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Special Issue

On

Ecological Systems

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SECTION I

Original Research Papers

NOTE FROM THE EDITOR-IN-CHIEF

In the first few volumes of this journal, the aim is to publish papers in mathematical sciences from authors of IIT Kanpur Research group in Mathematical Modeling and the Members of the Indian Academy of Mathematical Modeling and Simulation having Head Quarters at IIT Kanpur.

Researchers from elsewhere will be invited later to publish their research work in this journal. Self submission is not encouraged. But letters to the Editors are invited from researchers on ideas relevant to Nature and Society for publication in the journal.

In the present special volume of Ecological Systems, the four papers on the following aspects are included. They are, namely, Modeling The Depletion Of Forest Resources By Population And Industrialization And Their Conservation By Green Belts Plantation, Modeling the Depletion of Forest Resources due to Population and Industrialization and Its Effect on Survival of Wildlife Species, Modeling and Analysis of the Impact of Awareness Programs on Rural Population for Conservation of Forest Resources and Modeling the control of CO_2 from the atmosphere by using greenbelts plantation and seaweeds cultivation . These papers will provide young researchers a good idea as how to conduct research in these areas.

I also take this opportunity to thank the guest editor and all the authors for their cooperation in this venture.

J.B Shukla

NOTE FROM THE GUEST EDITOR

This special volume deals with the depletion of forest resources by population and industrialization in the first two papers. In the first paper , a new concept of conservation has been used in the modeling process where the rate of conservation has been assumed to be proportional to the rate of depletion. In the second paper, the effect of depletion on the survival and migration of wildlife has been studied, a very useful problem. In the third paper, impact of awareness programs on rural population living around the forest has been modeled and analysed to protect the forest. The final paper deals with the control of global warming by using green belts plantation and sea weed cultivation.

I find this volume very useful to resource modelers and have enjoyed editing this special issue.

Prof. Manju Agarwal

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Modeling the Depletion of Forest Resources by Population and Industrialization and Their Conservation by Green Belts Plantation

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Abstract

In developing countries forest land and forest resources are used for agriculture to fulfill the need of growing population and for industrialization related to mining and wood based industries. A system of nonlinear differential equations is proposed and analysed to study the depletion of forestry resources caused by population and industrialization. The conservation of the forest resources is also modelled and analysed by using green belts plantation, the growth rate of cumulative biomass density of which is assumed to be proportional to the rate of depletion of the cumulative biomass density of forest resources. Thus, the problem is governed by four nonlinearly interacting variables, namely, the cumulative biomass density of forest resources, the density of human population, the density of the industrialization and the cumulative biomass density of green belts plantation. The stability theory of differential equations and numerical simulation are used to analyze the model. The analysis shows that the forest resources are doomed to extinction in absence of conservation; however with conservation, under certain conditions, they can be maintained very close to the level before depletion.

Keywords: **Forestry Biomass, Population, Plantation, Conservation, Stability, Persistence.**

1. Introduction

The study of depletion of forest resources by population and industrialization is a very important problem to be investigated both theoretically as well as experimentally, (Rawat and Ginwal, 2009). Many researchers have used mathematical models to study the depletion of forest resources during last few decades, Shukla, et al. (1988, 1989, 2011), Dubey and Narayanan (2010), Agarwal and Devi (2012), Chaudhary (2013), Misra, et al (2014); etc. In particular, Shukla et al. (1989) have modeled and analysed a mathematical model for the depletion of forest resources by industrialization and population where they have shown that the forest resources may be doomed to extinction under growing population living in the forest (or under industrialization).

