Determinations , रज्जु Rope meaning Geometry, रासी Rule of Three, कलासवर्णा operations with fractions, यावत्तावत as many as meaning simple equations, वर्ग Square meaning Quadratic equations, घन Ghana Cube meaning cubic equations वर्ग वर्ग Biquadratic equations and विकल्प Permutations and Combinations

What was the Purpose to making Geometry ?

All the construction was made for the making of Cities. For the purpose of worship three types of agnis or fire sheltering them in certain altars of special design. The Agnis were called दक्षिण , गार्हपत्य , आहवनीय ।

दक्षिण = Semi Circular , आहवनीय = Square System , गार्हपत्य = Square in one system and Circular in other system

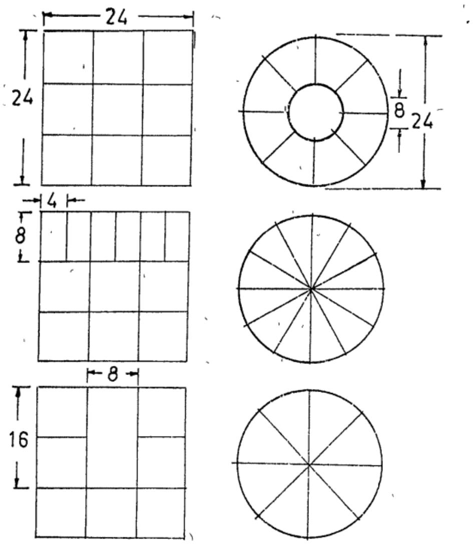
Therefore, for this construction sulba was used. Area of each had to be the same and equal to one square व्याम {१ व्याम = ९६ अङ्गुल = 2 yards} To construct this three alters involved three geometrical operations : 1) To construct a square on a given straight line 2) To circle a square and vice versa 3) To double a circle 4) Doublinga square and then circling it. or Evaluation of the surd root 2 {१ पुरुष = १२० अङ्गुल = 2½ yards} Vedic geometry also treated problems of Application of areas.

UNITS OF MEASUREMENT :

Śatapatha Brāhmana contains several measuring units and their length were the same as indicated by the units of length given in Śulbasūtras. As we find units in Śatapatha Brāhmana are Angula, Artni, Prādesa, Vitasti, Vyāma, Prakrama, Purusa, Yuga and Śamya.

1 Angula = 24 anus or 34 tilas
1 Aratni = 24 Angula
1 Prādesa = 12 Angula
1 Vitasti = 12 Angula
1 Vyāma = 96 Angula
1 Prakrama = 30 Angula
1 Purusa = 120 Angula
1 Yuga = 86 Angula
1 Śamya = 36 Angula

By seeing Śulbasūtras it seems that "Angula " is standard unit for measuring lengths of different vedis , whereas "Purusa" is the standard unit for measuring length of citis like Śyena and other.



Arrangement of bricks in different dhisnyas

एक विंशतिधा वा ह्वृच्यम् , एकशतमध्वर्य्यशाखाः । सहस्त्रवर्त्मा सामवेदः, नवधा आथर्व्वणो वेदः पञ्चदशभेदो वा। (पतञ्जलि १५० ई.पू.)

By Patañjali (150 BC) , There are 21 different schools of Ṛg-veda ; 101 schools of Yajur-veda; 1,000 of the Sāma-veda ; and 9 or 15 of Atharvaveda. Sulbas or as they are more commonly known at present amongst oriental scholars , the Śulba-sūtras, are manuals for the construction of altars. They are sections of the Kalpa-sūtras , which are from one of the Six Vedāṅgas and deal specially with rituals or ceremonials. Each Śrauta-sūtra seems to have its own Śulba section.

Kalpa - sūtras are broadly divided into two classes.
Gṛhya sūtras : " The rules for ceremonies relating to family or domestic affairs" such as marriage , birth , etc
Śrauta - sūtras : "The rules for ceremonies ordained by Veda" such as the preservation of sacred fires, performance of the sacrifices , etc.
Each school of Veda had its own Śrauta-Sūtra and hence probably its own Śulba. Thus it seems that there were numerous manuals of geometry in ancient India. But most of them are now lost.

BB Dutta lists at present we know however only 7 Śulba-sūtras : Baudhāyana, Āpastamba, Kātyāyana, Mānava, Maitrāyaṇa, Vārāha and Vādhula. Other two which are not available now but known : Maśaka Śulba and Hiraṇyakeśi Śulba Kṛṣṇa Yajur-veda = Baudhāyana, Āpastamba, Mānava, Maitrāyaṇa and Vārāha Śulba-sūtras Śukla Yajur-veda = Kātyāyana Śulba-sūtras.

Baudhāyana Śulba sūtra (B.Ś.S) :

Of all the extent Sulbas, that of the Baudhayana is the biggest and is also, perhaps the oldest . It is divided into 21 chapters 24 The first four chapters contain 116 sūtras of which the opening two are merely introductory, sūtras 3-21 define the various measures, ordinarily employed in the sulbas, sūtras 22-62 give the more important of the geometrical prepositions necessary for the construction of the sacrificial altars. And sūtras 63-116 deal briefly with the relative positions and spatial magnitudes of the various vedis (or altars) The 5th to 7th chapter consists of 86 sütras of which the major portion, sütras 1-61, is devoted to the description of the spatial relations in the different constructions of the Agnis in general and the remaining portion, sütras 62-86, elaborates the construction of the two simplest Agnis viz, the 'Garhaptya-citi and 'chandas citi' From 8th to 21st chapter, in altogether 323 sütras, describes the construction of as many as 17 different kinds of 'Kamya Agnis' of rather complex nature. In some the description is quite elaborate in details, but in other cases it is less so.

Āpastamba Śulba sūtra : (A.S.S)

It is broadly divided into six patalas of these the first, third and fifth are each subdivided again into these adhyayas and each of the remaining sections into four chapters So that altogether the work contains twenty one chapters and 223 sūtras. The first section of the manual, chapters I-III gives the important geometrical propositions required for the construction of altars.The second section IV-VII describe the relative positions of the various vedis and their spatial magnitudes Unlike Baudhayana, Apastamba here indicates briefly also the methods of their construction They are of course the particular applications of the general geometrical theorems taught in the earlier section The remaining sections of the Apastamba Sulbasûtra comprising the chapters- VII-XXI deal with the construction of the Kamya Agnis it is noteworthy that almost the same set of geometrical propositions are taught by both Baudhayana and Apastamba.

Kātyāyana Śulba sūtra : (K.S.S)

This sūtra consists of both prose and verse. 6 Chapters consist of 101 sūtras and 7th chapter consists of 39 verses.

[From book by S D. Khadilkar by Vaidik Samsodhana Mandal Pune]

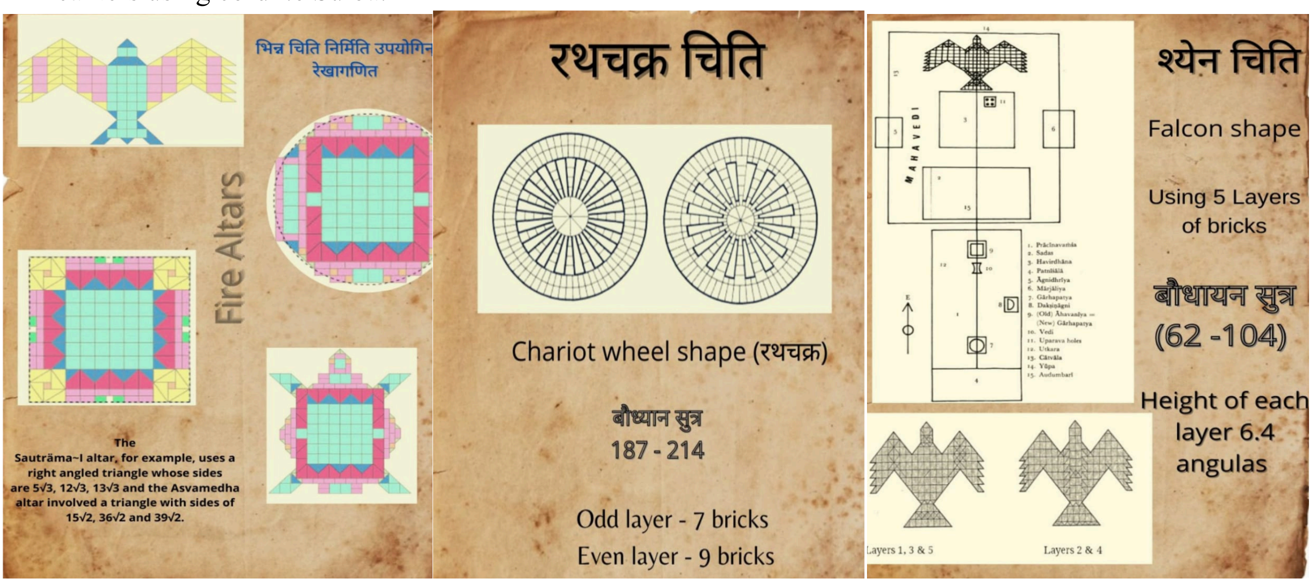
While performing various vedic sacrifices is scattered over the Brahmana and Samhita literature of Yajurveda, But the rules pertaining to the Mathematical side of it , especially the Geometrical portion involved which is required for constructing the various forms of alters, in shape square, triangle, trapezium or a circle of various cities shape of birds, spread wings , tail, etc given in only Śulba Sūtras. The system of measurement of various sacrificial places was being carried out by means of Śulba, a measuring cord and the word Sūtras denoting the rules in short forms.

Define Sūtra :

अल्पाक्षरम् असन्दिग्धं सारवत् विश्वतोमुखम् । अस्तोभम् अनवद्यं च सूत्रः सूत्रविदो विदुः ॥

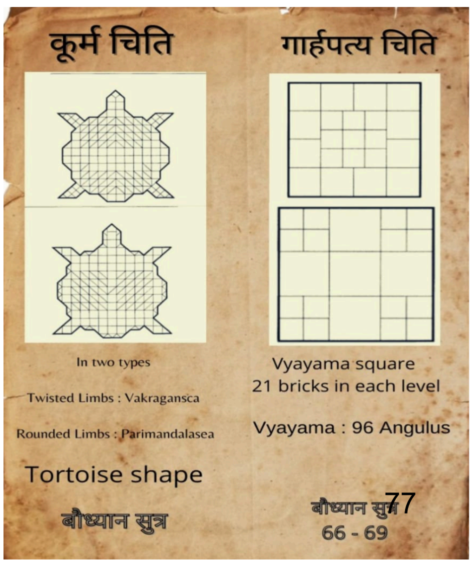
As Śrauta sūtras available only for Yajurveda therefore, work was assigned to Adhvaryu, so each belonging to Yajurveda was entitled to have its own Śulba sūrta. Out of the above mentioned sūtras Baudhayana, Apastamba and Kātyayana deal with the Mathematical portion namely Geometry in detail [Manava gives only description about Square and Circle].

From a research paper by John Price If mathematicians were asked to write manuals about mathematical geometry they likely would do two things : 1] Give construction procedure to a level of accuracy appropriate for the actual construction. 2] For their own enjoyment , show mathematical readers that they really understood that they were dealing with approximations. This is observed in Śulba-sūtras. General formats of main Śulba-sūtras are the same; each starts with a section on geometrical and arithmetical construction and ends with details of how to build citis ,etc. The measurements for geometrical construction are performed by drawing arcs with different radii and canters using cord i.e Śulba.



EXAMPLES of CITIS :

Citis making for yagya leads to Geometry in Ancient India



Chapter 2 - Sulba Sutra

What is Śulba ?

शुल्ब माने *: to measure*

भावव्युत्पत्ति शुल्बनम् शुल्बः । = *The act of measuring*

कर्मव्युत्पत्ति शुल्बयते इति शुल्बः ।= *to do i.e entity or result of measuring*

करणव्युत्पत्ति शुल्बयत्यनेन इति शुल्बः । = *instrument i.e instrument of measuring*

शुल्बनम् = शुल्बः *(*शुल्ब् + घञ्*)* ।

In Sanskrit Śulba or Rajju have got identical significance i.e. Ordinarily rope or cord. Mention of a linear measure called rajju in Śulba sūtra , Arthaśāstra , Śilpa sāstra. In Ancient India there were three kinds of measure : Linear , Superficial , and Voluminal.

Opening sūtra of Kātyāyana Śulba parisista i.e. : रज्जु समा समम् वक्ष्याम । {I shall speak of the collection of rules regarding the rajju } In work by Jainas (500 - 300 BC)
Rajju started a branch of mathematics.

Śilāṅka(862 A.D.) : रज्जु रिति रज्जुगणितम् क्षेत्रगणित मित्यर्थ ।
Abhayadeva Sūri : रज्ज्वा यत् सम्ख्यानम् तद् रज्जु रभिधीयते त्त्च क्षेत्रगणितम् ।
Rajju types by Jainas : सूची रज्जु (Needle like) , प्रतर रज्जु (Superficial) , घन रज्जु (Cubic)

Some important terms to be known :

करणी तत्करणी तिर्यङ्मानी पार्श्वमान्यक्ष्णय चेति रज्ज्वः । (कात्यायना शूल्ब सूत्र २.३)

Kātyāyana observes : The terms Karaṇi as Producer , Tat Karaṇī as That Producer, Tiryaṅmāni as Transverse measure , Pāṛśvamānī as Side measure and Akṣṇayā as Diagonal rajjus are Lines

The commentary by Mahīdhara (1589CE) explaining the origin of this names given in above sūtra :

- करणी क्रियते क्षेत्रपरिच्छेदः अनयेति करणी । : That which limits or produces the length or area
- तत्करणी तत्क्षेत्रं द्वैगुण्यादि क्रियते अनया सा तत्करणी , द्विकरणी ,त्रिकरणी , चतुः करण्यादिः। : That which limits produce the area that is twice etc.

- तिर्यङ्मानी तिर्यक् श्रोण्यंशस्वरूपं मीयतेऽनयेति सा तिर्यङ्मानी , प्राचीसूत्रान्तयोः तिर्यग्वर्तमानं रज्जुद्वयम् । : that which is measured Transverse / Horizontal measure
- पार्श्वमानी पार्श्वं मीयतेऽनया सा पार्श्वमानी, पार्श्वयोर्वर्तमानं पूर्वापरायतं रज्जुद्वयम् । : That which sides are measured refers to the cords on either sides that is stretched along the east - west direction
- अक्ष्णया अक्षिवत् क्षेत्रं नयतीति अक्ष्ण्या , कोणसूत्रभूता मध्यरज्जुः , तस्यां दत्तयां चतुरस्रं अक्षिद्वयसदृशं भवति, ततोक्ष्ण्येति कोण्सूत्ररज्जुः । : The mid cord that connects the corners, looks like eye

In Mānava Śulba(MāŚl) and Maitrāyaṇīya Śulba(MaiS) some important terms :

शुल्ब विज्ञान : Science of Geometry
शुल्ब विद् Expert in Śulba
शुल्ब परिप्रछका : The inquirer into the Śulba
संख्याज्ञ : The expert in Numbers
परिमणाज्ञ : The expert in measuring
सम सूत्र निराञ्छक : Uniform rope Stretcher

According to Sthanaga Sutra (300 BC)
परिकर्मरज्जः राशिः व्यवहारस्तथा कलासवर्णाश्च ।
पुट्गला: यावत्तावत् भवन्ति घन घनमूलम् वर्गः वर्गमूल।

The topics for discussion in mathematics
Samkhyana on the Science of numbers are ten in number *Parkarma* fundamental operation *Vyavahara* Subject of treatment , *rajju* rope geometry , *rasi* heap mensuration of solid bodies , *kalasavarna* fraction, *yavat tavat* simple equation, *varga* square quadratic equation, *ghana* cube cubic equation, *varga-varga* biquadratic eqn and , *vikalpa* permutation and combination

Geometrical Figures named as :

Samacaturasra = the equal four sided figure
Visamacaturasra = the unequal four sided figure
Samacatuskoṇa = the equiangular quadrilateral i.e. rectangle
Visamacatuskona = Quadrilateral with unequal angles
Samacakravāḷa = circle
Visamacakravāḷa = ellipse
Cakṛrdha cakravāḷa = semi - circles

The geometrical propositions involved in the construction are the following :

1. To divide a line into any number of equal parts
2. To divide a circle into any number of equal areas by drawing diameters {Bss 2.73-4 and Ass 7. 13-14}
3. To divide a triangle into a number of equal and similar areas {Bss 3.256} to draw a straight line at right angles to a given line. {Kss 1.3}
4. To draw a straight line at right angles to a given straight line from a given point on it. To construct a square on given side {Ass 8.8-10}
5. To construct a rectangle on the given side {Bss 1. 36-40} to construct an isosceles trapezium of given altitude , face and base. {Bss 1.41 Ass 5. 2-5}

6. To construct a parallelogram having given sides at a given inclination. {Ass 19 .5}
7. To construct a square equal to the sum of two different squares. {Bss 1. 51-52}
8. To construct a square equivalent to two given triangles.
9. To construct a square equivalent to two given pentagons. {Bss 3.68,288 ; Kss 4.8}
10. To construct a square equal to a given rectangle{Bss 1.58 ; Ass 2.7 ; Kss 3.2-3}
11. To construct a rectangle having a given side and equivalent to a given square{Ass 3.1;Bss 1.53}
12. To construct an isosceles trapezium having a given face and equivalent to a given square or rectangle. {Bss 1.55 }
13. To construct a triangle equivalent to a given square {Bss 1.56}
14. To construct a square equivalent to a given isosceles triangle.{Kss 4.5}
15. To construct a rhombus equivalent to a given square or rectangle.{Bss 1.57; Ass 9.9; Kss 4.4}
16. To construct a square equivalent to a given rhombus. {Kss 4.6}
17. Value of Pi and Root 2
18. Famous Baudhayana Sulba Sutra {BSS 1.12 , 1.13}

Above all are from BSS , KSS , ASS.(Mentioned important ones , other constructions as also mentioned)

Chapter 3 - Theorems and Postulates

Theorems :

The following theorems are either expressly stated or the results are implied in the methods of construction of the altars of different shapes and sizes:

(1) The diagonals of a rectangle bisect each other. They divide the rectangle into four parts, two and two (vertically opposite) of which are equal in all respects."

(2) The diagonals of a rhombus bisect each other at right angles.

(3) An isosceles triangle is divided into two equal halves by the line joining the vertex to the middle point of the base.

(4) The area of a square formed by joining the middle points of the sides of a square is half that of the original one.

(5) A quadrilateral formed by the lines joining the middle points of the sides of a rectangle is a rhombus whose area is half that of the rectangle.

(6) A parallelogram and rectangle on the same base and within the same parallels have the same area.

(7) The square on the hypotenuse of a right angled triangle is equal to the sum of the squares on the other two sides.

(8) If the sum of the squares on two sides of a triangle is equal to the square on the third side, then the triangle is right-angled.

Postulates :

a) A given finite straight line can be divided into any number of equal parts.

b) A circle can be divided into any number of parts by drawing diameters.

c) Each diagonal of a rectangle bisect it.

d) The diagonals of a rectangle bisect one another and they divide the rectangle into four parts two and two vertically opposite of which are equal in all respects.

e) The diagonals of a rhombus bisect each other at right angles.

f) A triangle can be divided into a number of equal and similar parts by dividing the sides into an equal number of parts and then joining the points of division two and two

g) An isosceles triangle is divided into two equal halves by the line joining the vertex with the middle point of the opposite side. Each of these has again been divided into six parts.

h) A triangle formed by joining the extremities of any side of a square to the middle point of the opposite side is equal to half the square.

i) A quadrilateral formed by the lines joining the middle points of the sides of a square whose area is half that of the original one.

j) A quadrilateral formed by the lines joining the middle points of the sides of a rectangle is a rhombus whose area is half that of the rectangle.

k) A parallelogram and a rectangle which are on the same base and within the same parallels are equal to one another.

l) The maximum square that can be described within a circle is the one which has its corner on the circumference of the circle.

Note : By prof. BB Datta : Described in Śulba , we find have tacitly assumed the truth of certain other results without any attempt to describe them beforehand or to indicate how they could be affected. They might not be postulates in Euclidean sense of the term but they can certainly be so called in accordance with the meaning given by Aristotle, namely "whatever is assumed , though it is a matter for proof and used without being proved"

Important Commentaries :

Śulbasūtra	Name of the commentaries	Author
Budhāyana	Śulbadīpikā Śulba - mīmāṃsā	Dvārakānātha Yajvā Venkateśvara Dikṣita

Āpastamba	Śulbavyākhyā Śulbapradīpikā Ṣulbapradīpa Śulbabhāṣya	Kapardisvāmin Karavindasvāmin Sundararāja Gopāla
Kātyāyana	Śulbasūtravivṛtti Śulbasūtravivaraṇa Śulbasūtrabhāṣya	Rāma/Rāmacandra Mahīdhara Karka

Mahīdhara (c. 17th cent) vivṛti on Kātyāyanaśulbasūtra describes the qualities of a śulbakāra (The person helps in construction of altars).

सङ्ख्याज्ञः परिमाणज्ञः समसूत्रनिरञ्छकः।
समसूत्रौ भवेद्विद्वान् शुल्बवित् परिपृच्छकः ॥
शास्त्रबुद्धिविभगज्ञः परशस्त्रकुतूहलः ।
शिल्पिभ्यः स्थपतिभ्यश्चाप्याददीत मतीः सदा॥
तिर्यङ्मान्याश्च सर्वार्थः पार्श्वमान्याश्च योगवित्।
करणीनां विभागज्ञः नित्योद्युक्तश्च सर्वदा॥

A śulbakāra must be versed in arithmetic versed in mensuration, must be an inquirer ,quite knowledgeable in one's discipline must be enthusiastic in learning other disciplines, always willing to learn from practising sculptors and architects and one who is always industrious.

Topic covered in Baudhāyana - Śulbasūtra:

TOPICS	MODERN TERMS
रेखामानपरिभाषा	Units of Measurement
चतुरश्रकरणोपायः	Construction of Square, rectangles , etc
करण्यानयनम्	Obtaining the surds/Theorem of the square of the diagonal

क्षेत्राकारपरिणामः	Transformation of geometrical figures
नानाविधवेदिविहरणम्	Plan for different sacrificial grounds (darśa,paśubandha,sutrāmaṇi,agniṣṭoma etc)
अग्नीनां प्रमाणक्षेत्रमानम्	Areas of the sacrificial fires/altars
इष्टकसङ्ख्यापरिमाणादिकथनम्	Specifying the number of bricks used in the construction of altars including their sizes and shapes
इष्टकोपधाने रीत्यादिनिर्णयः	Choosing clay, sand, etc in making bricks
इष्टकोपधानप्रकारः	Process of manufacturing the bricks
श्येनचिदद्याकारनिरुपणम्	Describing the shapes of śyenciti, etc

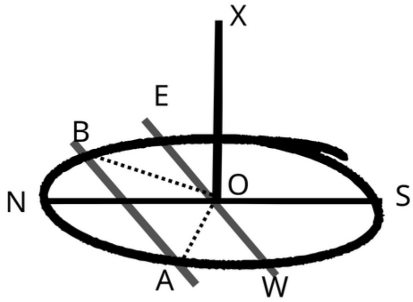
Chapter 4 - Construction and Proofs of Jyamiti

Finding the cardinal direction using Śaṅku , Determine East and West side :

समे शङ्कु निखाय शङ्कुसम्मिताय रज्ज्वा मण्डलं परिलिख्य |
यत्र लेखयोः शङ्कु ग्रच्छाया निपतति तत्र शङ्कु निहन्ति, सा प्रची || (KSS 1.2)

समे शङ्कु निखाय(Fixing a pin or Stick on Levelled ground)शङ्कुसम्मिताय रज्ज्वा मण्डलं (and Drawing a circle with a cord measured by fixed stick, fixing stick at points on the line of the circumference)शङ्कु ग्रच्छाया निपतति(where the shadow of the tip of stick falls). शङ्कु निहन्ति, सा प्रची || That gives the east - west line i.e prāci.

प्रची - line - East and West line
शङ्कु - stick or pin - XO
मण्डलं - Circle
रज्ज्वा - Rope
छाया - Shadow
AO = Forenoon (as shadow enter in circle)
BO = Afternoon (as shadow excess from circle)
निखाय - Placed



Perpendicular Bisector

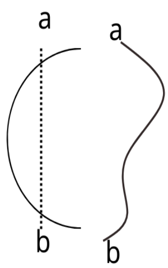
Two methods have been describes for obtaining the perpendicular bisector of a given straight line:

- रज्ज्वभ्यसनम् (Folding Cord) Fig 1
- मत्स्यचित्रणम् (Fish Figure) Fig 2

Methode by folding cord discussed by Kātyāyana, in the third sūtra right at the beginning of his text.

Having obtained prācī , getting udicī(the north - south line),

तदन्तरं रज्वाभ्यस्य , पाशौ कृत्वा , शङ्कोः पाशौ प्रतिमुच्य , दक्षिणायम्य मध्ये शङ्कु निहन्ति ।
एवमुत्तरतः , सोदीची |



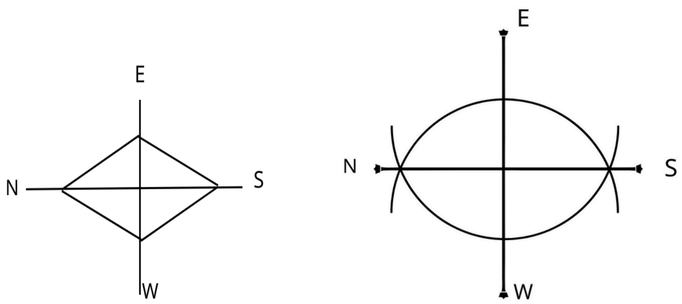


Fig : 1

Doubling the cord by a measure of distance between them (तदन्तरं रज्वाभ्यस्य) (sanku) and stretching the cord towards the south (दक्षिणायम्य) strikes a pin at the middle point (मध्ये शङ्कु निहन्ति).

In the above figure above,

a) A and B represent pins along the east - west direction to which the cord is tied.
b) We've doubled the cord AB
c) Stretching AB on both sides to get the north - south direction

Fig : 2

After getting direction from the sanku shadow method, we mark two points along the east-west line. [E and W]
With those points as centres and choosing an appropriate radius , circular arcs are drawn.
The line passing through the intersection points of these two arcs gives the north - south direction. [S and N]

Construction of Square using Nail and Cord :

As the use of Nail and Cord in Ancient Indian Mathematics is known to us till now Lets Learn construction through it and understand the construction.

Bodhayana’s Method of Constructing Square.

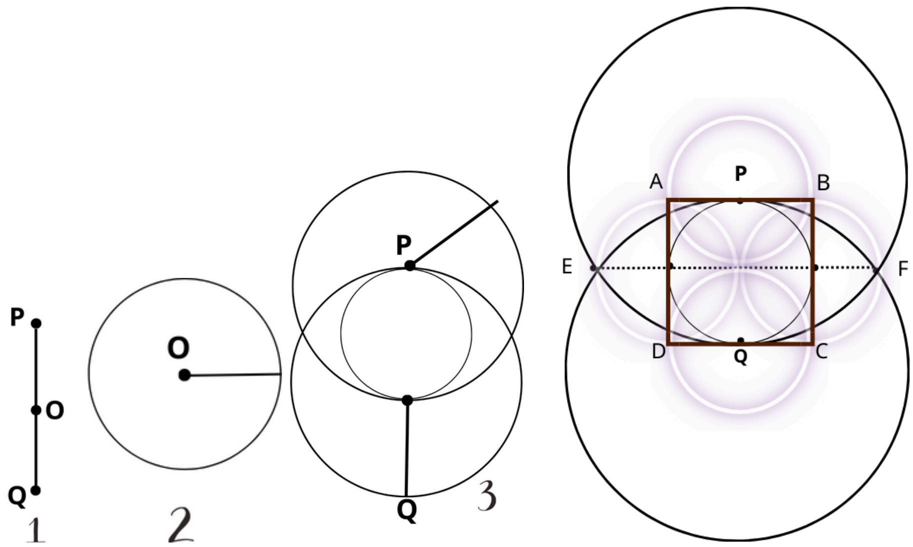
चतुरश्रं चिकीर्षन् यावच्चिकीर्षेत् तावर्ती रज्जुं उभयतः पाशं कृत्वा मध्ये लक्षणं करोति ।
लेखामालिख्य तस्य मध्ये शङ्कु निहन्यात् । तस्मिन् पाशौ प्रतिमुच्य लक्षणेन मण्डलं परिलिखेत्।
विष्कम्भान्तयोः शङ्कु निहन्यात् । पूर्वस्मिन् पाशं प्रतिमुच्य

पाशेन मण्डलम् परिलिखेत्। एवमपरस्मिन्। ते यत्र समेयातां तेन द्वितीयं विष्कम्भ आयच्छेत्।
विष्कम्भान्तयोः शङ्कु निहन्यात्। पूर्वस्मिन् पाशौ प्रतिमुच्य लक्षणेन मण्डलं परिलिखेत्। एवं दक्षिणेन एवं पखाल्देवमुत्तरतः । तेषां ये अन्त्याः संसर्गाः तच्चतुरश्रं सम्पदद्यते ।

चतुरश्रं चिकीर्षन् Desirous of constructing a square , take a cord of that length, उभयतः पाशं कृत्वा मध्ये लक्षणं करोति tie it at both the ends and mark its centre. Draw a line and मध्ये शङ्कु निहन्यात् fix a nail at its centre. पाशेन मण्डलम् परिलिखेत्। एवमपरस्मिन् Latching the ends may draw a circle. Similarly on the other side. From their points of intersection (E,F), तेन द्वितीयं विष्कम्भ आयच्छेत् obtain the second diameter (RS).

Desirous of constructing a Square,

Take a Rope of Length of You want Square Side.
Draw a Circle fixing nail as centre. O {fig2}
Draw two circle from circle arc taking point P and Q
Get two intersecting points E and F
Obtain new diameter RS for inner Circle
From the point at centre P,Q,R,S draw 4 Circles.
Intersection of each circle at each point give us SQUARE ABCD



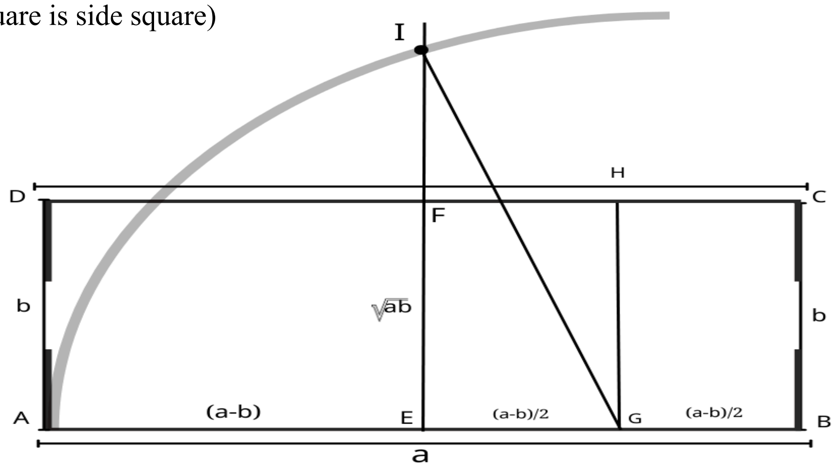
Construction of a square by nail and cord

Video Link:
https://drive.google.com/drive/folders/1jt0fhW3Wz5PMh_QbRvWo3bP75qfjL2T7?usp=drive_link

Rectangle to Square Construction :

दीर्घचतुरश्रं समचतुरश्रं चिकीर्षन् तिर्यङ्मानी करणीं कृत्वा शेषं द्विधा विभज्य, पार्श्वयोरुपदध्यात्। खण्डम् आवापेन तत्संपूरयेत्, तस्य निर्हार उक्तः |
BSS 1.54

• Thus, if ABCD is the given rectangle of length a units and breadth b units which is to be transformed into a square.
Then Baudhayana prescribes marking out AE of length b units along AB (and complete the square DAEF) and bisecting the remainder EB
With the midpoint G of EB as centre, an arc of radius AG is to be drawn which intersects the extension of the line EF at I.
• The segment El gives the desired side.
Note that GE = (a-b)/2 and GIGA = EA+GE= b+ (a-b)/2 = +b)/2 at so that El ² = {(a+b)/2}²-{(a-b)/2}² = ab. (As area of square is side square)



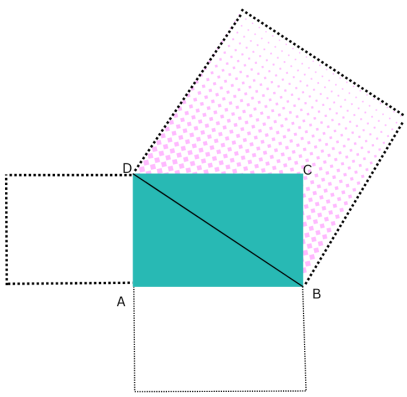
All the authors of Śulbasūtras , while treating the portion of Geometry, make a beginning with the propositions of square and rectangles. Their first object is to find out the side of a square which is equal in area to the sum of two squares or the difference of two squares, or to transform a circle, a triangle, a rectangle or a polygon into a square. For all these constructions, the principle of the theorem of Pythagoras is applied to get the solution.

BUDHĀYANA ŚULBAŚUTRA 1.12

So called 'Pythagorean' theorem called Bhujā-koṭi-karṇa-nyāya in later literature.

दीर्घचतुरश्रस्य अक्ष्णयारज्जुः पार्श्वमानी तिर्यङ्मानी च यत्
पृथग्भूते कुरुतः तदुभयं करोति। (BSS 1.12)

- दीर्घचतुरश्रस्य = Rectangle (longish 4 sided figure)
- अक्ष्णयारज्जुः = The Diagonal Rope
- पार्श्वमानी = The measure of the lateral side
- तिर्यङ्मानी = The measure of the perpendicular side.



तासां त्रिकचतुष्कयोः, द्वादशिकपञ्चिकयोः, पञ्चदशिकाष्टिकयोः, सप्तिकचतुर्विंशिकयोः, द्वादशिकपञ्चत्रिंशिकयोः, पञ्चदशिकषनिंशिकयोः इत्येतासु उपलब्धिः । (BSS 1.13)

This rule stated above is quite evident in this given pairs

$3^2 + 4^2 = 5^2$
$5^2 + 12^2 = 13^2$
$15^2 + 8^2 = 17^2$
$7^2 + 24^2 = 25^2$
$12^2 + 35^2 = 37^2$
$15^2 + 36^2 = 39^2$

Mahidhara in his Commentary explained in detail:
दीर्घचतुरश्रस्य तिर्यङ्मानीपार्श्वमान्यौ रज्जू पृथग्भूते सत्यौ यत्क्षेत्रं = यत्फलकं क्षेत्रं समचतुरश्रद्वयं कुरुतः, तदुभयमपि मिलितं दीर्घचतुरश्रस्य अरुणया = कोणसूत्रभूता रज्जूः करोतीति इति क्षेत्रज्ञानम् = क्षेत्रमानप्रकारो ज्ञातव्यः ।

Mānava Version of the sūtra differs from Budhāyana Version :
आयामं आयामगुणं विस्तारं विस्तरेण तु।
समस्य वर्गमूलं यत् तत् कर्णं तद्विदो विदुः॥

आयामं आयामगुणं = The length multiplied by itself
विस्तारं विस्तरेण तु = and indeed the breath by itself
समस्य वर्गमूलं = the square root of the sum
तत् कर्णम् = that is hypotenuse
तद्विदो विदुः = those versed in the discipline say so

Using Modern notation the result may be expressed as :

$\sqrt{(}$āyāma^2 + vistāni^2 = karṇa

Constructing a square that is sum of unequal Squares

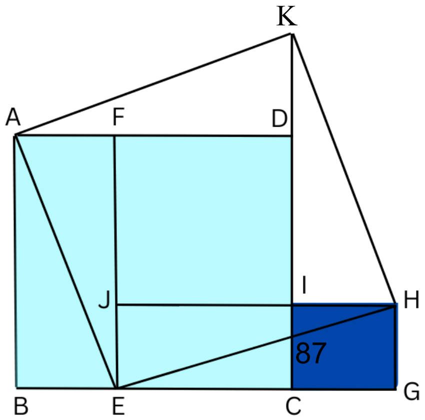
नानाचतुरश्रे समस्यन् कनीयसः करण्या वर्षीयसो वृध्रमुल्लिखेत्।
वृध्रस्य अक्षणयारज्जुः समस्यतोः पार्श्वमानी भवति। **BSS 1.50**

Desirous of combining different squares , may you mark the rectangular portion of the larger [square] with the sides of the smaller one. The diagonal of this rectangle is the side of the sum of the two [squares.]

CG = P = BE

$AB^2 + P^2 = AE^2$ (BSS)

⬛ABCD + ⬛CGHI = ▲ABE + ▲AEF + ▲EGH + ▲EHJ + ⬛DIJF

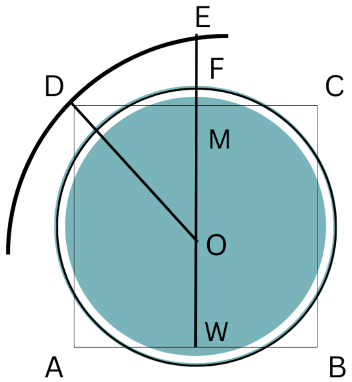


= ▲ABK + ▲AEF + ▲HIK + ▲EHJ + ⬛DIJF

= ⬛AEHK

Circling the Square

चतुरश्रं मण्डलं चिकीर्षन् अक्ष्णयार्धं मध्यात् प्राचीम् अभ्यपाच्चयेत् यद्यदतिशिष्यते तस्य सह तृतीयेन मण्डल परिलिखेत् | **(BSS)**



- अक्ष्णयार्धं = semi diagonal
- मध्यात् प्राचीम् = from center to east
- यद्यदतिशिष्यते = whatever portion remains
- तस्य सह तृतीयेन = with one third of that

"If you wish to circle a square , draw half its diagonal about the centre towards the east-west line ; then describe a circle together with the third part of that which lies outside the square"
Let ABCD be the square which is to be transformed into a circle.
Let O be the central point of the Square
Join OD.
With centre O and radius OD , describe a circle intersecting the east-west line EW at E.
Divide EM at F , such that EF = 2 FM. Then with centre O and radius OF describe a circle.
This circle is roughly equal in area to the square ABCD.
Let 2a denote a side of the given square and r the radius of the circle equivalent to it. Then

$OA = a\sqrt{2}$,

$ME = (\sqrt{2} - 1)a$

$r = a + a/3(\sqrt{2} - 1)$.

$= a/3(2 + \sqrt{2})$.

Āpastamba observes that the circle thus constructed will be inexact (anitya). Now, according to the Śulba,

$\sqrt{2} \approx 1 + 1/3 + 1/3*4 - 1/3*4*34$

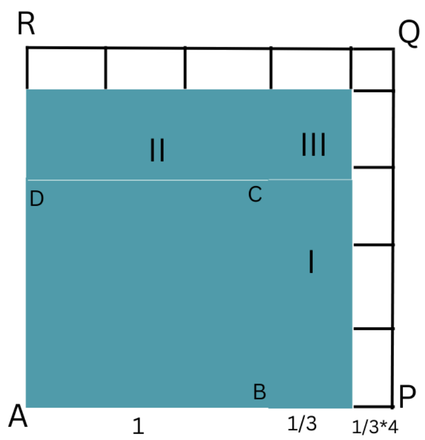
Therefore

$r = a \times 1.1380718\ldots$

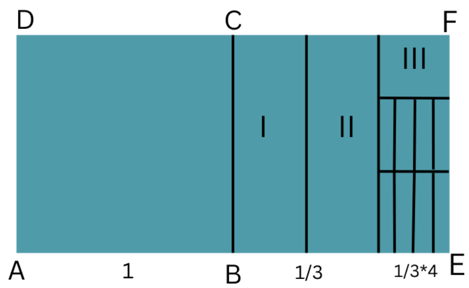
Chapter 4.1 - GEOMETRICAL CONSTRUCTION FOR ROOT 2 VALUE:

प्रमाणं तृतीयेन वर्धयेत्, तच्चतुर्थेन , आत्मचतुस्त्रिंशेनेन सविशेषः | **{BSS 2.12}**

- Consider two Square ABCD and BEFC with sides of unit length
- The second Square BEFC is divided into three stripes.
- The third strip is further divided into many parts and these parts are rearranged with void at
- Now each side of the new square APQR = 1+ 1/3 + 1/3*4



- Q is void
- Area of void Q is $(1/3*4)^2$
- Strip off segment of breath b from either side
- Such that area of striffed off portion is exactly equal to that of void Q



$2b(1+1/3+1/3*4) - b^2 = (1/3*4)^2$

neglect b^2 very small $b=\{(1/3*4)^2\}* 3*4/34 = 1/3*4*34$

hence the side of resulting square

$\sqrt{2} \approx$ **1 +1/3 +1/3*4 - 1/3*4*34**

Draw a Square equivalent to N times a given Square

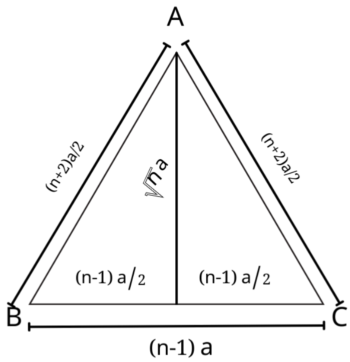
यावत्प्रमाणानि समचतुरश्राणि एकीकर्तुम् चिकीर्षेत् एकोनानि तानि भवन्ति तिर्यक् द्विगुणान्येकत एकाधिकानि त्र्यस्त्रिर्भवति । तस्यैषुः तत्करोति | {KSS 6}

'As many squares of equal size as you wish to combine into one, the transverse line will be equal to one less than that, thus forming a triangle. Its arrow i.e. altitude will do that.'

"Take transverse side equal to the measure of lengths of all squares to be added minus one and twice a side will be equal to the lengths of all the squares to which one more length of a square is added and a triangle formed then the perpendicular from the vertex to midpoint of opposite side of the triangle will serve the purpose"

That is, if the number of equal squares to be combined together into one, form the triangle ABC whose base BC is of length (n-1) times a side of a square and twice the sides AB and AC of which are severally equal to (n+1) times a side of a square.