The models for conservation of forest resources have also been proposed and analysed by many workers by using regeneration of forest resources by green belts plantation (Shukla, 1989, 2011).In most cases, the effort for plantation has been assumed to be proportional to the depletion level i.e. the difference between the maximum biomass density and the current biomass density of the resource. In these studies, the model analysis has shown that the biomass density of the resource can be restored to a desired level. Agarwal and Pathak (2015) have studied the conservation of forest biomass by using alternative resource. The effects of pollution and population pressure on the depletion of forest resources have also been studied by some workers, Dubey et al.(2009), Agarwal and Devi (2012), Mishra et al.(2014). Agarwal and Devi (2012) have modeled and analysed the effect of population pressure and augmented industrialization on the depletion of forest resource and have shown that these effects are very damaging. Mishra et al. (2014) have studied the effect of population pressure on depletion of forest resources and shown that population pressure has devasting effect on forestry resources. Dhar and Singh (2004) have studied the depletion of forest resources in two adjoining forest habitats.

It may be noted from the above investigations that for the conservation, the effort has not been assumed to be proportional to the rate of depletion of forest resources which is an important problem to be studied. Therefore our focus here is to model the conservation of forest resources by using green belts plantation in and around the forest region under conservation. It is assumed that the cumulative biomass density of resources in the green belts is proportional to the rate of depletion of forest resources in the region. This aspect of modeling is new and original.

2. Mathematical Model

To model the problem it is assumed that forest resources are exploited by human population and industrialization. For conservation of depleted resources, the effort is applied in the form of green belts plantation in and around the forest. Let B be the cumulative biomass density of forest resources, N be the human population density, I be density of Industrialization (i.e. the number of industries in a unit area) and F be the cumulative biomass density of biomass of resources in green belts plantation. In the modelling process, the growth rate of biomass density in green belts plantation is assumed to be proportional to the rate of depletion of cumulative biomass density of forestry resources. The dynamics of the problem is assumed to be governed by the following ordinary differential equations, Shukla,et al.(1989),Shukla et al.(2011).

$$\begin{aligned}
\frac{dB}{dt} &= sB - \frac{s_0 B^2}{L} - \alpha BN - \alpha_1 B^2 N - \beta\, BI - \beta_1 B^2 I + kF, \\
\frac{dN}{dt} &= rN - \frac{r_0 N^2}{K} + \alpha BN + \alpha_1 B^2 N, \\
\frac{dI}{dt} &= \beta BI + \beta_1 B^2 I - \beta_0 I^2 - \theta_0 I, \\
\frac{dF}{dt} &= \pi\alpha\, BN + \pi_1 \alpha_1 B^2 N + \phi\beta\, BI + \phi_1 \beta_1 B^2 I - \pi_0 F.
\end{aligned} \tag{2.1}$$

with initial conditions $B(0) = B_0 \geq 0, N(0) = N_0 \geq 0, I(0) = I_0 \geq 0, F(0) = F_0 \geq 0.$

Here $\alpha, \alpha_1, \beta, \beta_1, \beta_0, \theta_0, k, \pi, \pi_1, \phi, \phi_1$ and π_0 are positive constants, such that $0 \le \pi \le 1, 0 \le \pi_1 \le 1, 0 \le \phi \le 1, 0 \le \phi_1 \le 1, \pi_0 > k$. The depletion rates of forestry biomass by population and industrialization is represented by α and β respectively whereas, α_1 and β_1 are the depletion rate coefficients of forestry biomass due to crowding by these factors. The constants s and r are growth rate coefficients of B and N and the constants s_0, L, r_0 and K are interaction coefficients related to carrying capacity of B and N respectively. β_0 and θ_0 are intra specific interaction coefficient of industrialization and failure of industrialization.

3. Existence and Dissipativeness

To study of Dissipativeness of system (2.1) about different steady states, the following lemma is required.