The method of draw- ing the triangle, which will be consistent with the geo- metrical methods of the Sulba is this: Draw the line BC of length (n-1) times a side of a square. Fix two poles at B and C. Take a cord of length (n+1) times a side of a square. Fasten its two ends at the two poles and stretch the cord sidewise, having taken it by the middle point. Let A be the point reached. Bisect BC at D and join AD. Then the square on AD will be equivalent to the sum of n given squares.



Let, a be the side of a square

n = the number of squares to be added

BC = base of the isosceles ▲ABC = (n-1)a

$BD = DC = [(n-1)/2] \times a$

$AB = AC = [(n+1)/2] \times a$

$AB^2 = AD^2 + BD^2$

$AD^2 = AB^2 - BD^2$

$AD^2 = [(n+1)/2]^2 \times a^2 - [(n-1)/2]^2 \times a^2$

$a^2 \ [n^2 + 2n + 1 - n^2 + 2n - 1 /4] = 4n/4$

$a^2 = a^2 n$

$AD = a\sqrt{n}$

$AD^2 = na^2$

Chapter 4.2 - Value of Pi:

	Approximation to π	Accuracy (Decimal places)	Method Adopted
Rhind Papyrus - Egypt (Prior to 2000 BCE)	$\frac{256}{81} = 3.1604$	1	Geometrical
Babylon (2000 BCE)	$\frac{25}{8} = 3.125$	1	Geometrical
Śulvasūtras (Prior to 800 BCE)	3.0883	1	Geometrical
Jaina Texts (500 BCE)	$\sqrt{(10)} = 3.1623$	1	Geometrical
Archimedes (250 BCE)	$3\frac{10}{71} < \pi < 3\frac{1}{7}$	2	Polygon doubling ($6.2^4 = 96$ sides)
Ptolemy (150 CE)	$3\frac{17}{120} = 3.141666$	3	Polygon doubling ($6.2^6 = 384$ sides)
Lui Hui (263)	3.14159	5	Polygon doubling ($6.2^9 = 3072$ sides)
Tsu Chhung-Chih (480?)	$\frac{355}{113} = 3.1415929$ 3.1415927	6 7	Polygon doubling ($6.2^9 = 12288$ sides)
Aryabhata (499)	$\frac{62832}{20000} = 3.1416$	4	Polygon doubling ($4.2^8 = 1024$ sides)

Al Kasi (1430)	3.1415926535897932	16	Polygon doubling (6.2^{27} sides)
Francois Viete (1579)	3.1415926536	9	Polygon doubling (6.2^{16} sides)
Romanus (1593)	3.1415926535 . . .	15	Polygon doubling
Ludolph Van Ceulen (1615)	3.1415926535 . . .	32	Polygon doubling (2^{62} sides)
Wildebrod Snell (1621)	3.1415926535 . . .	34	Modified Polygon doubling (2^{30} sides)
Grienberger (1630)	3.1415926535 . . .	39	Modified Polygon doubling
Isaac Newton (1665)	3.1415926535 . . .	15	Infinite series

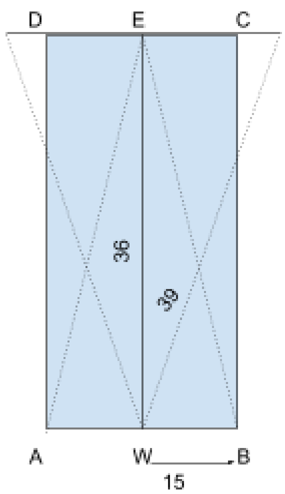
James Gregory (1671)	$tan^{-1}(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots$
Gottfried Leibniz (1674)	$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots$
Abraham Sharp (1699)	$\frac{\pi}{\sqrt{12}} = 1 - \frac{1}{3.3} + \frac{1}{3^2.5} - \frac{1}{3^3.7} + \ldots$
John Machin (1706)	$\frac{\pi}{4} = 4tan^{-1}\left(\frac{1}{5}\right) - tan^{-1}\left(\frac{1}{239}\right)$

Ramanujan (1914)

$$\frac{1}{\pi} = \frac{2\sqrt{2}}{9801} \sum_{k=0}^{\infty} \frac{(4K)!(1103 + 26390k)}{(k!)^4 396^{4k}}$$

Construction of Trapezium with Face, Base, and Altitude given :

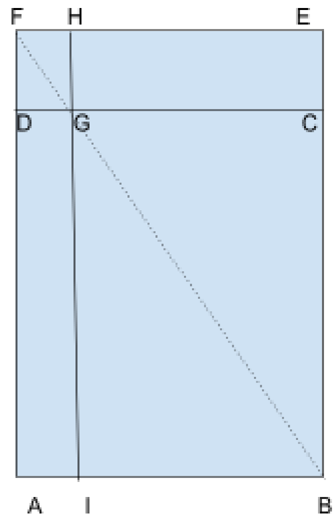
The name for a trapezium used by the Śulbasūtras is ekatoṇimat (smaller on one side).
The method used is essentially the same as construction of square and rectangle using a right angled triangle.
Once the perpendiculars at the ends of pṛsthyā (the line of symmetry or altitude) are drawn , lengths of the top and bottom sides can be marked off on them.



Since the measurements of the Mahāvedi are altitude = 36 prakrmas , base = 30 prakrmas and face = 24 prakrmas, the eligible right triangles, according to
Āpastamba , are :
36 , 15 , 39 (ĀSS. V.2)
3 , 4 , 5 multiplied by 4 & 5 (ĀSS. V.3)
12, 5 , 13 (ĀSS. V.4)
15, 8, 17 (ĀSS. V.5)
12 , 35 , 37 (ĀSS. V.6)

To Construct Square into Rectangle

यावदिच्छं पर्श्वमान्यौ प्राच्यौ वर्धयित्वा उत्तरपूर्वां कर्णरज्जुमायच्छेत्,
सा दीर्घचतुरश्रमध्यस्थायां समचतुरश्रतिर्यङ्मान्यां यत्र निपतति तत उत्तरं हित्वा
दक्षिणांशम् तिर्यङ्मानीं कुर्यात्, तद् दीर्घचतुरश्रं भवति ।
From Āpastamba's commentator Sundararāja



Producing the sides of the square eastward to the desired length of the lateral side , one should draw the north eastern diagonal.
The part of the transverse side to the north of the point where the diagonal
cuts it is to be discarded and its southern part is to be made the transverse side of the rectangle. That will be the rectangle.
Let ABCD be the given Square,
Let ABCD be the given square. Produce A D and BC to F and E so that AF BE the required side of the rectangle Complete the rectangle ABEF
and join the diagonal BF cutting CD in G. Through G draw a set line IH parallel to the sides of the square.
Then IBEH is the required rectangle

FAB = FEB
IGB = GCB
FDG = FHG
Hence rect. AIGD = rect. GCEH
Therefore, IBEH = rect. IBCG + rect. GCEH
= rect. IBCG + rect.AIGD
= sq. ABCD

To Convert Rectangle or Square into a Trapezium with the shorter parallel side given.

चतुरश्रमेकतोऽणिमच्चिकीर्षन् अणिमतः करणीं
तिर्यङ्मानीं कृत्वा शेषमक्ष्णया विभज्य
विपर्यस्येतरत्रोपदध्यात् ।

If one wishes to make a square or rectangle shorter on one side, one should cut off a portion by the shorter side. The remainder should be divided by the diagonal, inverted and attached on either side).

If ABCD is the given rectangle, let the rectangle A FED be cut off so that A F=DE= the given shorter side.

remaining rectangle EFBC is to be cut diagonally along B E
and
The portion BEC is to be inverted and attached to the
side AD of the rectangle in the position EA D.
Then DEBE is the equivalent trapezium.

Chapter 5 Areas and Volumes

Area of Triangle :

The method for finding the area of triangle that was known in Śulba was
Area = ½ (base × altitude)

Āryabhaṭa I says : "The area of a triangle is the product of the perpendicular and half the base" (Ā, i.6)

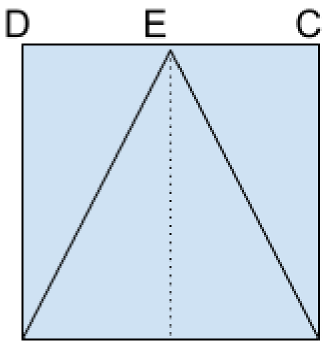
Brahmagupta :
"The product of half the sums of the sides and countersides of a triangle or a quadrilateral is the rough value of its area. Half the sum of the sides is severely lessened by the three or four sides, the square root of the product of the remainders is the exact area " (BrSpSi . xii. 21)
As a,b,c,d be four sides of a quadrilateral taken in order we have
Area = $c + d/2 \times a+b/2$, roughly
Area = $\sqrt{(s-a)(s-b)(s-c)(s-c)}$ exactly,
$s = ½ (a + b + c + d)$.
In case of Triangle $d = 0$; so that we get
$A = c/2 \times a+b/2$, roughly
$A = \sqrt{s(s-a)(s-b)(s-c)}$, exactly.

Mahāvira in GSS :
Area = $\sqrt{s(s-a)(s-b)(s-c)}$

TO CONSTRUCT ISOSCELES TRIANGLE EQUAL IN AREA TO A GIVEN SQUARE AND VICE VERSA:

यावानग्निस्सारन्निप्रादेशो द्विस्तावती भूमिं चतुरश्रं कृत्वा, पूर्वस्याः करण्या प्रर्धात् श्रोणी प्रत्यालिखेत् । सा नित्या प्रउगम् । (ASS. XII.5)

Making an area which is double as much as the fire-altar with the aratnis and pradešas, into a square, one should draw lines from



the middle point of the eastern side towards the bottom corners.
That is the equivalent prauga (isosceles triangle).
Let ABCD be a square of twice the required area. Let E be the middle point of C D. EA and EB are joined. Then A E B is the required triangle. For if the altitude EF is drawn, the square is divided into 2 equal rectangles AFED and FBC E
AFE = ½ rect. AFED
FBE = ½ rect. FBCE
AEB = ½ sq. ABCE

This construction leads to the Formula of Area of Triangle = ½ base × altitude.

Area of Quadrilateral :

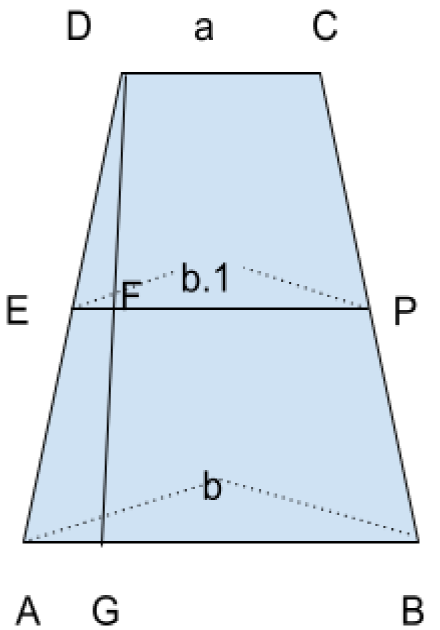
From GSS :
The product of halves of sum of opposite sides becomes quantitative measurement of the area of trilateral and quadrilateral figures.
Mahāvira lists five kinds of quadrilaterals.

1. Sama = Square and Rhombus (all sides equal)
2. Dvidvisama = with pairs of opposit sides equal , the rectangle and the parallelogram, though the latter does not get any notice in Mahāvira's work.
3. Dvisama = with two sides equal - the isosceles trapezium
4. Trisama = with three sides equal - the trapezium with three sides equal
5. Visama = with unequal sides, which most frequently , denotes the cyclic quadrilateral. Even the trapezium with unequal sides does not seem to be included under viṣama (probably because it is not cyclic).

Area of Trapezium :

It has been remarked that the trapezium , more especially the isosceles trapezium, had a place of honour both in the Vedic religion and in Jaina faith.
The usual expression for the area of a trapezium Mahāvīra says :
भुजयुत्यर्धचतुष्कात् भुजहीनात् घातितात् पदं सूक्ष्मम् ।
अथवा मुखतलयुति दलमवलम्बकगुणं न विषमचतुरश्रे ।   (GSS. vii, 50)
The square root from four sets of half the sum of the sides respectively diminished by the sides and multiplied together is the exact area. Or, half the sum of the base and the face multiplied by the altitude , but not in a viṣama quadrilateral. From this it is clear that Mahāvīra knew the formula $A = \sqrt{(s-a)(s-b)(s-c)(s-d)}$ was applicable to isosceles trapezium too.

Mahāvira has a rule for calculating the base and altitude of the sections with proportionate areas into which a given isosceles trapezium is divided by lines running parallel to its parallel sides.



If ABCD is a trapezium divided into parts with area in the ratio m/n by the line EP parallel to AB and if DG, the altitude from D is drawn cutting EP in F and meeting AB in G , from the similar triangles DEF and DAG , we have

$EF/AG = DF/DG$

I.e. $b1 - a / b - a = DF / DG$ where a and b are the face and base and b1 the intermediate base.

Or $(b1^2 - a^2) / b^2 - a^2 = DF(b1 + a) / DG\ (b + a) = m / m{+}n$

Then the altitude is easily calculated from the area. This solution is given in
खण्डयुति भक्ततलमुखकृत्यन्तरगुणित - खण्डमुखवर्गयुतम् ।
मूलमधस्तलमुखयुतदल्हृतलब्धं च लम्बकं क्रमशः ॥ (GSS vii 175 ½)

The part multiplied by the difference of the square of the base and the face and divided by the sum of the parts is combined with the square of the face . The square root of this is the base. The area divided by half the sum of the bottom side and the face is the perpendicular.

Circle Formulas :

Formula	Tiloyapannatti	Jambudvipapannati samgho	Triloksara	Ganitsarasanghrha
Circumference of Circle	$c = \sqrt{d^2 \times 10^2}$	$C = \sqrt{d^2 \times 10^2}$	$C = 3 \times d$ $C = \sqrt{10}$	$C = 3 \times d$ $C = \sqrt{10^2 \times d^2}$
Area of	$A = C \times d/4$	$A = C \times d/4$	$A = C \times$	$A = 3 \times (d/2)^2$

Circle		$A = \sqrt{((d/2)^2)^2 \times 10}$	d/4	
Chord in terms of h & d	$c = \sqrt{4}\ [(d/2)^2 - (d/2 - h)^2]c$	$c = \sqrt{(d - h) \times h \times 4c^2}$		
h in terms of c & d	$h = d/2-\sqrt{(d/2)^2-¼\ c^2h}$			
d in terms of c & h	$d = c^2/(4 \times h) + h$			
a in terms of c & h	$a^2 = 6h^2 + c^2$			

Above formulas are from different Jaina Mathematics books

Measurement of Segment of Circle

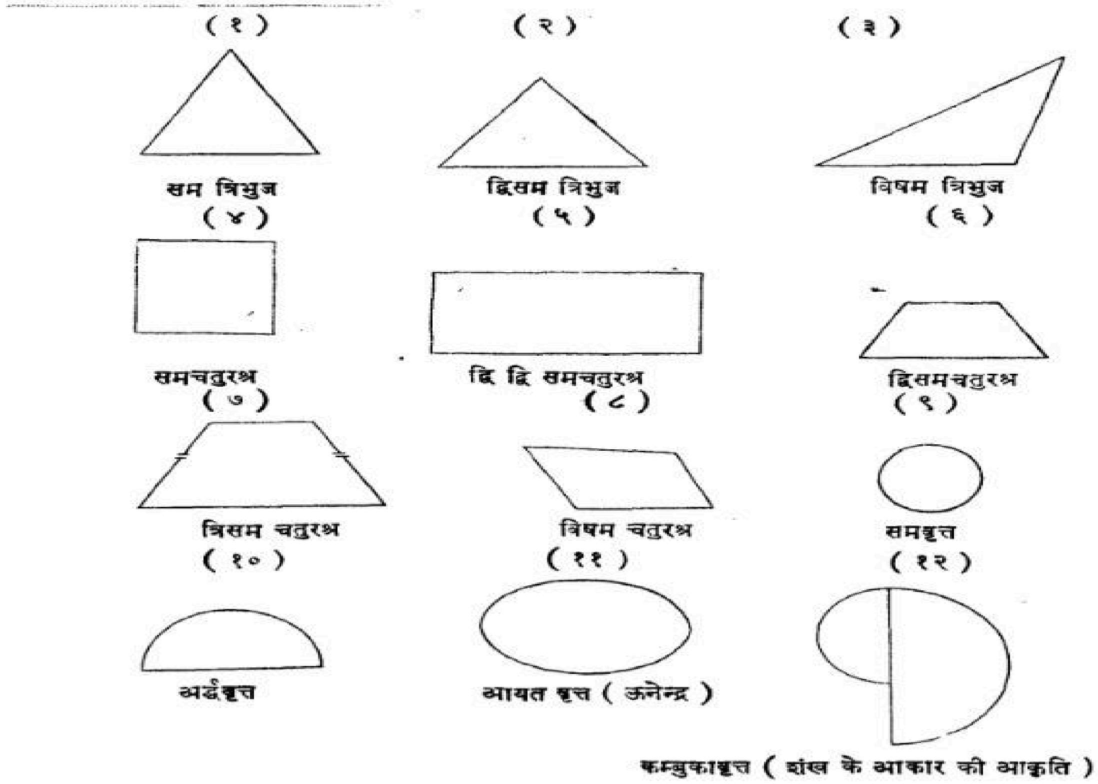
d = diameter of the circle , c = a chord of it, a = arc cut off by that chord , h = height of the segment or its arrow and a' = an arc of the circle lying between two parallel chords.

Mahāvira's Rules

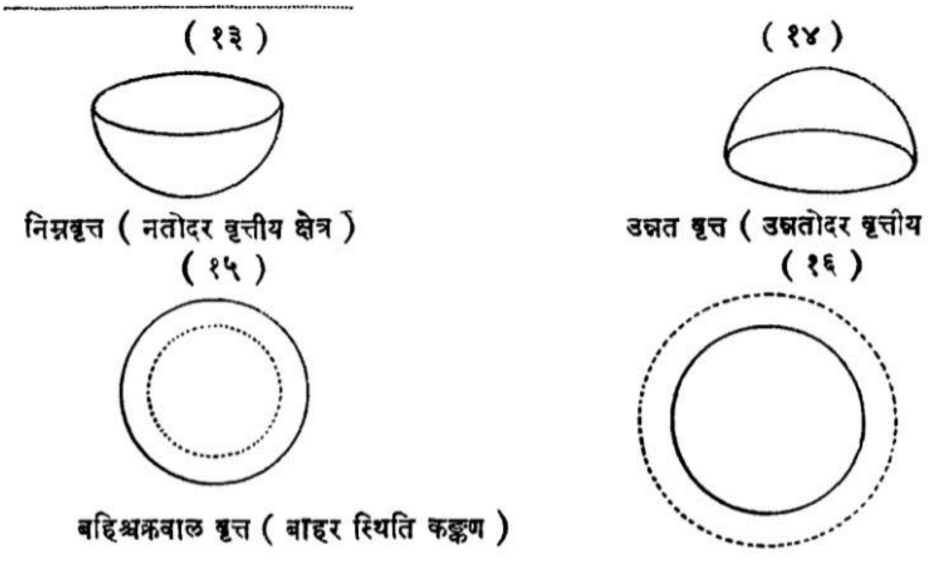
He gave two sets of formulae
1st set gives serving all practical purposes (vyāvahārika phala)
The 2nd set yields precise results (sūksma phala).

Rough Formulae are :
$A = ½\ h(c + h)$,
$h = \sqrt{a^2 - c^2/5}$
$c = \sqrt{a^2 - 5h^2}$
$a = \sqrt{5h^2 + c^2}$

Important Fig. from GSS

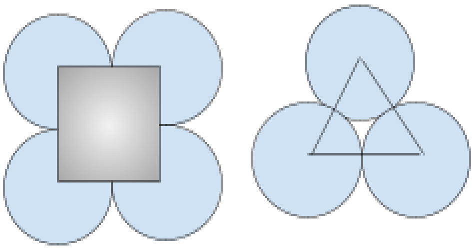


Samatribhuja = Equilateral Trilateral fig.
Dvisamatribhuja = Isoscles Trilateral fig.
Visamatribhuja = scalene trilateral



Now a days used lens shape , Convex and Concave :

Formulae for the area of space enclosed by three or more equal , mutually touching circles.