Lemma 3.1: The set $\Omega = \left\{ (B, N, I, F) : 0 \le B + F \le \frac{A}{\delta}, 0 \le N \le A_1 \text{ and } 0 \le I \le A_2 \right\}$, is the region of attraction for all solutions initiating in the interior of the positive octant,

where $A = \frac{\pi_0 (s+\delta)^2 L}{4ks_0}, A_1 = \frac{(r\delta^2 + \alpha A\delta + \alpha_1 A^2)K}{\delta^2 r_0}, A_2 = \frac{\beta A\delta + \beta_1 A^2 - \theta_0 \delta^2}{\delta^2 \beta_0}, \beta A\delta + \beta_1 A^2 > \theta_0 \delta^2$.

Proof: Let $W = \theta_1 B + \theta_2 F$, we may take $\theta_2 = 1$. $\theta_1 > 0, \theta_2 > 0$ are positive constants. Since B and F do not have the same dimension, θ_1 and θ_2 would compensate their dimensions.

Now

$$\frac{dW}{dt} = \theta_1 \frac{dB}{dt} + \frac{dF}{dt}, \tag{3.1}$$

Which an using (2.1), we get

$$\frac{dW}{dt} = \theta_1 \left\{ sB - \frac{s_0 B^2}{L} - \alpha BN - \alpha_1 B^2 N - \beta BI - \beta_1 B^2 I + kF \right\} + \left\{ \pi\alpha BN + \pi_1 \alpha_1 B^2 N + \right.$$

$$\left. \phi\beta BI + \phi_1 \beta_1 B^2 I - \pi_0 F \right\}, \tag{3.2}$$

or
$$\frac{dW}{dt} = \theta_1 \left(sB - \frac{s_0 B^2}{L} \right) + \theta_1 kF - \pi_0 F - \alpha BN(\theta_1 - \pi) - \alpha_1 B^2 N(\theta_1 - \pi_1) -$$
$$\beta BI(\theta_1 - \phi) - \beta_1 B^2 I(\theta_1 - \phi_1), \tag{3.3}$$

Choose $\theta_1 > \pi, \theta_1 > \pi_1, \theta_1 > \phi, \theta_1 > \phi_1$, i.e. $\theta_1 > \max(\pi, \pi_1, \phi, \phi_1)$.

Then from (3.3), we have

$$\frac{dW}{dt} \le \theta_1 \left(sB - \frac{s_0 B^2}{L} \right) + (\theta_1 k - \pi_0) F, \tag{3.4}$$

or $$\frac{dW}{dt} + \delta W \le \theta_1 (s + \delta) B - \frac{\theta_1 s_0 B^2}{L} + (\theta_1 k - \pi_0 + \delta) F, \tag{3.5}$$

Choose further

$$\delta < \pi_0 - \theta_1 k, \theta_1 < \frac{\pi_0}{k}. \tag{3.6}$$

Then, the equation (3.5) simplifies to

$$\frac{dW}{dt}+\delta W \leq \frac{\pi_0(s+\delta)}{k}B-\frac{\pi_0 s_0}{kL}B^2, \tag{3.7}$$

or $\frac{dW}{dt} \leq \frac{\pi_0}{k}\left[(s+\delta)B-\frac{s_0}{L}B^2\right]-\delta W,$

$$\leq \frac{\pi_0}{k}\left[\frac{(s+\delta)^2 L}{4s_0}\right]-\delta W.$$

This implies that

$$W_{\max}=\frac{\pi_0(s+\delta)^2 L}{4ks_0\delta}=\frac{A}{\delta}. \tag{3.8}$$

where $A=\frac{\pi_0(s+\delta)^2 L}{4ks_0}$.

Thus $0 \leq B \leq \frac{A}{\delta}, 0 \leq F \leq \frac{A}{\delta}$. (3.9)

Now from (2.1), we get

$$\frac{dN}{dt} \leq \left(r+\frac{\alpha A}{\delta}+\frac{\alpha_1 A^2}{\delta^2}\right)N-\frac{r_0}{K}N^2, \tag{3.10}$$

which on the comparison principle, we have

$$N_{\max}=\frac{(r\delta^2+\alpha A\delta+\alpha_1 A^2)K}{\delta^2 r_0}=A_1. \tag{3.11}$$

Further from the system (2.1), we find

$$\frac{dI}{dt} \le \left(\frac{\beta A}{\delta} + \frac{\beta_1 A^2}{\delta^2} - \theta_0 \right) I - \beta_0 I^2, \tag{3.12}$$

which gives

$$I_{\max} = \frac{\beta A \delta + \beta_1 A^2 - \theta_0 \delta^2}{\delta^2 \beta_0} = A_2. \tag{3.13}$$

This completes the proof of lemma.