This Formula has been accurately given by Mahāvira and Nārāyaṇa Paṇḍita
Fig 1 : If the diameter of the circles is d, the area enclosed by 4 circles = area of a square with side equal to the diameter minus the area of one circle.
(82 ½ GSS) : If the minutely accurate measure of the area of any one circle is subtracted from the quantity which forms the square of the diameter of the circle there results in the value of the area of the interspace included within four equal circles touching each other.
Fig 2 : In case of Three circles , the enclosed area = area of an equilateral triangle with side equal to the diameter minus half the area of one circle. Here since the angles of the three sectors are 60 degree each, each of the sectors = ⅙ the circle.
Therefore, 3 sectors together = ½ the circle. Hence the Rule.
(84 ½ GSS) : The minutely accurate measure of the area of an equilateral triangle, each side of which is equal in measure to the diameter of the circle, is diminished by half the area of any of the three equal circles. The remainder happens to be the measure of the interspace area caused by three manually touching equal circle

Chapter 6 - Examples

Example Questions from Ganitsarasanghra and Lilavati

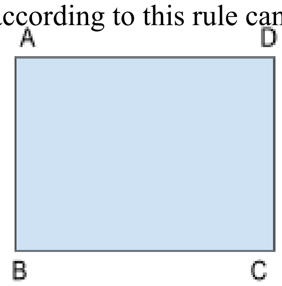
क्षेत्रव्यवहार गणित -

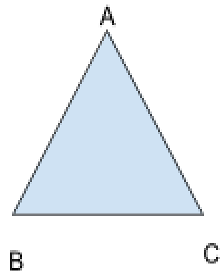
त्रिभुजचतुर्भुजबाहुप्रतिबाहुसमासदलहतं गणितम् ।
नेमेर्भुजयुत्यर्ध व्यासगुणं तत्फलर्धमिह बालेन्दो ॥७॥ (GSS , vii .7)

The product of halves of sum of opposite sides becomes quantitative measurement of the area of trilateral and quadrilateral figures. …1

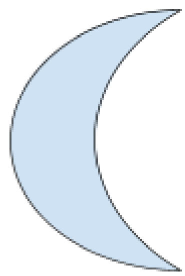
A trilateral figure is here conceived to be formed by making the topside, s.e., the side opposite to the base, of a quadrilateral so small as to be neglected. Then the two lateral sides of the trilateral figure become the opposite sides, the topside being taken to be nil in value. Hence it is that the rule speaks of opposite sides even in the case of a trilateral figure. As half the sum of the two sides of a triangle is, in all cases, bigger than the altitude, the value of the area arrived at according to this rule cannot be accurate in any instance.

In regard to quadrilateral figures the value of the area arrived at according to this rule can be accurate in the case of a square and an oblong, but only approximate in other cases.





In Circular annulus like the rim of wheel , half of the sum of the inner or outer circumference multiplied by the measure of the breadth of the annulus gives quantitative measure of the area…..2

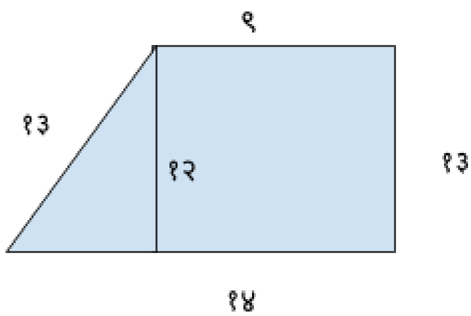


Therefore, Result here happens to be the area of a fig. It Resembles the crescent moon.
Nemi is the area enclosed between the circumferences of two concentric circles;
and the rule here stated for finding out the approximate measure of the area of a Nemiksētra happens to give the accurate measure thereof.

In the case of a figure resembling the crescent moon, it is evident that the result arrived at according to the rule gives only an approximate measure of the area.

For Triangle the value of area rule cannot be accurate in any instance
For Quadrilateral fig. Accurate in case of square and oblong approx in case of other fig.

Lilavati from Dwitiyakhand kshetravyvhar



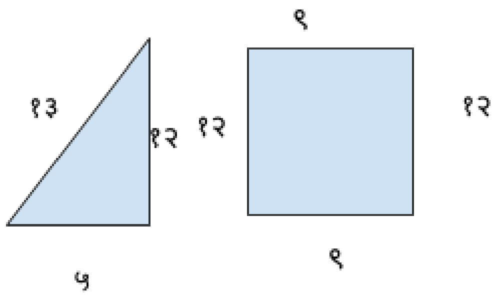
चतुर्भुजे त्रिभुजे चास्पष्ट स्पष्टफलानयने करणसूत्रं । वृत्तम्

सर्वदोर्युतिदलञ्चतुः स्थितं बाहुभिर्विरहितं च तद्वधात् ||
मूलमस्फुटफलं चतुर्भुजे सपष्टमेवमुदितं त्रिबाहुके ||१९||
(L. ii.19)

(भूमि + मुखम्) / २ × लम्ब
१४ + ९ / २ = २३/२ = ११.५ × १२

= १३८

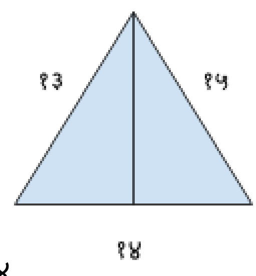
If we calculate by dividing fig.



with same process we get
30 and 108 area respectively
There addition 138 i.e total area.

Process for Triangle (त्रिभुज)

सर्वदीर्युतिदलमित्यादिना त्रिभुजे स्पष्टफलानयनाय अत्र त्रिभुजस्य पूर्वोदाहृतस्य न्यासः
Sum of All sides , dividing it by two , the number produced places them in different places .
Subtract each by each side value, then multiply all numbers and take square root.



भूमिः १४
भुजौ १३ । १५

१३ । १५ । १४
योग = ४२
अर्ध = २१

२१	२१	२१	२१
१३	१५	१४	००

=

८	६	७	२१

घात (Product) - product of 4 of this
७०५६

मूल = ८४ क्षेत्रफल

Abbreviations

ŚBr = Śatapatha Brāhmana
KSS = Kātyāyana Śulba
GSS = Gaṇita sāra saṃgraha
L = Lilāvatī
ASS = Āpastamba Śulba
BSS = Baudhāyana Śulba
MāŚl = Mānava Śulba
MaiS = Maitrāyaṇīya Śulba

Section 5 - Proofs (उपपत्ति)

Chapter 1 - Introduction to the Proofs in Indian Mathematics

While there have been several extensive investigations on the history and achievements of Indian mathematics, there has not been much discussion on its methodology, the Indian mathematicians' and philosophers' understanding of the nature and validation of mathematical results and procedures, their views on the nature of mathematical objects, and so on.

Traditionally, such issues have been dealt with in the detailed bhāṣyas or commentaries, which continued to be written till recent times, and played a vital role in the traditional scheme of learning.

It is in such commentaries that we find detailed upapattis or 'proofs' of the results and procedures, apart from a discussion of methodological and philosophical issues .

- Early European Scholars Were Aware of Proof

In the early stages of modern scholarship on Indian mathematics, we find references to the methods of demonstration found in texts of Indian mathematics.

In 1817, H. T. Colebrooke referred to them in his pioneering and widely circulated translation of Lilavati and Bijaganita and the two mathematics chapters of Brahmasphuta-siddhanta.
"On the subject of demonstrations, it is to be remarked that the Hindu mathematicians proved propositions both algebraically and geometrically: as is particularly noticed by Bhaskara himself, towards the close of his algebra, where he gives both modes of proof of a remarkable method for the solution of indeterminate problems, which involve a factum of two unknown quantities."
Similarly, Charles Whish, in his seminal article on Kerala School of Mathematics of 1835, referred to the demonstrations in Yuktibhāṣā.

"A further account of the Yuktibhasa, the demonstrations of the rules for the quadrature of the circle by infinite series, with the series for the sines, cosines, and their demonstrations, will be given in a separate paper:
I shall therefore conclude this, by submitting a simple and curious proof of the 47th proposition of Euclid [the so called Pythagoras theorem], extracted from the Yuktibhāṣa.“

Whish does not seem to have written any further papers on the demonstrations of the infinite series as given in Yuktibhāṣā.

Whish's paper was widely noticed in the scholarly circles of Europe in the second quarter of nineteenth century.

But it was soon forgotten and there was no study of Yuktibhāṣā till 1940s, when C. T. Rajagopal and his collaborators wrote pioneering articles on the proofs outlined in that seminal text .

- *The Alleged Absence of Proofs in Indian Mathematics*

It has been the scant attention paid, by the modern scholarship of the last two centuries, to this extensive tradition of commentaries which has led to a lack of comprehension of the methodology of Indian mathematics. This is reflected in the often-repeated statements on the absence of logical rigour in Indian mathematics in works on history of mathematics such as the following:

"As our survey indicates, the Hindus were interested in and contributed to the arithmetical and computational activities of mathematics rather than to the deductive patterns. Their name for mathematics was ganita, which means 'the science of calculation'. There is much good procedure and technical facility, but no evidence that they considered proof at all. They had rules, but apparently no logical scruples. Moreover, no general methods or new viewpoints were arrived at in any area of mathematics ."

"Indian mathematics had no proof."
Here, in NCERT Book of Class IX
It is said that,
"While mathematics was central to many ancient civilisations
like Mesopotamia, Egypt, China, and India,
*there is no clear evidence that they used proofs***".**
(p. 287, Class IX, Appendix 1)

Chapter 2 - Proof in Nyaya Sutra of Gotama

प्रमाणप्रमेयसंशयप्रयोजनदृष्टान्तसिद्धान्तावयवतर्कनिर्णयवादजल्पवितण्डाहेत्वाभासच्छलजातिनिग्रहस्था नानाम् तत्त्वज्ञानात्निःश्रेयसाधिगमः ।। १ ।।
1. Supreme felicity is attained by the knowledge about the true nature of sixteen categories, viz., means of right knowledge (pramâna), object of right knowledge (pra- meya), doubt (saṁśaya), purpose (prayojana), familiar instance (drṣṭânta), established tenet (siddhanta), members (avayava), confutation (tarka), ascertainment (nirnaya), discussion (vâda), wrangling (jalpa), cavil (vitanda), fallacy (hetvâbhûsa), quibble (chala), futility (jati), and occasion for rebuke (nigrahasthâna). Knowledge about the true nature of sixteen categories means true knowledge of the "enunciation," "definition" and "critical examination" of the categories. Book I (of the Nyaya-Sûtra) treat of "enunciation" and "definition," while the remaining four Books are reserved for "critical examination." The attainment of supreme felicity is preceded by the knowledge of four things, viz., (1) that which is fit to be abandoned (viz., pain), (2) that which produces what is fit to be abandoned (viz., misapprehension, etc.), (3) complete destruction of what is fit to be abandoned and (4) the means of destroying what is fit to be abandoned (viz., true knowledge).

प्रत्यक्षानुमानोपमानशब्दाः "प्रमाणानि" ।। २ ।।
2. Perception, inference, comparison and word (verbal testimony)-these are the means of right knowledge.
[The Charvakas admit only one means of right knowledge, viz., perception (pratyaksas, the Vaiserikas and Bauddhas admit two, vis., perception and inference (anumana), the Säfikhyas admit three, viz., perception, inference and verbal testimony (ngama or sabda) while the Naiyayikas whose fundamental work is the Nyaya-sûtra admit four, viz., perception, inference, verbal testimony and comparison (upamâna). The Prabhākrras admit a fifth means of right knowledge called presumption arthapatti),

the Bhattas and Vedantins admit a sixtlı, viz., non-existence (abhava) and the Pauranikas recognise a seventh and eighth means of right knowledge, named probability (samblava) and rumour (aitihya)].

इन्द्रियार्थसन्निकर्षोत्पन्नं ज्ञानमव्यपदेश्यमव्यभिचारि व्यवसायात्मकं "प्रत्यक्षम्" ।। ३ ।।

3. Perception is that knowledge which arises from the contact of a sense with its object, and which is determinate, unnameable, and non-erratic.

Determinate. — This epithet distinguishes perception from indeterminate knowledge; as for instance, a man looking from a distance cannot ascertain whether there is smoke or dust.

Unamenable. — Signifies that the knowledge of a thing derived through perception lias' no connection with the name which the thing bears.

Non-erratic. — Tn summer the sun's rays coming in contact with earthly heat quiver and appear to the eyes of men as water. The knowledge of water derived in this way is not perception. To eliminate such cases the epithet non-erratic has been used.

[This aphorism may also be translated as follows: — Perception is knowledge, and which arises from the contact of a sense with its object and which is non-erratic being either indeterminate (nirvikalpaka as " this is something") or determinate vsavikalpaka as " this is a Brahmana ")].

वृत्तं भ्रमेण साध्यं त्रिभुजं च चतुर्भुजं च कर्णाभ्याम् ।
साध्या जलेन समभूरधऊध्वं लम्बकेनैव ॥ १३ ॥

A circle should be constructed by means of a pair of compasses; a triangle and a quadrilateral by means of the two hypotenuses (Karna). The level of ground should be tested by means of water; and verticality by means of a plumb .

अथ तत्पूर्वकं "त्रिविधमनुमानं" पूर्ववत्शेषवत्सामान्यतोदृष्टं च ।। ४ ।।

4. Inference is knowledge which is preceded by perception, and is of three kinds, viz., a priori, a posteriori and ' commonly seen. '

A priori is the knowledge of ellect derived from the perception of its cause, e. </., one seeing clouds infers that there will be rain.

A posteriori is the knowledge of cause derived from the perception of its effect, e. g., one seeing a river swollen infers that there was rain.

[' Commonly seen ' is the knowledge of one thing derived from the perception of another thing with which it is commonly seen, e. g., one seeing a beast possessing horns, infers that it possesses also a tail, or one seeing smoke on a hill infers that there is fire on it].

Vatsyayana takes the last to be " not commonly seen " which he interprets as the knowledge of a tiling which is not commonly seen, e. g., observing affection, aversion, and other qualities one infers that there is a substance called soul.

वृत्त अपञ्जरमध्ये कच्च्यापरिवेष्टितः खमध्यगतः ।
सृज्जलशिखिवायुमयो भूगोलः सर्वतो वृत्तः ।। ६ ।।

The globe of the Earth stands (supportless) in space at the centre of the circular frame of the asterisms (i.e., at the centre of the Bhagola) surrounded by the orbits (of the planets); it is made up of water, earth, fire and air and is spherical (lit. circular on all sides) .

प्रसिद्धसाधर्म्यात्साध्यसाधनम् "उपमानम्" ।। ५ ।।

5. Comparison is the knowledge of a thing through its similarity to another thing previously well known.

A man hearing from a forester that a bos gavaeus is like a cow resort to a forest where he sees an animal like a cow. Having recollected what he heard he institutes a comparison, by which he arrives at the conviction that the animal which he sees is bos gavaeus. This is knowledge derived through comparison. Sonic hold that comparison is not a separate means of knowledge, for when one notices the likeness of a cow in a strange animal one really performs an act of perception. In reply it is urged that we cannot deny comparison as a separate means of knowledge, for how does otherwise the name bos gavaeus signify the general notion of the animal called bos gavaeus. That (he names bos gavaeus signifies one and all members of the bos gavaeus class is not a result of perception bnt the consequence of a distinct knowledge called comparison.

यद्वत् कदम्बपुष्पग्रन्थिः प्रचितः समन्ततः कुसुमैः ।
तद्वदधि सर्वसत्त्वैर्जलजैः स्थलजैश्च भूगोलः ॥ ७ ॥
Just as the bulb of a Kadamba flower is covered all around by blossoms, just so is the globe of the Earth surrounded by all creatures, terrestrial as well as aquatic .44

आप्तोपदेशः "शब्दः" ।। ६ ।।
6. Word (verbid testimony) is the instructive assertion of a reliable person.
A reliable person is one — may be a lisi, Arya or mleccha, who as an expert in a certain matter is willing to communicate his experiences of it.
[Suppose a young man coming to the side of a river cannot ascertain whether the river is fordable or not, and immediately an old, experienced man of the locality, who has no enmity against him, comes and tells him that the river is easily fordable: the word of the old man is to be accepted as a means of right knowledge called verbal testimony].

"तद विदो विदुः" "so say the learned".

"स द्विविधो" दृष्टादृष्टार्थत्वात् ।। ७ ।।
7. It is of two kinds, viz., that which refers to matter which is seen and that which refers to matter which is not seen.
The first kind involves matter which can be actually verified. Though we are incapable of vcrifiying the matter involved in the second kind, we can somehow ascertain it by means of inference.
[Matter, which is seen, e.g., a physician's assertion that physical strength is gained by taking butter].
[Matter, which is not seen, e.g., a religious teacher's assertion that one conquers heaven by performing horse-sacrifices].

Chapter 3 - Use of Tarka in Indian Proofs

The method of "proof by contradiction" is referred to as tarka in Indian logic. We see that this method is employed in order to show the non-existence of an entity.
For instance, Kṛṣṇa Daivajña essentially employs tarka to show the non-existence of the square-root of a negative number while commenting on the statement of Bhaskara that a negative number has no root.
वर्गस्य हि मूलं लभ्यते । ऋणाङ्कस्तु न वर्गः कथमतस्तस्य मूलं लभ्यते । ननु ऋणाङ्कः कुतो वर्गो न भवति न हि राजनिर्देशः ।... सत्यम्। ऋणाङ्कं वर्ग वदता भवता कस्य स वर्ग इति वक्तव्यम् । न तावद्धनाङ्कस्य "समद्वि घातो हि वर्गः" तत्र धनाङ्केन धनाङ्के गुणिते यो वर्गो भवेत् स धनमेव "स्वयोर्वधः स्वम्"

इत्युक्तत्वात् । नाप्यृणाङ्कस्य । तत्रापि समद्विघातार्थ- मृणाङ्केनर्णाङ्कगुणिते धनमेव वर्गो भवेत् "अस्वयोर्वधः स्वम्" इत्युक्त- त्वात् । एवं सति कथमपि तमङ्कं न पश्यामो यस्य वर्गः क्षयो भवेत् ।
Thus, according to Krsna
"The square-root can be obtained only for a square. A negative number is not a square. Hence how can we consider its square-root? It might however be argued: 'Why will a negative number not be a square? Surely it is not a royal fiat... Agreed. Let it be stated by you who claim that a negative number is a square as to whose square it is; surely not of a positive number, for the square of a positive number is always positive by the rule... not also of a negative number. Because then also the square will be positive by the rule... This being the case, we do not see any such number whose square becomes negative..."
While the method of "proof by contradiction" or reduction ad absurdum has been used to show the non-existence of entities, the Indian mathematicians do not use this method to show the existence of entities, whose existence cannot be demonstrated by other direct means. They have a "constructive approach" to the issue of mathematical existence.
a general principle of Indian logic that tarka is not accepted as an independent pramana, but only as an aid to other pramāṇas .