It is assumed that the (2.1) persists even without conservation. Then from the above and (2.1) it is noted that

$$s > \left(\alpha + \frac{\alpha_1 A}{\delta} \right) A_1 + \left(\beta + \frac{\beta_1 A}{\delta} \right) A_2. \tag{3.14}$$

4. Equilibrium Analysis of the System

The system (2.1) has six nonnegative equilibria namely, $E_0(0,0,0,0)$, $E_1\left(\frac{sL}{s_0},0,0,0\right)$, $E_2\left(0,\frac{rK}{r_0},0,0\right)$, $E_3\left(\hat{B},\hat{N},0,\hat{F}\right)$, $E_4\left(\overline{B},0,\overline{I},\overline{F}\right)$, and $E_5\left(B^*,N^*,I^*,F^*\right)$ .The existence of equilibrium points $E_0(0,0,0,0)$, $E_1\left(\frac{sL}{s_0},0,0,0\right)$ or $E_2\left(0,\frac{rK}{r_0},0,0\right)$, is obvious. We show the existence of other equilibria as follows.

Existence of $E_3\left(\hat{B},\hat{N},0,\hat{F}\right)$

Here $\hat{B}, \hat{N}$ and $\hat{F}$ are the positive solutions of the following algebraic equations

$$s\hat{B} - \frac{s_0 \hat{B}^2}{L} - \alpha \hat{B}\hat{N} - \alpha_1 \hat{B}^2 \hat{N} + k\hat{F} = 0, \tag{4.1}$$

$$r - \frac{r_0 \hat{N}}{K} + \alpha \hat{B} + \alpha_1 \hat{B}^2 = 0, \tag{4.2}$$

$$\pi\alpha \hat{B}\hat{N} + \pi_1 \alpha_1 \hat{B}^2 \hat{N} - \pi_0 \hat{F} = 0. \tag{4.3}$$

From equations (4.2) and (4.3), we get

$$\hat{N} = \frac{K\left\{r + \left(\alpha + \alpha_1 \hat{B}\right)\hat{B}\right\}}{r_0}, \; \hat{F} = \frac{\left(\pi\alpha + \pi_1 \alpha_1 \hat{B}\right)\hat{B}K\left\{r + \left(\alpha + \alpha_1 \hat{B}\right)\hat{B}\right\}}{\pi_0 r_0}. \tag{4.4}$$

Using the value of $\hat{N}$ and $\hat{F}$ in the equation (4.1), function $H_1\left(\hat{B}\right)$ can be defined

$$\begin{aligned} H_1\left(\hat{B}\right) = {} & \alpha_1^2 KL(\pi_0 - \pi_1 k)\hat{B}^3 + \alpha_1 \alpha KL\{(\pi_0 - \pi k) + (\pi_0 - \pi_1 k)\}\hat{B}^2 + \left[\pi_0 r_0 s_0 + L\alpha_1 Kr(\pi_0 - \pi_1 k) + KL\alpha^2 \right. \\ & \left. (\pi_0 - \pi k)\right]\hat{B} - Lr_0 s\pi_0 + \alpha KLr(\pi_0 - \pi k) = 0. \end{aligned} \tag{4.5}$$

Now from (4.5) we get

$$H_1(0) = -\pi_0 L(sr_0 - \alpha rK) - \alpha KLr\pi k < 0, \text{ due to (3.14)} \tag{4.6}$$

$$\begin{aligned} H_1\left(L^*\right) = {} & \alpha_1^2 KL(\pi_0 - \pi_1 k)L^{*3} + \alpha_1 \alpha KL\{(\pi_0 - \pi k) + (\pi_0 - \pi_1 k)\}L^{*2} + \left[\pi_0 r_0 s_0 + L\alpha_1 Kr(\pi_0 - \pi_1 k) + KL\alpha^2 \right. \\ & \left. (\pi_0 - \pi k)\right]L^* - Lr_0 s\pi_0 + \alpha KLr(\pi_0 - \pi k) > 0. \end{aligned} \tag{4.7}$$

where $L^* = \dfrac{sL}{s_0}$.

This implies that there is a root $\hat{B}$ in the region $(0, L^*)$ such that $H_1(\hat{B}) = 0$.

Now, the sufficient condition for the uniqueness of E_3 is $H_1'(\hat{B}) > 0$.For this we find $H_1'(\hat{B})$ from (4.5) as follows.

$$\begin{aligned} H_1'(\hat{B}) = {} & 3LK\alpha_1^2(\pi_0 - \pi_1 k)\hat{B}^2 + 2\alpha_1\alpha KL[(\pi_0 - \pi_1 k) + (\pi_0 - \pi k)]\hat{B} \\ & + \pi_0 r_0 s_0 + L\alpha_1 Kr(\pi_0 - \pi_1 k) + \alpha^2 KL(\pi_0 - \pi k) > 0. \end{aligned} \tag{4.8}$$

Using (4.5) we can check that $H_1'(\hat{B})$ is positive.

This completes the proof of existence of E_3.

Existence of $E_4(\overline{B}, 0, \bar{I}, \overline{F})$

Here $\overline{B}, \bar{I}$ and $\overline{F}$ are the positive solutions of the following algebraic equations

$$s\overline{B} - \frac{s_0 \overline{B}^2}{L} - \beta \overline{B}\bar{I} - \beta_1 \overline{B}^2 \bar{I} + k\overline{F} = 0, \tag{4.9}$$

$$\beta\overline{B} + \beta_1 \overline{B}^2 - \beta_0 \bar{I} - \theta_0 = 0, \tag{4.10}$$

$$\phi\beta \overline{B}\bar{I} + \phi_1\beta_1 \overline{B}^2 \bar{I} - \pi_0 \overline{F} = 0. \tag{4.11}$$

From equations (4.10) and (4.11), we get

$$\bar{I} = \frac{\beta\overline{B} + \beta_1\overline{B}^2 - \theta_0}{\beta_0}, \; \overline{F} = \frac{(\phi\beta + \phi_1\beta_1\overline{B})\overline{B}(\beta\overline{B} + \beta_1\overline{B}^2 - \theta_0)}{\pi_0\beta_0}.$$

With condition $\beta \overline{B} + \beta_1 \overline{B}^2 > \theta_0$. (4.12)

Using the value of $\bar{I}$ and $\overline{F}$ in the equation (4.9), we define the function $H_2'(\overline{B})$ as follows,

$$H_2(\overline{B}) = \beta_1^2 L(\pi_0 - \phi_1 k)\overline{B}^3 + L\beta\beta_1[(\pi_0 - \phi_1 k) + (\pi_0 - \phi k)]\overline{B}^2 + [\pi_0\beta_0 s_0 + L\beta^2(\pi_0 - \phi k) - \beta_1\theta_0 L(\pi_0 - \phi_1 k)]\overline{B}$$
$$- L\pi_0 \beta_0 s - \beta_1\theta_0 L(\pi_0 - \phi k) = 0. \tag{4.13}$$