Chapter 4 - Indian Logic Excludes Prasiddha Entities from Logical Discourse

Naiyayikas or Indian logicians do not grant any scheme of inference, where a premise which is known to be false is used to arrive at a conclusion, the status of an independent pramāṇa or means of gaining valid knowledge.
In fact, they go much further in exorcising the logical discourse of all aprasiddha terms or terms such as "rabbit's horn" (sasasrniga) which are empty, non-denoting or unsubstantiated.
"Nyaya...(excludes) from logical discourses any sentence which will ascribe some property (positive or negative) to a fictitious entity. Vacaspati remarks that we can neither affirm nor deny anything of a fictitious entity, the rabbit's horn. Thus nyaya apparently agrees to settle for a superficial self-contradiction because, in formulating the principle that nothing can be affirmed or denied of a fictitious entity like rabbit's horn, nyaya, in fact violates the same principle. Nyaya feels that this superficial self-contradiction is less objectionable (than admitting fictitious entities in logical discourse)... (This can be seen from the discussion in) Udayana's Atmatattvaviveka... "

Chapter 5 - Kṛṣṇa Daivajña on the Importance of Upapatti

The following passage from Krsna Daivajña's commentary on Bijaganita brings out the general understanding of the Indian mathematicians that citing any number of favourable instances (even an infinite number of them) where a result seems to hold, does not amount to establishing it as a valid result in mathematics. Only when the result is supported by an upapatti or demonstration can the result be accepted as valid.
ननुपपत्त्या विना वर्गयोगो द्विघातेन युतो हीनो वा युतिवर्गोऽन्तरवर्गो वा भवतीत्येतदेव कथम्? क्वचिद्दर्शनं त्वप्रयोजकम् । अन्यथा चतुर्गुणो राशिघातो यतिवर्गो भवतीत्यपि सुवचम् । तस्यापि क्वचित्तथा दर्शनात् । तथाहि राशी २, २ अनयोर्घातः ४ चतुर्गुणः १६ अयं जातो युतिः ४ वर्गः १६ वा राशी ३,३ अनयोर्घातश्चतुर्गणः ३६ अयमेव यति ६ वर्गश्च ३६ वा राशी ४, ४ अनयोर्घातः १६ चतुर्गुणः ६४ अयमेव युति ८ वर्गः ६४ इत्यादिषु । तस्मात् क्वचिद्दर्शनम् अप्रयोजकं क्वचिद्वाभिचारस्यापि संभवात् । अतो वर्गयोगो द्विघ्नघातयुतोनो युतिवर्गोऽन्तरवर्गश्च भवतीत्यत्र युक्तिर्वक्तव्येति चेत् सत्यम्। इयमुपपत्तिरेकवर्णमध्यमाहरणान्ते ।

"How can we state without proof (उपपत्ति) that twice the product of two quantities when added or subtracted from the sum of their squares is equal to the square of the sum or difference of those quantities? That it is seen to be so in a few instances is indeed of no consequence. Otherwise, even the statement that four times the product of two quantities is equal to the square of their sum, would have to be accepted as valid. For, that is also seen to be true in some cases. For instance, take the numbers 2, 2. Their product is 4, four times which will be 16, which is also the square of their sum 4. Or take the numbers 3, 3. Four times their product is 36, which is also the square of their sum 6. Or take the numbers 4, 4. Their product is 16, which when multiplied by four gives 64, which is also the square of their sum 8. Hence the fact that a result is seen to be true in some cases is of no consequence, as it is possible that one would come across contrary instances also. Hence it is necessary that one has to provide a proof (yukti) for the rule that twice the product of two quantities when added or subtracted from the sum of their squares results in the square of the sum or difference of those quantities. We shall provide the proof (उपपत्ति) at the end of the section on ekavarna-madhyamaharana. "

Chapter 6 - Bhāskara I on उपपत्ति

In his discussion of Aryabhata's approximate value of the ratio of the circumference and diameter of a circle, Bhaskara I notes that the approximate value is given, as the exact value cannot be given. He then goes on to argue that other values which have been proposed are without any justification:
एवं मन्यन्ते स उपाय एव नास्ति येन सूक्ष्मपरिधिरानीयते ।
ननु चायमस्ति विक्खंभवग्गदसगुणकरणी वट्टस्स परिरओ होवि।
(विष्कम्भवर्गदशगुणकरणी वृत्तस्यपरिणाहो भवति)
इति । अत्रापि केवल एवागमः नैवोपपत्तिः । रूपविष्कम्भस्य दशकरण्यः परिधिरिति । अथ मन्यन्ते प्रत्यक्षेणैव प्रमीयमाणो रूपविष्कम्भ क्षेत्रस्य परिधिर्दशकरण्य इति । नैतत् अपरिभाषितप्रमाणत्वात् करणीनाम् ।
एकत्रिविस्तारायामायतचतुरश्रक्षेत्रकर्णेन दशकरणिकेनैव तद्विष्कम्भ- परिधिर्वेष्ट्यमाणः स तत्प्रमाणो भवतीति चेत्तदपि साध्यमेव ।
परिधि :- circumference ; करणी :- surds ; विष्कम्भ :- diameter

Chapter 7 - Bhāskara II on Upapatti

In Siddhantasiromani, Bhaskaracarya II (1150) presents the raison d'être of upapatti in the Indian mathematical tradition:
मध्याद्यं द्युसदां यदत्र गणितं तस्योपपत्तिं *(तस्योपपत्तिं = तस्य+ उपपत्ति)*विना प्रौढिं प्रौढसभासु नैति गणको निःसंशयो न स्वयम् ।
गोले सा विमला करामलकवत् प्रत्यक्षतो दृश्यते तस्मादस्म्युपपत्तिबोधविधये गोलप्रबन्धोद्यतः ॥
Without the knowledge of उपपत्ति, by merely mastering the calculations (gunita) described here, from the madhyamadhikara (the first chapter of Siddhantasiromani) onwards, of the [motion of the] heavenly bodies, a mathematician will not be respected in the scholarly assemblies; without the उपपत्ति he himself will not be free of doubt (nihsamsaya). Since उपपत्तिis clearly perceivable in the (armillary) sphere like a berry in the hand, I therefore begin the Goladhyaya (section on spherics) to explain the उपपत्ति.
The same has been stated by Ganesa Daivajña in the introduction to his commentary Buddhivilasinī (c. 1540) on Lilavati of Bhaskaracārya

व्यक्ते वाव्यक्तसंज्ञे यदुदितमखिलं नोपपत्तिं विना तत् ,
निर्भ्रान्तो वा ऋते तां सुगणकसदसि प्रौढतां नैति चायम्।
प्रत्यक्षं दृश्यते सा करतलकलितादर्शवत् सुप्रसन्ना ,
तस्मादग्र्योपपत्तिं निगदितुमखिलम् उत्सहे बुद्धिवृद्धौ ॥

व्यक्ते :- express; सुगणकसदसि :- assembly of scholars

Whatever is stated in the vyakta or avyakta branches of mathematics, without upapatti, will not be rendered nirbhranta (free from confusion); will not have any value in an assembly of mathematicians. The upapatti is directly perceiv-able like a mirror in hand. It is therefore, as also for the elevation of the intellect (buddhi-vrddhi), that I proceed to enunciate upapatti-s in entirety.

Thus, according to the Indian mathematical texts, the purpose of upapatti is mainly:

(i) To remove confusion and doubts regarding the validity and interpretation of mathematical results and procedures; and,

(ii) To obtain assent in the community of mathematicians.

This is very different from the ideal of "proof" in the Greco-European tradition which is to irrefutably establish the absolute truth of a mathematical proposition.

In his Brjaganita-vasana, Bhaskaracārya II (c.1150) refers to the long tradition of upapattis in Indian mathematics.

अस्योपपत्तिः । सा च द्विधा सर्वत्र स्यादेका क्षेत्रगताऽन्या राशिगतेति। तत्र क्षेत्रगतोच्यते । ... अथ राशिगतोपपत्तिरुच्यते सापि क्षेत्रमूला

न्तर्भूता । इयमेव क्रिया पूर्वाचार्यैः संक्षिप्तपाठेन निबद्धा । ये क्षेत्रगतां उपपत्तिं न बुद्ध्यन्ति तेषामियं राशिगता दर्शनीया।

"The demonstration follows. It is twofold in each case: One geometrical and the other algebraic. There, the geometrical one is stated... Then the algebraic demonstration is stated, that is also geometry-based. This procedure [of upapatti] has been earlier presented in a concise instructional form by ancient teachers. For those who cannot comprehend the geometric demonstration, to them, this algebraic demonstration is to be presented ."

Chapter 8 - Pythagorus Theorem

भुजाकोटिवर्गयोः योगः

भुजाकोटिवर्गयोः योगः also can be says as “sum of squares of perpendicular and base of a triangle”(Popularly Known as Pythagoras theorem) is a proof demonstration of statement from Yuktibhāṣā also known as Gaṇita-yukti-bhāṣā, which is a major treatise on mathematics and astronomy, written by the Indian astronomer Jyesthadeva of the Kerala school of mathematics around 1530. Yuktibhāṣā mainly gives rationale for the results in Nilakantha's Tantrasamgraha.

इह कोटितुल्यमेकं समचतुरश्रक्षेत्रमुत्पाद्य भुजातुल्यं च । इत्थं समचतुरश्रद्वयमुत्पादय । अनन्तरं भुजाचतुरश्रमुत्तरभागे, कोटिचतुरश्रं दक्षिणभागे। उभयोश्च पूर्वभागमेकसूत्रे आगमन समये मिथो योजय । भुजायाः दक्षिणभागं कोट्याः उत्तरपार्श्वेन संयुज्यमाने। इह कोट्याः उत्तरपार्श्व भुजापार्श्वं अवसानान्तरं पश्चिमपार्श्वभवं शेषितं भवति । अनन्तरं भुजाया उत्तरपूर्वकोणस्य दक्षिणकोट्यः दीर्घसदृशं ईयात् । तत्र एक बिन्दु निधाय एतस्माद् दक्षिणपार्श्व भुजासमं दीर्घं भवति । अनन्तरमेतद्विन्दोः कोट्याः दक्षिणपश्चिमकोणसदृशं भुजायाः उत्तरपश्चिमकोणसदृशं च विद्यमानरेखामार्गेण [विदारय] द्वयमपि विशेषेण [संज्ञा ? सन्धि] परित्यागं विना भवति । अनन्तरं बिन्दोः लघु [विदारण] द्वयं च (संज्ञा ? सन्धि) विशेषबिन्दौ संयुक्तरेखाग्राणि द्वयं च मिथः कोट्याः उत्तरपश्चिमसन्धेः मानं कुरु दृष्टमहच्चतुरश्रस्य भागद्वयं च रज्वीकृत्य सङ्गमय । अन्तर्बहिश्चास्ये विन्यासशिष्टयोजितव्यं तु । तत् एकं समचतुरर्थं भवति । एतस्य बाहवः भुजाकोट्याः कर्णेन समं भवति । किन्तु भुजाकोट्योः वर्गयोगं कर्णवर्गम्, कर्णवर्गात् भुजाकोटिमध्ये एकस्य वर्गत्यागे भुजाकोटिषु अन्यस्य वर्गमिति संस्थितमासीदिदानीम् । इदं सर्वत्र ज्ञातव्यं भवति ।।

इह कोटितुल्यमेकं समचतुरश्रक्षेत्रमुत्पाद्य भुजातुल्यं च ।

इह – here ; कोटितुल्यमेकं(कोटि+तुल्य+एकं) – equivalent to the koṭi ; समचतुरश्रक्षेत्रमुत्पाद्य (सम+चतुर्भुज+क्षेत्र+उत्पाद्य) – obtain area of square ; भुजातुल्यं(भुजा+तुल्यं) – equivalent to base
Here, draw a square (with its side) whose sides are equivalent to the koṭi and another square whose sides are equivalent to the bhuja.
अब, कोटि के बराबर एक वर्ग (उसकी भुजा सहित) और भुजा के बराबर एक और वर्ग बनाएं।

Here, koti is referred to perpendicular and bhuja as base .
इत्थं समचतुरश्रद्वयमुत्पादय ।
इत्थं - so on ; समचतुरश्रद्वयमुत्पादय (द्वौ समचतुर्भुज उत्पादितौ) – two squares obtained
And so on, two squares are obtained.
इस प्रकार दो वर्ग बनाएं।

अनन्तरं भुजाचतुरश्रमुत्तरभागे, कोटिचतुरश्रं दक्षिणभागे।
अनन्तरं – after ; भुजाचतुरश्रमुत्तरभागे(भुजा+तुल्यं+चतुर्भुज+उत्तर+भागे) – square equivalent to bhuja on north side ; कोटिचतुरश्रं दक्षिणभागे (कोटि+तुल्यं चतुर्भुज+ +दक्षिण+भागे) – square of area equivalent to koti on the south side
After that place the bhuja-square be on the northern side and the koti-square on the southern side.
बता दें कि भुज-वर्ग उत्तरी तरफ और कोटि-वर्ग दक्षिणी तरफ है |

उभयोश्च पूर्वभागमेकसूत्रे आगमन समये मिथो योजय ।
उभयोश्च(उभयो:+ च) – and both ; पूर्वभागमेकसूत्रे(पूर्व+भागे+एक+सूत्र)- one side from the east section; आगमन समये – that time;मिथो – together;योजय - add
in such a way that the eastern side of both the squares fall on the same line
इस प्रकार कि दोनों वर्गों का पूर्वी भाग एक ही रेखा पर पड़े |

भुजायाः दक्षिणभागं कोट्याः उत्तरपार्श्वेन संयुज्यमाने।
भुजायाः दक्षिणभागं – bhuja-square on southern side; कोट्याः – koti-square उत्तरपार्श्वेन (उत्तर+पार्श्वे) -on the north side; संयुज्यमाने – being combined
and in such a manner that the southern side of the bhuja-square being combined on the northern side of the koti-square.
और इस प्रकार कि भुज-वर्ग का दक्षिणी भाग उत्तरी भाग पर पड़े

इह कोट्याः उत्तरपार्श्वं भुजापार्श्वं अवसानान्तरं पश्चिमपार्श्वभवं शेषितं भवति ।
इह – here; अवसानान्तरं (अवसान+ अनन्तरं)– after the end ; भवं - produce ; शेषितं - exploited ; भवति – happen
Here, this (northern) side (of the koti-square) will be further extended in the western-side than the bhuja.
यह (उत्तरी) भाग (कोटि-स्क्वायर का) पश्चिमी भाग की तुलना में और अधिक बढ़ाया जाएगा

अनन्तरं भुजाया उत्तरपूर्वकोणस्य दक्षिणकोट्यः दीर्घसदृशं ईयात् ।
अनन्तरं – afterwards;दीर्घसदृशं (दीर्घ+सदृशं) – similar to a long ; स्यात् – should
From the north-east corner of the bhuja-square, measure southwards a length equal to the koṭi and mark the spot with a point.
भुज-वर्ग के उत्तर-पूर्व कोने से, दक्षिण की ओर कोटि के बराबर लंबाई मापें और उस स्थान को एक बिंदु से चिह्नित करें।

तत्र एक बिन्दु निधाय एतस्माद् दक्षिणपार्श्वं भुजासमं दीर्घं भवति ।

तत्र – there ; निधाय – putting ; एतस्माद् – from this ; भुजासमं - equal to bhuja; दीर्घं – long; भवति - happen
From this (point) the (remaining) line towards the south will be of the length of the bhuja.
इस (बिंदु) से दक्षिण की ओर (शेष) रेखा भुज की लंबाई की होगी।

अनन्तरमेतद्विन्दोः कोट्याः दक्षिणपश्चिमकोणसदृशं भुजायाः उत्तरपश्चिमकोणसदृशं च विद्यमानरेखामार्गेण [विदारय] द्वयमपि विशेषेण [संज्ञा ? सन्धि] परित्यागं विना भवति ।
अनन्तरमेतद्विन्दोः (अनन्तरम+एतद+बिन्दोः)– after this point ; कोट्याः -koti-square ; दक्षिणपश्चिमकोणसदृशं (दक्षिण -पश्चिम + कोणसदृशं) – similar to the south-west corner ;भुजायाः – bhuja-square ; उत्तरपश्चिमकोणसदृशं(उत्तर-पश्चिम+कोण+सदृशं) – similar to the north-west corner ; च - and ; विद्यमानरेखामार्गेण (विद्यमानरेखा+मार्गेण) – through where the line exist ; { विदारय } – parting; द्वयमपि(google) both of them ; विशेषेण – property ; परित्यागं विना भवति - happen without changing anything
Then cut along the lines starting from this point towards the south-west corner of the koti- square and the north-west corner of the bhuja-square. Allow a little clinging at the two corners so that the cut portions do not fall away.
फिर इस बिंदु से शुरू होकर कोटि-स्क्वायर के दक्षिण-पश्चिम कोने और भुज-स्क्वायर के उत्तर-पश्चिम कोने की ओर जाने वाली रेखाओं के साथ काटें। दोनों कोनों को थोड़ा सा चिपकने दें ताकि कटे हुए हिस्से दूर न गिरें |

अनन्तरं बिन्दोः लघु [विदारण] द्वयं च विशेषबिन्दौ संयुक्तरेखाग्राणि द्वयं च मिथः कोट्याः उत्तरपश्चिमसन्धेः मानं कुरु दृष्टमहच्चतुरश्रस्य भागद्वयं च रज्वीकृत्य सङ्गमय ।
लघु – small ; द्वयं च - and both ; विशेषबिन्दौ - special point ; संयुक्तरेखाग्राणि(संयुक्त+रेखा+अग्रणि) – combined lines tips ; मिथः – among themselves ; उत्तरपश्चिमसन्धेः(उत्तर-पश्चिम+सन्धेः) – convergence at north-west ; मानं – value ; कुरु -do it ; दृष्टमहच्चतुरश्रस्य(दृष्ट+महच्चतुरश्र) – the great square seen ; भागद्वयं च – and two parts ; रज्वीकृत्य(sfdff+कृत्य) – done by rope ; सङ्गमय - combined
Now break off the two parts (i.e., the triangles) from the marked point, turn them round alongside the two sides of the bigger (i.e., Koti) square, so that the corners of the triangles, which met at that point earlier, now meet in the north-west direction, and join them so that the cut portions form the outer edges.
अब चिह्नित बिंदु से दो हिस्सों (यानी, त्रिकोण) को तोड़ें, उन्हें बड़े (यानी, कोटि) वर्ग के दोनों किनारों के साथ गोल करें, ताकि त्रिकोण के कोने, जो पहले उस बिंदु पर मिलते थे, अब उत्तर-पश्चिम दिशा में मिलें और उन्हें इस प्रकार जोड़ दें कि कटे हुए हिस्से बाहरी किनारों का निर्माण करें।

अन्तर्बहिश्चास्ये विन्यासशिष्टयोजितव्यं तु । तत् एकं समचतुरर्थ भवति ।
अन्तर्बहिश्चास्ये = अन्तः बहि श्च)-inside and outside ; विन्यासशिष्टयोजितव्यं (विन्यास+शिष्टयोजितव्यं)– configuration discipline should be added तत् – that; भवति - happen
The figure formed thereby will be a square.
इससे बनी आकृति एक वर्ग होगी।
एतस्य बाहवः भुजाकोट्याः कर्णेन समं भवति ।
एतस्य – of this ; भुजाकोट्याः bhuja and koti-square ; कर्णेन समं - equal to Karna ; भवति – happen
And the sides of this square will be equal to the Karna associated with the (original) bhuja and Koti.
और इस वर्ग की भुजाएँ (मूल) भुज और कोटि से जुड़े कर्ण के बराबर होंगी।

किन्तु भुजाकोट्योः वर्गयोगं कर्णवर्गम्, कर्णवर्गात् भुजाकोटिमध्ये एकस्य वर्गत्यागे भुजाकोटिषु अन्यस्य वर्गमिति संस्थितमासीदिदानीम् । इदं सर्वत्र ज्ञातव्यं भवति ।
किन्तु – but ; वर्गयोगं - square sum ;कर्णवर्गम् - square of hypotenuse,; कर्णवर्गात् - in the diagonal class ; भुजाकोटिमध्ये - in bhuja koti-square ;एकस्य वर्गत्यागे - discarding square ; भुजाकोटिषु अन्यस्य

-other bhuja-koti square ; वर्गमिति – square measure; संस्थितमासीदिदानीम् (संस्थित+ आसीद+ इदानीम्) – it was settled now ; इदं – this is ; सर्वत्र - everywhere ; ज्ञातव्यं – it should be known; भवति - happen
Hence it is established that the sum of the squares of the bhuja and koṭi is equal to the square of the karna and it also follows that if the square of one of them is deducted from the square of the karna, the square of the other will be the result. This is to be understood in all cases.

इसलिए यह स्थापित है कि भुज और कोसी के वर्गों का योग कर्ण के वर्ग के बराबर है और यह भी इस प्रकार है कि यदि उनमें से एक का वर्ग कर्ण के वर्ग से घटा दिया जाए, तो दूसरे का वर्ग होगा परिणाम हो. इसे सभी मामलों में समझा जाना चाहिए।

EXPLANATION:-
This is to be understood in all cases.
In any rectangle, the longer side (taken as the lateral side) is the koti and the shorter side (taken as the vertical side) is the bhuja. Then the square of the karna, the diagonal, is equal to the sum of the squares of the bhuja andthe koti.
In order to prove this bhuja-koti-karna-nyaya, consider Figure. Here ABCD and BPQR, are squares with sides equal to the bhuja and koti respectively. The square BPQR is placed on the south such that the easternsides of both the squares fall on the same line and the south side of the bhuja-square lies along the north side of the koti-square. As stated above, itis assumed that the bhuja is smaller than the koti.
Mark M on AP such that

AM = BP = Koti.

Hence,

MP = AB = Bhuja, and MD = MQ = Karna

Cut along MD and MQ, such that the triangles AMD and PMQ just cling at D, Q respectively. Turn them around to coincide with DCT and QRT. Thus, is formed the square DTQM, with its side equal to the karna. It is thus seen that. Karna-square DTQM = Bhuja-square ABCD+ Koti-square BPQR.

Chapter 9 - The area of the circle (वृत्तक्षेत्रे)

वृत्तक्षेत्रे or the area of the circle is also from Yuktibhāṣā of Jyesthadeva .

वृत्तक्षेत्रे वर्गफलस्य उत्पादनप्रकारं अनन्तरम्।
वृत्तक्षेत्रे – circular area वर्गफलस्य – of the square root ; उत्पादनप्रकारं – production type ; अनन्तरम् – after or subsequently
Now is stated the method to derive the square of the area of a circle.
अब वृत्त के क्षेत्रफल का वर्ग निकालने की विधि बताई गई है।
वृत्तक्षेत्रं व्यासार्धमार्गेण योजय समं सत्।
व्यासार्धमार्गेण योजय(व्यास+अर्ध+मार्गेण+ योजय) – add through half the diameter ; समं – equal ; सत् - shown

Cut the circle equally into two across a diameter.
वृत्त को एक व्यास में समान रूप से दो भागों में काटें।

पुनः संयोगद्वयेऽपि केन्द्रादारभ्य नेमिपर्यन्तं विदारयेत्।
पुनः – again ; संयोगद्वयेऽपि – even in both combinations ; केन्द्रादारभ्य -beginning from centre ; नेमिपर्यन्तं – up to rim ; विदारयेत् – in a split
In both halves cut (equal) sections from the centre to the circumference.
दोनों हिस्सों में केंद्र से परिधि तक (बराबर) खंड काटें।
नेमिभवप्रदेशः सर्वस्मादपि विस्तृतं सत्, केन्द्रभवप्रदेशः तनुमान् एवं भवेत्।
नेमिभवप्रदेशः(नेमि+भव+प्रदेशः) – area originated by circumference ; सर्वस्मादपि(सर्वस्मात+अपि) – from everything else; विस्तृतं – detailed ; सत् – truth ;केन्द्रभवप्रदेशः(केन्द्र+भव+प्रदेशः) – area originated by centre of circle; तनुमान्(तत्+अनुमान्) – base estimate; एवं – and; भवेत् – it may be
The divisions would be spread out at the circumference and pointed at the centre.
विभाजन परिधि पर फैले होंगे और केंद्र की ओर इंगित किए जाएंगे।
अनन्तरं वृत्तखण्डद्वयमपि नेमेः शिरोद्वयश्च गृहीत्वा प्रसार्य मिथो योजय। केन्द्रे तनुप्रदेशः इतराभ्यन्तरे व्यापतं सत् भवेत्।
वृत्तखण्डद्वयमपि(वृत्त+खण्ड+द्वय+अपि) – also the two circular section ; नेमेः - शिरोद्वयश्च(शिरो+द्वय+च) – and the two heads ; गृहीत्वा – taking ; प्रसार्य – spreadable ; मिथो योजय – add with ; केन्द्रे तनुप्रदेशः – thin area of centre ; इतराभ्यन्तरे(इतय+अभ्यन्तरे) – it's inside ; व्यापतं – covers ; सत् भवेत् – it will be true
Then, taking hold of the ends of the two halves, straighten them up and join one into the other, so that the pointed parts of one go into the cavities of the other.
फिर दोनों हिस्सों के सिरों को पकड़कर सीधा करें और एक को दूसरे से जोड़ दें, ताकि एक के नुकीले हिस्से दूसरे की गुहाओं में चले जाएं।

तदा वृत्तार्धं दीर्घविशिष्टं व्यासार्धं विस्तारं सदेकं आयतचतुरश्रक्षेत्रं भवेत्।
तदा – then ; वृत्तार्धं(वृत्त+अर्धं) – half- circle ; दीर्घविशिष्टं(दीर्घ+विशिष्टं) -long specific ; व्यासार्धं(व्यास+अर्धं) – halved-diameter ; विस्तारं – expansion ; सदेकं(स+एकं) – this one; आयतचतुरश्रक्षेत्रं(आयत+चतुरश्र+क्षेत्रं) – rectangular square area
This arrangement will result in the shape of a rectangle having half the circumference of the circle as length and the radius as breadth.
इस व्यवस्था के परिणामस्वरूप एक आयत का आकार मिलेगा जिसमें वृत्त की आधी परिधि लंबाई के रूप में और त्रिज्या चौड़ाई के रूप में होगी।

एवश्चेत् वृत्तार्धं व्यासार्धश्च मिथो गुणितश्चेत् वृत्तक्षेत्रीयचतुरश्रफलं भवति ।
एवश्चेत् – if so, then ; वृत्तार्धं व्यासार्धश्च – half-circle and half-diameter ; मिथो – with ; गुणितश्चेत् – if multiplied ; वृत्तक्षेत्रीयचतुरश्रफलं भवति - it will be area of circle obtained by rectangle
Thus, by multiplying half the circumference by the radius is obtained the area of the circle.
इस प्रकार, आधी परिधि को त्रिज्या से गुणा करने पर वृत्त का क्षेत्रफल प्राप्त होता है।

Obtaining the area of a circle
► In the figure we have indicated a circular slice being turned into a rectangle by appropriately sectioning it and inserting one half of the circular slice into the other.
► The length of this rectangular strip corresponds to half the circumference C. If the radius is r', then the area of this slice is
Area = 1/2 C × r^'

Chapter 10 - Sum of A.P. series(**श्रेढीव्यवहार)**

Series of A.P or **श्रेढीव्यवहारे** is from the book ‘Patiganita of Sridharacarya ’. Sridhara was an Indian mathematician who wrote on practical applications of algebra and was one of the first to give a formula for solving quadratic equations. Sridhara is known as the author of two mathematical treatises, namely the Trisatika (sometimes called the Patiganitasara) and the Patiganita. However at least three other works have been attributed to him, namely the Bijaganita, Navasati, and Brhatpati.

तत्रादो श्रेढीव्यवहारे श्रेढीस्वरूपं तावदाह-
विस्तारोऽल्पोऽधस्तादुपरि महान् स्याद्यथा' शरावस्य । श्रेढीक्षेत्रस्य तथा

मृत्पात्रस्याधस्ताद् भूम्युपवेशभागेऽल्पो विस्तारस्ततः क्रमेणोपर्युपरि महान् विस्तारो भवत्येव तद्वद्यस्य भूप्रदेशस्य स श्रेढीसंज्ञः, द्विसमचतुरश्रभेदप्रायोऽस्य क्षेत्र विशेषस्य सन्निवेशोऽभि- हितो भवति । तथा हि तस्य भुजौ नियमतः समावेव भवतो भूमुखे तु मिथो विषमे एव । श्रत एव श्रेढीगणितादभिन्नमेवास्य गणितं, 'श्रेढीक्षेत्रे तु फलं भूमुखयोगार्धलम्बहति" रिति क्षेत्रगणितं च 'भूवदनसमासार्धं मध्यमलम्बेन संगुणित' मिति, न चानयोः सूत्रयोरस्ति कश्चिदर्थे फले वा भेदः । किमर्थं हि श्रेढीक्षेत्रगणितमारभ्यते' ? भूमुखयोरिह नियसेन व्यवस्थापयिष्यमाणत्वाज्जात्यक्षेत्रादिभ्यः पृथक्करणाद्राम्येत्कश्चिदिह फलानयनार्थमिति शिष्यहितायाचार्यः सूत्रमारभते । अथ किमर्थं भूमुखयोरिह नियमेन व्यवस्थापनम् ? इष्टगच्छाद्येनेष्टलम्बे प्रतिहस्तमायुत्तरनियमेनापद्यमानधनरक्षणार्थं, क्षेत्रफलं तु भूवदनसमा- सार्धेऽपि स्यात्, प्रतिहस्तं न स्यात् ।
ततश्चात्र गच्छसमलम्बक इह लक्ष्यते । आदिप्रचयात्मके क्षेत्रेऽस्मिन् लम्बकस्तावद् गच्छसम एव नान्यथा कल्पनीयो भूमुखभुजवदित्याह--

गच्छसमो लम्बकस्तस्य ॥ ७९ ॥

विस्तारोऽल्पोऽधस्तादुपरि(विस्तार+ऽल्पो +अधस्तात+तदुपरि) – expansion slightly below and above; महान् स्याद्यथा – it would be like; मृत्पात्रस्याधस्ताद् – from under the soil vessel; भूम्युपवेशभागेऽल्पो - small part from the bottom; विस्तारस्ततः – then the expansion; क्रमेणोपर्युपरि – gradually over and above; महान् विस्तारो – great expansion; भवत्येव – it does happen; तद्वद्यस्य – of the same; नियमतः – coordinate; समावेव – include; भवतो भूमुखे तु मिथो विषमे – in the odd; भूमुखयोरिह नियसेन व्यवस्थापयिष्यमाणत्वाज्जात्यक्षेत्रादिभ्यः – from the field and others born of being to be arranged; पृथक्करणाद्राम्येत्कश्चिदिह – some part would be attracted to separation; फलानयनार्थमिति – for the purpose to obtaining result; नियमेन – as a rule; इष्टगच्छाद्येनेष्टलम्बे प्रतिहस्तमायुत्तरनियमेनापद्यमानधनरक्षणार्थ – to proceed the result that arises by the rule of multiplication; क्षेत्रफलं तु भूवदनसमासार्धेऽपि – even for the compound of the bottom; स्यात् – should; ततश्चात्र – then here; गच्छसमलम्बक - by equal length; इह लक्ष्यते – it is noticed here; आदिप्रचयात्मके क्षेत्रेऽस्मिन् लम्बकस्तावद् गच्छसम – like; एव नान्यथा कल्पनीयो – imaginations; गच्छसमो लम्बकस्तस्य

भूमध्याद्वदन मध्यस्पर्षांशसूत्रमिह लम्बकः, स च तस्य श्रेढीक्षेत्रस्य गच्छसमः पदतुल्यः । यथा--द्विके आदौ त्रिके प्रचये पञ्चके पदे श्रेढीक्षेत्रस्य कि फलं भवति, किविधभूवदनभुजलम्बकं च तत्क्षेत्रं स्यादिति पृष्टे लम्बकस्तावदकल्पितसिद्धः पञ्चकप्रमाण एवेति" ज्ञेयम् ।
इदानीमन्यक्षेत्रेभ्योऽस्य पृथक्करणहेतुविशेषमाह—
लम्बककरे पृथक् पृथगिष्टादिचयेन तत्फलं भवति ।

भूमध्याद्वदन - intermediate; मध्यस्पर्षांशसूत्रमिह लम्बकः- the middle tangent is perpendicular here; स च तस्य – and that’s it; गच्छसमः- going equal; पदतुल्यः- equivalent to; यथा--द्विके आदौ – as in the beginning of the two; ज्ञेयम् - knowable;इदानीमन्यक्षेत्रेभ्योऽस्य – now from the other fields of; पृथक्करण – separation; हेतु – for the purpose; विशेषमाह – specific;लम्बककरे – make longer; पृथक् पृथगिष्टादिचयेन – by a collection of separate values; तत्फलं भवति – then there is the result

लम्बकस्य पञ्चकादिप्रमाणस्य हस्तो लम्बो यो भवति तत्र पृथक् पृथक् तस्य श्रेढीक्षेत्रस्य फलं भवति । ननु च क्षेत्रान्तराणामपि लम्बे हस्तशो विभज्यमाने प्रतिहस्तं क्षेत्रफलं विद्यते । सत्यम्, इह त्विष्टादिचयेनेत्येष विशेषः, प्रथमे लम्बहस्ते श्रादिसम्मितमेव क्षेत्रफलं, द्वितीये सत्रचयादिसम्मितं, तृतीये द्विगुणप्रचययुतादिसम्मितं, चतुर्थादौ त्रिगुणादिप्रचययुतादि- सम्मितमिति । एतच्चाये दर्शयिष्यते ।
इदानीं क्षेत्रस्वरूपरचनामाह –
तभूमुखमितिसिद्ध्यै करणमहं सम्प्रवक्ष्यामि ॥ ८० ॥

ननु च क्षेत्रान्तराणामपि लम्बे – even in the difference in length of the field; हस्तशो विभज्यमाने – when divided by hand; प्रतिहस्तं क्षेत्रफलं विद्यते – there is an area pre hand; इह त्विष्टादिचयेनेत्येष – aim with clarity; विशेषः – special; सम्मितमेव क्षेत्रफलं – the measured field result; सत्रचयादिसम्मितं – in the appearance of; एतच्चाये – and this is the next; दर्शयिष्यते – will be shown; तभूमुखमितिसिद्ध्यै – to prove that it is face of earth(starting point)

तस्य श्रेढीक्षेत्रस्य भूमुखयोः परिमाणं साधयितुं कर्म सम्प्रवक्ष्यामि, तत्सिद्धी भुजयोरत्र सिद्धेः लम्बस्य च साध्यत्वात् । तदाह –
पदमेकं तल्लम्बश्चयदलहीनं मुखं धरा भवति ।
सचया सा स्याद्वक्त्रं कुर्यात्सूत्रेण तच्चिह्नम् ॥ ८१ ॥

साधयितुं कर्म – action to accomplish; सम्प्रवक्ष्यामि – I will tell you; पदमेकं – one step; सचया सा – like truth; स्याद्वक्त्रं – it would be a face; तच्चिह्नम् – that's the sign

यद्यपि पञ्चकादिकं पदं प्राश्निकोक्तं भवति तथापि तत्पदं रूपमेव ग्रहीतव्यं, तस्मिंश्च गृहीते 'गच्छसमो लम्बकस्तस्ये 'ति न्यायेन रूपमेव लम्बोऽपि ग्रहीतव्यस्तावत्क्षेत्रस्वरूपकरणार्थः तत्पा रतन्त्र्येणैवोत्तरत्रपदेषु स्वरूपव्यवस्थानात् । तत्र प्रथमपदक्षेत्रे" चयस्य दलेन" हीनं मुखं धरा भवति । तद्यथा-चयस्य त्रिकस्य दलेनाध्यर्धेन मुखं द्विकं हीनमर्धं भूर्भवति । सा एव भूश्चयेन सहिता प्रथमपदक्षेत्रस्य वक्त्रं स्यात्, यथा संवाधिकी भूश्चयेन त्रिकेन सहिताऽर्धचतुर्थप्रमाणं वक्त्रम् ।

'कुर्यात् सूत्रेण तच्चिह्नम्। यावांल्लम्बो यावती भूर्यावच्च वक्त्रं " तावत्सूत्रेण तच्चि तं कुर्यात् । श्रभिम (त) दिवसाम्येन प्रागपरायतं हास्तिकं लम्बसूत्रं निपात्य तत्प्रान्तद्वये चिह्न कुर्यात् । ततः पश्चिमचिह्न मध्ये कृत्वा दक्षिणोत्त (र) माधिकं भूसूत्रं प्रसार्य तत्प्रान्तद्वये चिह्न कुर्यात् । ततश्च प्राचिह्नं मध्ये कृत्वोदग्दक्षिणायतमर्धचतुर्थप्रमाणं मुखसूत्रं प्रसार्य तत्प्रान्त- द्वयमपि चिल्लयेत् । प्रतिसूत्रं रेखा एव वा कुर्यादिति प्रयोगः ।
भूमुखरेखाग्रस्पृक्" प्रसारयेत्" सूत्रमुभयतो बाहू ।
सूत्रप्रसूतिर्वज्रवद्द्रणगतभूमौ भवेदित्यम् ।। ८२ ॥

यद्यपि – though; पञ्चकादिकं पदं – five and so on terms; प्राश्निकोक्तं – dependent; तथापि – however; तत्पदं – that term; ग्रहीतव्यं – to be taken; तस्मिंश्च – and in that; गच्छसमो – it was going to be; लम्बकस्तस्ये न्यायेन रूपमेव लम्बोऽपि ग्रहीतव्यस्तावत्क्षेत्रस्वरूपकरणार्थः तत्पा रतन्त्र्येणैवोत्तरत्रपदेषु स्वरूपव्यवस्थानात् । तत्र – there; प्रथमपद – first term; चयस्य – of the choice; दलेन" हीनं - devoid;मुखं धरा भवति तद्यथा – for example; सा एव – that's it; भूश्चयेन – and the base by which; सहिता – including; कुर्यात् – do it; तच्चिह्नम् – that's the sign; यावती – as long as; भूर्यावच्च – and to the base; तावत्सूत्रेण – by the same formula; तच्चि – but; तं – that's it; कुर्यात् – should do it; साम्येन – equally; निपात्य – dropping; कुर्यात् – do it; ततः – then; तत्प्रान्तद्वये चिह्न – then mark the two regions; कुर्यात् in the matter; ततश्च – and then; प्राचिह्नं – symbol; प्रसार्य – spread; रेखा एव वा – or the line itself; कुर्यादिति – to do so; भूमुखरेखाग्रस्पृक् प्रसारयेत् – in circulation; सूत्रमुभयतो – sutra on both side; सूत्रप्रसूतिर्वज्रवद्द्रणगतभूमौ– it spread like a thunderbolt on the ground; भवेदित्यम् – it should be

भूरेखाग्राद् दक्षिणान्मुखरेखाग्रं' दक्षिणं यावत् सूत्रं प्रसारयेदेष दक्षिणो भुजः, ततो वामाद् भूरेखाग्राद् वाममेव मुखरेखाग्रं यावत् सूत्रं प्रसारयेदेष वामो भुज' इति लम्बभूवदनभुजनिय - न्त्रितप्रथमपदश्रेढीक्षेत्रं सन्निवेशितसंसर्गतो भवति । अपवादस्तु सूत्रप्रसूतिवं नवदृणगत भूमौ " भवेत् इत्यम्' । 'चयदलहीनं मुखं घरा भवती' ति कर्मणा साधिताया भुव ऋणात्मकत्वे ज। ते सति स एव हास्तिको लम्बकः स एवं धरासूत्रपातः स एव च वक्त्रसूत्रविन्यासः भुजसूत्रमेव त्वन्यथा भवति, भूरेखाग्राद्दक्षिणाद् वामं मुखरेखाग्रं यावन्नीतसूत्रमुपरि श्यश्रे वामो भुजो जायते अधस्त्र्यश्रे दक्षिणो भुजो जायते, बामभूसूत्राग्राद् दक्षिणवक्त्रसूत्रप्रान्तं (यावन्) नीतं सूत्रमुपरि त्र्यश्रे दक्षिणो भुजः अधस्त्र्यश्रे वामो भुजो जायते। एवं सति त्र्यश्रद्वयात्मकं श्रेढीक्षेत्रं भवति ।

इदानीमस्मिनृणगतभूमिवशाद् वज्रवद्द्भुजसूत्रपातजनितत्र्यश्रद्वयात्मके श्रेढीविशेषे स हास्तिको लम्बस्त्र्यश्रयोविभज्यते—
उपरि श्यने लम्बो भूमितिरहितेन भाजितं वदनम् ।

'सचया सा स्याद् वक्त्र' मिति कर्मणा साधितं वदनं स्वात्मनैव भूपरिमाणरहितेन भाजितं सदुपरि स्थिते व्यश्रे लम्बो भवति वक्त्ररेखामध्यात् भुजसूत्रसम्पातस्थानं यावत्, ततः प्रभृति भूरेखामध्यं यावत्तदधस्थ्यश्रे" लम्बो भवति, तस्य करणम् –
रूपात्तस्यापगमेऽधस्त्रयों जायते लम्बः ।। ८३ ।।

वामाद् – from the left; भूरेखाग्राद् – from the tip of the ground (starting) line; वाममेव मुखरेखाग्रं – tip of the starting line; यावत् – until; प्रसारयेदेष – extended; अपवादस्तु – exception; सूत्रप्रसूतिवं – in process; नवदृणगत – newly obtained; कर्मणा - by action; साधिताया – to achieve; स एव च – and that's it; त्वन्यथा – otherwise; यावन्नीतसूत्रमुपरि – as far as formula given; जायते – it is obtained; अधस्त्र्यश्रे – inside; भाजितं – divided; सा स्याद् – it would be; स्वात्मनैव – by himself; परिमाणरहितेन – without dimensions सदुपरि – on top of that; स्थिते – positioned; सम्पातस्थानं – the place of conjunction; ततः – then; प्रभृति – etc.. etc..; यावत्तदधस्थ्यश्रे - and as it is.

तस्योपरित्र्यश्रसम्बन्धिनो लम्बस्य रूपाच्छुद्धी शिष्टमधस्तनत्र्यश्रलम्बकप्रमाणं भवति ।

इत्थ प्रथमपदे हास्तिकलम्बे धनर्णविभागेन भुजभेदाद द्विविधं श्रेढीक्षेत्रं कृत्वा परपदावधिकं क्षेत्रं कथं कार्यमित्याह—
इत्थं श्रेढीक्षेत्रं कृत्वेष्टलम्बके मुखं कल्प्यम् ।
इष्टावलम्बगुणितं" घरोनमुखमवनियुग्वदनम् ।। ८४ ।।

तस्योपरित्र्यश्रसम्बन्धिनो – related to its surrounding; रूपाच्छुद्धी शिष्टमधस्तनत्र्यश्रलम्बकप्रमाणं – the remaining clearance form from measurement of the lower three values; इत्थ प्रथमपदे - it is in the first step; द्विविधं – there are two types; श्रेढीक्षेत्रं कृत्वा – having done so; परपदावधिकं क्षेत्रं कथं – how about the field of the next step

अनेन प्रकारेण हास्तिकं लम्बश्रेढी क्षेत्रं विस्चय्याभिमते पञ्चकादिलम्बके समस्तश्रेढी- क्षेत्रसम्बन्धिवदनं साध्यम् । सचया घरा हास्तिकश्रेढीक्षेत्र सम्बन्धिवदनं, (तत्) तत्सम्बन्धिन्यैव धरया ऊनमीप्सितनम्बगुणितं तत्प्रागानीत भूमिप्रमाणेन" युतमिष्टलम्बकश्रेढीवदनं भवति । तदत्र भूमिर्येव प्रथमपदश्रेढधाः " सेव । पदान्तरेषु" लम्बान्यथात्वे वदनप्रमाणभेदः प्रवर्तत एवेति क्षेत्रं प्रकल्प्यम् ।

इदानीमायुत्तरिकायाः क्षेत्रगतेर्भेदेन गणितमाह-
व्येकपदार्धघ्नचयः सादिः' पदसङ्गुणो भवेद् गणितम् ।
श्रेढीक्षेत्रे तु फलं भूमुखयोगार्बलम्बहतिः ।। ८५ ।।

लम्बश्रेढी क्षेत्रं - long range area; विस्चय्याभिमते – disturbed; पञ्चकादिलम्बके – five and other length;; समस्तश्रेढी- entire series; साध्यम् – achievable;, (तत्) तत्सम्बन्धिन्यैव – it is related to that; धरया – hold; ऊनमीप्सितनम्बगुणितं – multiplied by the length; तत्प्रागानीत – that's right; भूमिप्रमाणेन – the base by proof; युतमिष्टलम्बकश्रेढीवदनं भवति – it becomes a combined long series; तदत्र – where it is; भूमिर्येव – the Bhumi(starting point) itself; प्रथमपदश्रेढधाः – first term series; पदान्तरेषु – in other terms; लम्बान्यथात्वे वदनप्रमाणभेदः – difference in face size(values of faces); प्रवर्तत – activate; प्रकल्प्यम् – it is conceivable; सादिः – achieved; पदसङ्गुणो – the term is multiplied

विरूपस्य पदस्यार्धेन हतश्चय श्रादिना सहितः पदेन सङ्गुणितः सङ्कलितं भवति । तथा भूमेर्मुखस्य च यो योगस्तदर्धस्य लम्बस्य च घातः श्रेढीक्षेत्रफलं भवति । किमाद्युत्तर- पदप्रकृतीदमिति तन्न ज्ञायते, केवलं तु श्रेढीक्षेत्रमित्येव ज्ञायते, ज्ञायमानं भूवदनलम्बं, तदा न 'व्येकपदे'त्यादिना सिद्धिरस्तीति लक्षणान्तराररम्भः ।

विरूपस्य – of the disfigured; पदस्यार्धेन - by half of the term; हतश्चय - determined; सहितः – including; पदेन सङ्गुणितः – multiplied by the term; सङ्कलितं – compiled;; तथा भूमेर्मुखस्य and the part facing the ground; च यो योगस्तदर्धस्य - and the sum of that half; लम्बस्य च and of the length; पदप्रकृतीदमिति – this is the nature of the term; तन्न ज्ञायते – knowing that; केवलं तु – only it is

79. As in the case of an earthen drinking glass (sarâva) the width at the base is smaller and at the top greater, so also is the case with a series-figure (średhi-kşetra).
The altitude (lambaka) of that (series-figure) is equal to the number of terms (gaccha) of the (corresponding) series.

The series-figure contemplated here is a plane figure resembling a trapezium with equal flank sides
If a series be
a+(a+d)+(a+2d) +... to n terms,
then, according to the second part of the verse, the altitude of the corresponding series-figure =n units, say n cubits.

80(i). The (partial) areas (phala) of the series-figure for the successive cubits (kara) of the altitude form a series which begins with the given âdi ('first term of the series') and increases successively by the given caya ('common difference of the series').

That is, the area of the series-figure for the first cubit of the altitude =a, i.e., the first term of the series; the area of the series-figure for the second cubit of the altitude = a+d, i.e., the second term of the series; the area of the series-figure for the third cubit of the altitude=a+2d; and so on.

Construction of a series-figure:

80(ii). I shall now describe the method for finding the lengths of the base (i.e., lower side, bhû) and the face (i.e., upper side, Mukha) of the series-figure (corresponding to the first term of the series).

81. The number of terms (pada), i.e., one, is the altitude of the (corresponding) series-figure; the first term of the series (Mukha) as diminished by half the common difference of the series is the base (Dhara); and that (base) increased by the common difference of the series is the face (vaktra). All these should be shown by means of threads.
That is,
base a- d/2 ,
and, face = (a- d/2)+d, i.e., a + d/2

82. (Two) threads should then be stretched out, one on either side, joining the extremities of those base and face: these are the flank sides (bâhu) of the series-figure.

When the base is negative, these threads should be stretched out crosswise.

Thus the series-figure will be of one of the following two forms:

Forms (ii) corresponds to the negative base.

83. (When the base is negative the series-figure reduces to two triangles situated one over the other.) In the upper triangle, the altitude is equal to the face as divided by face minus base; and that subtracted from one gives the altitude in the lower triangle.

That is,

(i) altitude of the upper triangle = face/(face-base), i.e., $(2a+d)/2d$,

(ii) altitude of the lower triangle =1- face/(face-base) ,i.e., $(d-2a)/2d$

Rule for finding the face of the series-figure corresponding to the given series:

84. Having constructed the series-figure (for altitude unity) in this manner, one should determine the face for the desired altitude (i.e., for the desired number of terms of the series) (by the following rule): The face (for altitude unity) minus the base (for altitude unity), multiplied by the desired altitude, and then increased by the base (for altitude unity), gives the face (for the desired altitude).

This rule, on simplification, reduces to the following formula:

face for altitude $n = a+(n-1) d$.

Rule for finding (i) the sum of a series in A. P. (interpreted geometrically by a series-figure), and (ii) the area of the corresponding series-figure:

85. The common difference as multiplied by one-half of the number of terms minus one, being increased by the first term, and then multiplied by the number of terms, gives the sum of the series. And the area of the (corresponding) series-figure is equal to the product of one-half of the sum of the base and the face, and the altitude.

That is, the sum of the series

$a+(a+d)+(a+2d)+\ldots$ to n terms

is equal to $\{(n-1)/2\ d+a\}\ n$; (1)

and the area of the corresponding series-figure is equal to

(base+face)/2 x altitude, (2)

where, according to vv. 80(ii) to 84,

base = $a-d/2$,

face = $a+(n-1/2)d$,

altitude = п.

and

It may be easily seen that (1) and (2) are the same.

79. जैसे मिट्टी के पीने के गिलास (सारवा) के मामले में आधार पर चौड़ाई छोटी होती है और शीर्ष पर अधिक होती है, वैसे ही श्रृंखला-आकृति (श्रेधि-क्षेत्र) के मामले में भी ऐसा ही होता है।

उस (श्रृंखला-आकृति) की ऊंचाई (लम्बका) (संबंधित) श्रृंखला के पदों (गच्चा) की संख्या के बराबर है।

यहां पर विचार की गई श्रृंखला-आकृति एक समतल आकृति है जो समान पार्श्व भुजाओं वाले समलंब के समान है
यदि कोई शृंखला हो
a+(a+d)+(a+2d) +... n पद,
फिर, श्लोक के दूसरे भाग के अनुसार, संबंधित श्रृंखला-आकृति की ऊंचाई = n इकाई, मान लीजिए n हाथ।

80(i). ऊंचाई के क्रमिक क्यूबिट्स (कारा) के लिए श्रृंखला-आकृति के (आंशिक) क्षेत्र (फला) एक श्रृंखला बनाते हैं जो दिए गए आदि ('श्रृंखला का पहला पद') से शुरू होता है और दिए गए काया (' श्रृंखला का सामान्य अंतर') द्वारा क्रमिक रूप से बढ़ता है।

अर्थात्, ऊँचाई के पहले हाथ के लिए श्रृंखला-आकृति का क्षेत्रफल =a, अर्थात, श्रृंखला का पहला पद; ऊँचाई के दूसरे हाथ के लिए श्रृंखला-आकृति का क्षेत्रफल = a+d, अर्थात, श्रृंखला का दूसरा पद; ऊँचाई के तीसरे हाथ के लिए श्रृंखला-आकृति का क्षेत्रफल=a+2d; और इसी तरह।

एक श्रृंखला-आकृति का निर्माण:
80(ii). अब मैं श्रृंखला-आकृति (श्रृंखला के पहले पद के अनुरूप) के आधार (यानी, निचला पक्ष, भु) और चेहरे (यानी, ऊपरी पक्ष, मुख) की लंबाई खोजने की विधि का वर्णन करूंगा।

81. पदों की संख्या (पद), यानी, एक, (संबंधित) श्रृंखला-आकृति की ऊंचाई है; श्रृंखला का पहला पद (मुख) श्रृंखला के सामान्य अंतर के आधे से कम होने पर आधार (धारा) है; और वह (आधार) श्रृंखला के सामान्य अंतर से बढ़ा हुआ फलक (वक्त्र) है। इन सभी को धागों के माध्यम से दर्शाया जाना चाहिए।
वह है,
आधार a- d/2 ,
और, फलक = (a- d/2)+d, अर्थात, a + d/2
2. फिर (दो) धागों को फैलाया जाना चाहिए, दोनों तरफ एक-एक, उन आधार और चेहरे के छोरों को जोड़ते हुए: ये श्रृंखला-आकृति के पार्श्व पक्ष (बाहु) हैं।
जब आधार ऋणात्मक हो तो इन धागों को आड़ा-तिरछा फैलाना चाहिए।
इस प्रकार श्रृंखला-आकृति निम्नलिखित दो रूपों में से एक होगी:
फॉर्म (ii) नकारात्मक आधार से मेल खाता है।

83. (जब आधार ऋणात्मक होता है तो श्रृंखला-आकृति एक के ऊपर एक स्थित दो त्रिभुजों में बदल जाती है।) ऊपरी त्रिभुज में, ऊंचाई चेहरे के बराबर होती है जिसे चेहरे से आधार घटाकर विभाजित किया जाता है;

Chapter 11 - Quadratic formula Proof (मध्यमाहरणम्)

Bhāskara II (c. 1114–1185), also known as Bhāskarāchārya and as Bhāskara II to avoid confusion with Bhāskara I, was an Indian mathematician, astronomer and inventor. From verses, in his main work, Siddhānta Shiromani (सिद्धांतशिरोमणी). He has been called the greatest mathematician of medieval India. His main work Siddhānta-Śiromaṇi, is divided into four parts called Līlāvatī, Bījagaṇita, Grahagaṇita and Golādhyāya, which are also sometimes considered four independent works. These four sections deal with arithmetic, algebra, mathematics of the planets, and spheres respectively. He also wrote another treatise named Karaṇā Kautūhala.
Sridhara was one of the first mathematicians to give a rule to solve a quadratic equation. Unfortunately, as we indicated above, the original is lost and we have to rely on a quotation of Sridhara's rule from Bhaskara II:-

अव्यक्तवर्गादि यदावशेषं पक्षौ तदेष्टेन निहत्य किंचित् ।
क्षेप्यं तयोर्येन पद्मदः स्याद्व्यक्तपक्षस्य पदेन भूयः ॥
व्यक्तस्य पक्षस्य समक्रियैवमव्यक्तमानं खलु लभ्यते तत् ।
न निर्वहश्चेद्यनवर्गवर्गेष्वेवं तदा शेयमिदं स्वबुद्ध्या ।।
अव्यक्त मूलर्णगरूपतोऽल्पं व्यक्तस्य पक्षस्य पदं यदि स्यात् ।
ऋणं धनं तच्च विधाय साध्यमव्यक्तमानं द्विविधं क्वचित्तत् ।। ११५ ।।
यदावशेषं – whatever remains; पक्षौ – side; तदेष्टेन – by that desired; निहत्य – extremely; किंचित् - slightly;क्षेप्यं तयोर्येन – by the two; व्यक्तस्य – of the known;पक्षस्य – of the side समक्रियैवमव्यक्तमानं – the same action is taking place; खलु – indeed; लभ्यते – available; तत् न – that's not it; तदा – then; शेयमिदं –this is to be known; स्वबुद्ध्या – by his own intelligence; अव्यक्त मूलर्णगरूपतोऽल्पं – smallest character in the form of root; व्यक्तस्य पक्षस्य पदं यदि स्यात् – should; ऋणं धनं – addition and subtraction; तच्च – that's it; विधाय साध्यमव्यक्तमानं – the achievable is expressed; द्विविधं – two types; क्वचित्तत् – somewhere that

Taking the quadratic and first-degree term on one side we are to multiply both sides by some number and add some number in order to complete the square. After that square roots are equated, and the value of the unknown is obtained.
If the third power and the fourth power of the unknown is present this device does not work. We have in that case, to adopt some artifice of our own.
In a quadratic equation if the number in the square root of the unknown side be negative and smaller than the number in the square root of the other side, then we should assume plus and minus for that and get two values for the unknown. In some questions both values are admissible.

'चतुराहतवर्गसमै रूपैः पक्षद्वयं गुणयेत्
पूर्वाव्यक्तस्य कृतेः समरूपाणि क्षिपेत्तयोरेव' इति ।। ११६ ।।
चतुराहतवर्गसमै – equally multiply by four; रूपैः – form; पक्षद्वयं – two sides; गुणयेत् - multiply; पूर्वाव्यक्तस्य – of the previously unknown; कृतेः – for; समरूपाणि – even form; इति – so far

Multiply both sides by four times the coefficient of the square of the unknown. Add to both sides the square of the coefficient of the unknown.
This is the device. [It is known as Shvidhara's method.]

EXPLATION: -
Multiply both sides by four times the coefficient of the square of the unknown. Add to both sides the square of the coefficient of the unknown.
$x^2+ bx+c =0$
$x^2+ bx = -c$
Multiplying both sides by 4 times the coefficient of the square of the unknown i.e 4a
$4 a^2 x^2+4abx = -4ac$
3
Then,
Adding both sides the square of the coefficient of the unknown
$4 a^2 x^2+4abx + b^2+ = b^2- 4ac$
$[(2ax+b)]^2 = b^2- 4ac$
$2ax+b = \pm\sqrt{b^2-4ac}$
$x = (- b \pm\sqrt{b^2-4ac})/2a$

Chapter 12 - Kuttaka Process to solve one linear equation of two variable.

Kuttaka Process is a statement of proof mainly from the book Bijaganita of Bhaskar II. Bhāskara II (c. 1114–1185), also known as Bhāskarāchārya and as Bhāskara II to avoid confusion with Bhāskara I, was an Indian mathematician, astronomer and inventor. From verses, in his main work, Siddhānta Shiromani (सिद्धांतशिरोमणी). He has been called the greatest mathematician of mediaeval India. His main work Siddhānta-Śiromaṇi, is divided into four parts called Līlāvatī, Bījagaṇita, Grahagaṇita and Golādhyāya, which are also sometimes considered four independent works. These four sections deal with arithmetic, algebra, mathematics of the planets, and spheres respectively. He also wrote another treatise named Karaṇā Kautūhala.

भाज्यो हारः क्षेपकश्चापवर्त्यः केनाप्यादौ संभवे कुट्टकार्थम् ।
येनच्छिन्नौ भाज्यहारौ न तेन क्षेपश्चैतद् दुष्टमुद्दिष्टमेव ॥ ५० ॥
भाज्यो हारः- divisible ; संभवे – possibly; भाज्यहारौ न – not the divisors ; तेन – then; येनच्छिन्नौ – here cut off; न तेन – these isn't

Here 'a' is dividend, 'b' is divisor, 'c' is remainder. To solve a Kuttaka कुट्टक(pulveriser) first we should ask if a, b, c have a common divisor. In that case let us remove the common divisor and simplify the equation. If the H. C. F. of a and b does not divide 'c' then the example is improper.

यहां 'ए' लाभांश है, 'बी' भाजक है, 'सी' शेषफल है। कुट्टक को हल करने के लिए पहले हमें पूछना चाहिए कि क्या ए, बी, सी में कोई उभयनिष्ठ भाजक है। उस स्थिति में आइए हम उभयनिष्ठ भाजक को हटा दें और समीकरण को सरल बनाएं। यदि a और b का H. C. F. 'c' को विभाजित नहीं करता है तो उदाहरण अनुचित है।

परस्परं भाजितयोर्ययोर्यः शेषस्तयोः स्याद् अपवर्तनं सः ।
तेनापवर्तेन विभाजितौ यौ तौ भाज्यहारी दृढसंशकौ स्तः ॥
मिथो भजेत्ती दृढभाज्यहारी यावद्विभाज्ये भवतीह रूपम् ।
फलान्यधोऽधस्तदधो निवेश्यः क्षेपस्तथाऽन्ते खमुपान्तिमेन ।
स्वोर्ध्वहतेऽन्त्येन युते तदन्त्यं त्यजन्मुहुः स्याद् इति राशियुग्मम् ।
ऊर्ध्वो विभाज्येन दृढेन तष्टः फलं गुणः स्याद् अपरो हरेण ॥ ५१ ॥
परस्परं – each other ; भाजितयोर्ययोर्यः – divided by the two ; शेषस्तयोः – the rest of them; स्याद् – maybe ; अपवर्तनं सः – it is the refraction ; तेना – by that; यौ तौ – that's it; स्तः - they are; मिथो – together; इति – it may be that ; तष्टः फलं गुणः - satisfied by the result of multiply तदन्त्यं - that end ; त्यजन्मुहुः – leaving again and again; ऊर्ध्वो – upwards; विभाज्येन – by division; तष्टः – satisfied; स्याद् – it would be; अपरो – another

By the continued division method of finding the HCF we find the common divisor of a and b if any. Having removed the common factors of a and b if any, our a and b are now pucca for the process. Now we carry the continued division method with दृढभाज्य and दृढहार i.e. a, b till we arrive at remainder 1. The quotients are placed one below the other in succession, in a vertical column and below them Chhep (क्षेप) i.e. c and zero at the end. Rule for the process is: the penultimate number is to multiply the number (quotient) over it and to this product the ultimate number is added, and the sum is put above i.e. in the row of the multiplicand. The last number is discarded. Continuing this process, we arrive at two numbers at the top rows. Dividing the upper number by a we get the remainder as the value of y (लब्धि). And dividing the other number by b we get the remainder as the value of x (गुण).

H.C.F ज्ञात करने की निरंतर विभाजन विधि द्वारा हम ए और बी का उभयनिष्ठ भाजक ज्ञात करते हैं, यदि कोई हो। ए और बी के सामान्य कारकों को हटाने के बाद, यदि कोई हो, तो हमारा ए और बी अब प्रक्रिया के लिए पक्का है। अब हम दृढभाज्य और दृढहार यानी ए, बी के साथ निरंतर विभाजन विधि को तब तक जारी रखते हैं जब तक कि हम शेष 1 पर नहीं पहुंच जाते। भागफल को एक ऊर्ध्वाधर स्तंभ में क्रमिक रूप से एक के नीचे एक रखा जाता है और उनके नीचे छप्प (क्षेप) यानी सी और शून्य पर रखा जाता है। अंत। प्रक्रिया के लिए नियम यह है: अंतिम संख्या को उस संख्या (भागफल) से गुणा करना है और इस उत्पाद में अंतिम संख्या जोड़ दी जाती है, और योग को से ऊपर रखा जाता है। इ। गुणक की पंक्ति में. अंतिम संख्या को हटा दिया गया है. इस प्रक्रिया को जारी रखते हुए, हम शीर्ष पंक्तियों में दो नंबरों पर पहुंचते हैं। ऊपरी संख्या को a से विभाजित करने पर शेषफल y (लब्धि) के मान के रूप में प्राप्त होता है। और दूसरी संख्या को b से विभाजित करने पर हमें x (गुण) के मान के रूप में शेषफल प्राप्त होता है।

एवं तदेवात्र यदा समास्ताः स्युर्लब्धयश्चेद् विषमास्तदानीम् ।
यथागतौ लब्धिगुणी विशोध्यौ स्वतक्षणाच्छेषमितो तु तौ स्तः ॥ ५२ ॥
एवं – and ; तदेवात्र – that's it here ; यदा – when ; समास्ताः(समस्त) – all ; विषमास्तदानीम् - odds then; यथागतौ – as in the last; लब्धिगुणी – obtained multiplier; स्वतक्षणाच्छेषमितो – rest at that moment; तु तौ - they both; स्तः – they are
When the number of quotients is even the process gives गुण and लब्धि correctly. But when the number is odd, values obtained must be subtracted from b and a respectively to get the correct values.

जब भागफल की संख्या सम होती है तो प्रक्रिया गुण और लब्धि को सही ढंग से देती है। लेकिन जब संख्या विषम हो, तो सही मान प्राप्त करने के लिए प्राप्त मानों को क्रमशः b और a से घटाया जाना चाहिए।

भवति कुट्टविधेर्युतिभाज्ययोः समपवर्तितयोरपि वा गुणः ।
भवति यो युति भाजकयोः पुनः स च भवेद् अपवर्तनसंगुणः || ५३ ॥
भवति – happen ; कुट्टविधेर्युतिभाज्ययोः – in the light divisions of the kuttaka method ; समपवर्तितयोरपि वा – or even of the two equally reversed ; यो युति – this union ; भाजकयोः पुनः – the divisors again ; स च – it and ; भवेद् – it will be
If we divide c and a by a common factor and then adopt the कुट्टक process, we get correct value for x but not for y. To get the value for y for the original equation, the value got by the process should be multiplied by the common factor.

यदि हम c और a को एक सामान्य कारक से विभाजित करते हैं और फिर कुट्टक प्रक्रिया अपनाते हैं, तो हमें x के लिए सही मान मिलता है, लेकिन y के लिए नहीं। मूल समीकरण के लिए y का मान प्राप्त करने के लिए, प्रक्रिया द्वारा प्राप्त मान को सामान्य कारक से गुणा किया जाना चाहिए।

योगजे तक्षणाच्छुद्धे गुणाप्ती स्तो वियोगजे ।
धनभाज्योद्भवे तद्वद् भवेताम् ऋणभाज्यजे ॥ ५४ ॥
तक्षणा – instantly; गुणाप्ती स्तो – multiplication is ; वियोगजे – separation ; धनभाज्योद्भवे - positive divisible obtained; तद्वद् – similarly; भवेताम् - happens
The values of x and y obtained when c is positive must be subtracted from b and a respectively for the case when c is negative. In the same way the values of x and y when a is positive must be subtracted from b and a for the case when a is negative.

जब c धनात्मक हो तो प्राप्त x और y के मान को उस स्थिति के लिए क्रमशः b और a से घटाया जाना चाहिए जब c ऋणात्मक हो। उसी प्रकार जब a धनात्मक हो तो x और y के मान को उस स्थिति में b और a से घटाया जाना चाहिए जब a ऋणात्मक हो।

गुणलब्ध्योः समं ग्राह्यं धीमता तक्षणे फलम् ॥। ५५ ।।
गुणलब्ध्योः - to obtain the multiplication ; समं – equal ; धीमता – slowness ; तक्षणे फलम् – instant result
In making a selection of proper pairs (x, y) the intelligent person will take care to see that the values correspond with each other.

उचित जोड़ी (x, y) का चयन करते समय बुद्धिमान व्यक्ति इस बात का ध्यान रखेगा कि मान एक-दूसरे के अनुरूप हों।

हरतष्ठे धनक्षेपे गुणलब्धी तु पूर्ववत् ।
क्षेपतक्षणलाभाढया लब्धिः शुद्धौ तु वर्जिता ।। ५६ ।।
हरतष्ठे – every aspect ; गुणलब्धी तु – multiply the result ; पूर्ववत् – undone; तक्षण – instantly; वर्जिता – forbidden

When c > b divide c by b and take the remainder as new c and calculate (x, y) as before. Value of x will be correct. To get the correct value of y, to its calculated value add the quotient obtained when b divides c. If c is negative this quotient should be subtracted from the calculated value.

जब c > b, c को b से विभाजित करें और शेष को नए c के रूप में लें और पहले की तरह (x, y) की गणना करें। x का मान सही होगा. y का सही मान प्राप्त करने के लिए, इसके परिकलित मान में b द्वारा c को विभाजित करने पर प्राप्त भागफल को जोड़ें। यदि c ऋणात्मक है तो इस भागफल को परिकलित मान से घटाया जाना चाहिए।

अथवा भागहारेण तष्टयोः क्षेपभाज्ययोः ।
गुणः प्राग्वत् ततो लब्धिर् भाज्याद्धत युतोद् धृतात् ॥ ५७ ॥
अथवा – or ;क्षेप – parts added but other to the original composition ; भाज्ययोः – factors ; गुणः - प्राग्वत् – as before ; ततो – it ; भाज्याद्धत – factorial half ; युतोद् – joining ; धृतात् - from the hold

EXPLANATION: -

Let the two remainders be such that $r_1 > r_2$ so that (a, b) be the divisors corresponding to the greater and smaller remainders respectively.
Let $c = r_1 - r_2$
We write down the procedure, when the number of quotients (ignoring the first one q) is even.

a (q
bq
r_1) b (q_1
$r_2 q_1$
r_2) 〚 r〛 _1 (q_2
$r_2 q_2$
.
.
.
r_{2n}) $r_{(2n-1)}$ (q_{2n}
$r_{2n} q_{2n}$
$r_{(2n+1)}$

Arranging the quotients and the choice of mati
The
prescription for the choice of the optimal number t (mati) is:

$r_{(2n+1)} t + c$ should be divisible by r_{2n}. (quotients even)
$r_{2n} t - c$ should be divisible by $r_{(2n-1)}$ (quotients odd)
Let's be the quotient
Having found t and s, we have to arrange them in the form of a valli (column), to generate successive columns.

q_1	q_1	q_1	………	$q_1\beta_{(2n-1)} + \beta_{(2n-2)}$
q_2	q_2	q_2	……....	$\beta_{(2n-1)}$
.	.	.		
.	.	.		
.	.	.		
$q_{(2n-1)}$	$q_{(2n-1)}$	$q_{(2n-1)} \beta_1 + t = \beta_2$		
q_{2n}	$q_{2n} t + s = \beta_1$			
t	t			

Divide $q_1\beta_{(2n-1)} + \beta_{(2n-2)}$ by b. The remainder is x and $N = ax + r_1$

Expiation of Kuttaka Process by Krsna Daivajña
As an example of an upapatti which proceeds in a sequence of steps, we may briefly consider the detailed upapatti for the kuttaka procedure given by Krsna Daivajña (c.1600) in his commentary Bijapallava on Bījaganita of Bhāskara.
The kuttaka procedure is for solving first order indeterminate equations of the form
$((ax+c))/b=y$

Here, a, b, c are given integers (called bhājya, bhajaka and kşepa) and x, y are to be solved for in integers.
Kışņa first shows that the solutions for x, y do not vary if we factor all three numbers a, b, c by the same common factor.
He then shows that if a and b have a common factor then the above equation will not have a solution unless c is also divisible by the same.
He then gives the upapatti for the process of finding the apavartanka (greatest common divisor) of a and b by mutual division (the so-called Euclidean algorithm).
Krsna then provides a detailed justification for the kuttaka method of finding the solution by making a valli (table) of the quotients obtained in the above mutual division, based on a detailed analysis of the various operations in reverse (vyasta-vidhi).
In doing the reverse computation on the valli (vallyupasamhara) the numbers obtained, at each stage, are shown to be the solutions to
the kuttaka problem for the successive pairs of remainders (taken in reverse order from the end) which arise in the mutual division of a and b.
After analysing the reverse process of computation with the vallī, Krsna shows how the solutions thus obtained are for positive and negative kşepa, depending upon whether there are odd or even number of coefficients generated in the above mutual division.
And this indeed leads to the different procedures to be adopted for solving the equation depending on whether there are odd or even number of quotients in the mutual division.
As an illustration, Krsna considers the equation $((173x+3))/71=y$ with bhājya 173, bhājaka 71 and ksepa 3.

In the mutual division of 173 and 71 we get the quotients 2, 2, 3 and 2 and remainders 31, 9, 4 and 1.

If we do the reverse computation on the valli formed by 2, 2, 3, 2, 1, 3 and 0, we first get 6, 3 as the labdhi and guna, which satisfy the equation.
((9.3-3))/4=6 , with the remainders 9, 4 serving as bhājya and bhājaka.

In the reverse computation on the valli, we then get 21, 6 as labdhi and guna, which satisfy the equation ((31.6+3))/9=21, with the remainder 31, 9 serving as bhajya and bhajaka.

And so on, till we get 117 and 48 as labdhi and guna, which satisfy the equation ((178.48+3))/71=117.

वर्गप्रकृतिः

वर्गप्रकृतिः or Equation of the form $ax^2 + b = y^2$ is a process is a statement of proof from the book Bijaganita of Bhaskar II. Bhāskara II (c. 1114–1185), also known as Bhāskarāchārya and as Bhāskara II to avoid confusion with Bhāskara I, was an Indian mathematician, astronomer and inventor. From verses, in his main work, Siddhānta Shiromani (सिद्धांतशिरोमणी). He has been called the greatest mathematician of medieval India. His main work Siddhānta-Śiromaṇi, is divided into four parts called Līlāvatī,
Bījagaṇita, Grahagaṇita and Golādhyāya, which are also sometimes considered four independent works. These four sections deal with arithmetic, algebra, mathematics of the planets, and spheres respectively. He also wrote another treatise named Karaṇā Kautūhala.

STATEMENT: -

इष्टं ह्रस्वं तस्य वर्गः प्रकृत्या क्षुण्णौ युक्तो वर्जिता वा स येन ।
मूलं दद्यात् क्षेपकं तं धनर्ण मूलं तच्च ज्येष्ठमूलं वदन्ति ।। ७० ।।
तस्य – his; प्रकृत्या – nature; युक्तो – crushed and fitted; वा – or; स – he; येन – by which; मूलं – whose root; दद्यात् – should give; क्षेपकं तं – that's it; तच्च –that's; ज्येष्ठमूलं – first root

What is desired is x, the first variable. By multiplying the square of the desired by प्रकृति and adding or subtracting something we get a square number whose root is the second variable. The augment b may be positive or negative.

जो वांछित है वह x है, पहला चर। प्रकृति द्वारा वांछित के वर्ग को गुणा करने और कुछ जोड़ने या घटाने से हमें एक वर्ग संख्या प्राप्त होती है जिसका मूल दूसरा चर होता है। संवर्द्धन बी सकारात्मक या नकारात्मक हो सकता है।

ह्रस्वज्येष्ठक्षेपकान् न्यस्य तेषां तानन्यान्वाऽधो निवेश्य क्रमेण ।
साध्यान्येभ्यो भावनाभिर्बहूनि मूलान्येषां भावना प्रोच्यतेऽतः ।।
वज्राभ्यासौ ज्येष्ठलध्वोस्तदैक्यं ह्रस्वं लघ्वोराहतिश्च प्रकृत्या ।
क्षुण्णा ज्येष्ठाभ्यासयुग्ज्येष्ठमूलं तत्राभ्यासः क्षेपयोः क्षेपकः स्यात् ।।
ह्रस्वं वज्राभ्यासयोरन्तरं वा लध्वोर्घातो यः प्रकृत्या विनिघ्नः ।
घातो यश्च ज्येष्ठयोस्तद्वियोगो ज्येष्ठं क्षेपोऽत्रापि च क्षेपघातः ॥७१।।
ह्रस्वज्येष्ठक्षेपकान् - ;न्यस्य - placed;तेषां -theirs;तानन्यान्वाऽधो – or the others below; क्रमेण - in order;साध्यान्येभ्यो – from the simple ones;भावनाभिर्बहूनि -;मूलान्येषां – the roots of others; प्रोच्यतेऽतः – it is therefore pronounced;वज्राभ्यासौ -;ज्येष्ठलध्वोस्तदैक्यं -; ह्रस्वं -;लघ्वोराहतिश्च -;प्रकृत्या – by nature;क्षुण्णा -;ज्येष्ठाभ्यासयुग्ज्येष्ठमूलं -;तत्राभ्यासः – practice there;क्षेपयोः क्षेपकः -;स्यात् -

should;ह्रस्वं -;वज्राभ्यासयोरन्तरं -;वा - or;लध्वोर्घातो -;यः - who;प्रकृत्या -;विनिघ्नः -;घातो -;यश्च – and who;ज्येष्ठयोस्तद्वियोगो -;ज्येष्ठं -;क्षेपोऽत्रापि -;च क्षेपघातः -

Put down x, y, b in this order. Below them write the same or other ह्रस्व, ज्येष्ठ and क्षेपक satisfying a similar equation with the same प्रकृति, a, from these by a process called भावना we can have many values for x, y. This is why the process is called bhavana (generator). By cross-multiplying and adding the two products of x and y we get a new व्हस्व. By multiplying the product of two first variables with प्रकृति and adding to it the product of the second variables we get new y. Product of two augments (i.e. क्षेप) gives new value for b. another method. Take x_1 y_2 x_2 y_1 as new 〖ax〗_1 x_2- y_1 y_2, as new y and b_1 b_2as new b.

इस क्रम में x, y, b को नीचे रखें। उनके नीचे समान प्रकृति के साथ समान समीकरण को संतुष्ट करने वाले समान या अन्य ह्रस्व, ज्येष्ठ और क्षेपक लिखें, इनसे भावना नामक प्रक्रिया द्वारा हम x, y के लिए कई मान प्राप्त कर सकते हैं। यही कारण है कि इस प्रक्रिया को भावना (जनरेटर) कहा जाता है। क्रॉस-गुणा करने और x और y के दो उत्पादों को जोड़ने पर हमें एक नया व्हस्व प्राप्त होता है। पहले दो चरों के गुणनफल को प्रकृति से गुणा करने और उसमें दूसरे चरों के गुणनफल को जोड़ने पर हमें नया y मिलता है। दो संवर्द्धनों का गुणनफल (अर्थात गति) b के लिए नया मान देता है। दूसरी विधि. x_1 y_2 x_2 y_1 को नए 〖ax〗_1 x_2- y_1 y_2, नए y के रूप में और b_1 b_2 को नए b के रूप में लें।

इष्टवर्गप्रकृत्योर् यद् विवरं तेन वा भजेत्।
द्विघ्नमिष्टं कनिष्ठं तत्पदं स्याद् एकसंयुतौ ।
ततो ज्येष्ठ मिहानन्त्यं भावनातस्तथेष्टतः ।। ७३ ।।
इष्टवर्गप्रकृत्योर् – nature of the desired (suitable) multiplate;यद् – if;विवरं -details;तेन -then;वा - or;भजेत् – it will happen;द्विघ्नमिष्टं – twice the desired ;कनिष्ठं -smaller;तत्पदं – that term;स्याद् – it would be;एकसंयुतौ -together;ततो - then;ज्येष्ठ -bigger

If the absolute number is 1, then 2x/(a-x^2) may be taken as new x and from that new y can be obtained. From these values we can get infinite values for the triad (x, y, b) by the application of bhavana process and इष्ट as given in stanzas 71 and 72 .
यदि पूर्ण संख्या 1 है, तो 2x/(a-x^2) को नए x के रूप में लिया जा सकता है और उससे नया y प्राप्त किया जा सकता है। इन मानों से हम भावना प्रक्रिया और इष्ट के अनुप्रयोग द्वारा त्रिक (x, y, b) के लिए अनंत मान प्राप्त कर सकते हैं जैसा कि श्लोक 71 और 72 में दिया गया है।

EXPLANATION: -
Method of composition (Bhavana)
Brahmagupta was the first one who discovered the method to find the solution of the special case (2) of the equation *D*x^2 + 1 = y^2 called varga-prakrti. This method, known as bhavana or the method of composition, enabled one to produce many solutions for certain difficult cases in a few simple steps, starting off from one solution. He saw that if one could find one solution to this equation, then he could find others. Brahmagupta gave the rule of this method and illustrated it with examples for some special values for D, like *D* = 83 or *D* = 92. The rule of bhavana is as follows.
“Put down twice the square root of a given square multiplied by a multiplier and increased or diminished by an arbitrary number. The product of the first pair, multiplied by the multiplier, with the product of the last pair is the last computed. The sum of the thunderbolt product is the first. The

additive is equal to the product of the additive. The two square roots, divided by the additive or the subtractive, are the additive unity."

Chapter 13 - Chakravala method to solve one equation of two variable both having second degree

Both Acarya Jayadeva (c.1000) and Bhaskara II, studied Pell's equation and discovered the complete solution of it. Bhaskara's method is easier to follow, and it determines the smallest solution of the equation, as well as all the solutions. It is known as the cyclic method, referring to the cyclic nature of the steps, where the same procedure can be repeated again and again in a circle, otherwise called as chakravala. According to Selenius (1975), there is a remarkable interaction between the Pell's equation and the Kuttaka, that Bhaskara II in his Bijaganita made use of the Kuttaka in his procedure to get the desired solution of the equation. The main idea of this method is to use the principle of composition, given by Brahmagupta, to repeatedly obtain solutions to $Dx^2 + k = y^2$ (for different values of k). Eventually, this method will introduce a solution to $Dx^2 + 1 = y^2$ and furthermore, the smallest solution of the given equation will be obtained. He writes that to deal with the quantities in the equation $Dx^2 + 1 = y^2$, one uses the following terms: x is the lesser root, y is the greater root, D is the multiplier or prakrti, and k is the interpolator or additive. 1 For the kuttaka equation $(ax + c)/b = y$ the considered terms are: x is the multiplier, y is the quotient, a is the dividend, d is the divisor and c is the additive (interpolator).

"Making the smaller and larger roots and the additive into the dividend, the additive and the divisor, the multiplier is to be imagined. When the square of the multiplier is subtracted from the "nature" or is diminished by the "nature" so that the remainder is small, that divided by the additive is the new additive. It is reversed if the square of the multiplier is subtracted from the "nature". The quotient of the multiplier is the smaller square root; from that is found the greatest root.Then it is done repeatedly, leaving aside the previous square roots and additive. They call it the chakravala (circle). Thus, there are two integer square roots increased by four, two or one. The supposition for the sake of an additive one is from the roots with four and two as additives."

According to Bhaskara II, every cycle in chakravala procedure consists of four steps:

1) The first step begins with finding (u, v), for some additive k, satisfying the auxiliary equation $Dx^2 + k = y^2$ (the same procedure as in the composition method), then forming the kuttaka´s equation by taking the lesser root u to be the dividend, the greater root v to be the additive, and the additive k to be the divisor.

2) The kuttaka equation $(um+v)/k = n$ (that was formed in the first step), has more than one solution. In the second step, (m, n) should be chosen such that the square of m (the multiplier in the kuttaka equation), is as close to the multiplier D (the nature) as possible. In other words, we want $|D - m^2|$ to be minimal. The value of n that obtained from the kuttaka equation (corresponding to that value of m), is the new lesser root u1. The new first root is $u_1 = (um+v)/k$.

3) The third step is to find the new additive or interpolator k_1. It is the result obtained by dividing the residue of ($D - m^2$ or $m^2 - D$) by the original divisor in the kuttaka equation k (if $D > m^2$ one takes $-(D - m^2)$ but if $D < m^2$ then one takes $m^2 - D$). The new additive is $k_1 = \pm (D-m^2)/k$.

4) The last step is to determine the new greater root v_1. Selenius (1975), presents the three paths that the Indian mathematicians used to find v_1.

4.1 The value of v_1 can be naturally obtained from the new auxiliary equation ${Du_1}^2 + k_1 = {v_1}^2$ where D is the original multiplier and both u_1, k_1 have been obtained in the second and the third steps, respectively.

4.2 According to the method given by the Indian mathematician Narayana, the value of the greater root v_1, is determined in a very different rule "…and that [the new lesser root] multiplied by the 23

multiplier [m] and diminished by the product of the previous lesser root [u] and (new) interpolator [k_1] will be its greater root." Then the greatest root is $v_1 = u_1 m - u k_1$
4.3 The last path to find v_1 is by performing Bhaskara's II rule." The original greatest root v multiplied by the multiplier m, is added to the least root u multiplied by the given coefficient D; and the sum is divided by the additive k". Then the greatest root is $v_1 = (vm+Du)/k$.
As soon as the four steps in chakravala procedure are done, then the first cycle is complete, and from initial value (u, v, k) we obtain new quantities (u_1,v_1,k_1). The same procedure is repeated again and again until one achieves the integral solution for an equation with the additive $k = \pm 1, \pm 2, \pm 4$. Applying the principle of composition, the same integral roots corresponding to the additive 1 will be derived with much fewer steps than continuing with the method of chakravala.
Interpretation of the chakravala rules
The clear of the previous procedure depends basically on the principle of composition given by Brahmagupta. That is, if (u, v) and $(1, m)$ are solutions to two auxiliary equations with additive k and $D - m^2$ respectively, such that
$Du^2 + k = v^2$ (1)
$D \cdot 1^2 - m^2 = D - m^2$ (2)
Applying the method of composition on the two solutions, the new equation is obtained $D(um + v)^2 + k(D - m^2) = (Du + vm)^2$. Dividing by k^2 the new equation $D (um+v)^2/k^2 + k(D-m^2)/k^2 = (Du+vm)^2/k^2$ and thus $D ((um+v)/k)^2 + (D-m^2)/k = ((Du+vm)/k)^2$. It is required to obtain integral solutions, so Bhaskara's method now is to choose m such that the greater root $(Du+vm)/k$ is an integer (in other words $(Du + vm)$ is divisible by k). Thus, the two other terms $(D-m^2)/k$ and $(um+v)/k$ will be both integers. It should also pick up m such that $|m^2 - D|$ is minimal, thus $(m^2 - D)/k$ will be minimal. This process ensures the access to a smaller additive in every cycle. The procedure can be applied again and again until getting the additive $k = \pm 1, \pm 2, \pm 4$ then the Pell's equation is solved by the method of composition.
Example: Solve Pell`s equation $67x^2 + 1 = y^2$
1) We start with choosing a solution for the auxiliary equation $67u^2 + k = v^2$. We begin with considering $u = 1$, then it is easy to see that the other quantities are $v = 8$ and the additive $k = -3$. Thus the auxiliary equation is $67 \cdot 1^2 - 3 = 8^2$. We form the kuttaka equation $1 \cdot m + 8 = -3n$.
2) The previous equation gives $m = 1 + 3t$ and $n = -3 - t$ for any integer t, ($1m + 8$ must be divisible by 3). The values of m can be (1, 4, 7, 10, 13…) and we choose the value that minimises the value of $|67 - D^2|$. The desired value of m = 7, and then $u_1 = (1 \cdot 7+8)/(-3) = -5$ is an integer. Because the roots are always squared then we can take $u_1 = 5$. At the same time $|67 - 7^2|$ is minimal.
3) The third step by Bhaskara is to determine the new additive k_1 and the solution of the new auxiliary equation. The new additive is $(D-m^2)/k = (67-49)/(-3) = -6$ and because $D > m^2$ we take $k_1 = 6$.
4) To compute the new greater root, we use the first rule in the fourth step to get $v_1 = \sqrt{67.25 + 6} = \sqrt{1681} = 41$. From the four previous steps west, (5,41,6) is solution for the auxiliary equation, thus we can form the new auxiliary equation now: $67 \cdot 5^2 + 6 = 41^2$.
The cycle is complete now, and we repeat the same procedure. We should find the solution for the new equation. We form the kuttaka equation $5 \cdot m + 41 = 6 \cdot n$ and $m = 6t + 5$. The set of the values of m is (5, 11, 17…) and we will choose $m = 5$.
The new additive is $k_1 = (67-5^2)/6 = 7$ and because $D > m^2$ we take $k_1 = -7$. The new lesser root is $u_2 = (5 \cdot 5+4.1)/6 = 11$, and the new greater root is $v_2 = \sqrt{67 \cdot 11^2 - 7} = \sqrt{8100} = 90$. The new derived auxiliary equation is $67 \cdot 11^2 - 7 = 90^2$. Then the next triple is (11, 90, −7).
By the same procedure we obtain the next triple (27, 221, −2). Since the value of the additive is equal to −2 then either the method of Brahmagupta can be applied or the chakravala method. Combining the last solution with itself we get (11934, 97684, 4). Finally, the solution for the additive one can be

obtained by dividing the last solution by 2, and then the desired solution to the original equation $x^2-67y^2 = 1$ is $x = 5967$ and $y = 48842$.

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