From (4.13) we have,

$$H_2(0) = -L\pi_0\beta_0 s - \beta_1\theta_0 L(\pi_0 - \phi k) < 0, \tag{4.14}$$

and

$$H_2(L^*) = (\pi_0 - \phi_1 k)\beta_1 L L^*[\beta_1 L^{*2} - \theta + \beta L^*] + (\pi_0 - \phi k)L[\beta_1(L^{*2}\beta - \theta_0) + \beta L^{*2}]. \tag{4.15}$$

with conditions $\pi_0 > \phi_1 k, \pi_0 > \phi k, L^{*2}\beta > \theta_0, L^{*2}\beta_1 > \theta_0$. (4.16)

Thus, there exists a $\overline{B}$, $0 < \overline{B} < L^*$, such that $H_2(\overline{B}) = 0$.

Now, the sufficient condition for the uniqueness of E_4 is $H_2'(\overline{B}) > 0$. For this, we find $H_2'(\overline{B})$ as follows which is positive,

$$H_2'(\overline{B}) = (\pi_0 - \phi_1 k)\beta_1 L(2\beta\overline{B} - \theta + 3\beta_1\overline{B}^2) + L\beta(\pi_0 - \phi k)(2\beta_1\overline{B} + \beta) > 0. \tag{4.17}$$

This completes the proof of existence of E_4.

Existence of $E_5(B^*, N^*, I^*, F^*)$

To determine B^*, N^*, I^*, F^* we get the following equations from (2.1)

$$sB^* - \frac{s_0 B^{*2}}{L} - \alpha B^* N^* - \alpha_1 B^{*2} N^* - \beta\, B^* I^* - \beta_1 B^{*2} I^* + kF^* = 0, \tag{4.18}$$

$$r - \frac{r_0 N^*}{K} + \alpha B^* + \alpha_1 B^{*2} = 0, \tag{4.19}$$

$$\beta B^* + \beta_1 B^{*2} - \beta_0 I^* - \theta_0 = 0, \tag{4.20}$$

$$\pi\alpha\, B^* N^* + \pi_1 \alpha_1 B^{*2} N^* + \phi\beta\, B^* I^* + \phi_1 \beta_1 B^{*2} I^* - \pi_0 F^* = 0. \tag{4.21}$$

From equations (4.19), (4.20) and (4.21), we get

$$N^* = \frac{K\left(r + \alpha B^* + \alpha_1 B^{*2}\right)}{r_0},\ I^* = \frac{\left(\beta\, B^* + \beta_1 B^{*2} - \theta_0\right)}{\beta_0}, \tag{4.22}$$

$$F^* = \frac{K\left(\pi\,\alpha\, B^* + \pi_1 \alpha_1 B^{*2}\right)\left(r + \alpha\, B^* + \alpha_1\, B^{*2}\right)}{\pi_0 r_0} + \frac{\left(\phi\beta\, B^* + \phi_1 \beta_1 B^{*2}\right)\left(\beta\, B^* + \beta_1 B^{*2} - \theta_0\right)}{\pi_0 \beta_0}. \tag{4.23}$$

Using the value of N^*, I^* and F^* in the equation (4.18), we get

$$\begin{aligned} H_3\left(B^*\right) = {} & \left[KL\alpha_1^2 \beta_0 \left(\pi_0 - \pi_1 k\right) + \beta_1^2 L r_0 \left(\pi_0 - \phi_1 k\right)\right] B^{*3} + \left[\alpha\alpha_1 KL\beta_0 \left(\pi_0 - \pi\, k\right) + \alpha\alpha_1 KL\beta_0 \left(\pi_0 - \pi_1\, k\right)\right. \\ & \left. + \beta\beta_1 L r_0 \left(\pi_0 - \phi\, k\right) + \beta\beta_1 L r_0 \left(\pi_0 - \phi_1\, k\right)\right] B^{*2} + \left[s_0 r_0 \beta_0 \pi_0 + KLr\alpha_1 \beta_0 \left(\pi_0 - \pi_1 k\right) + KL\alpha^2 \beta_0 \left(\pi_0 - \pi\, k\right) + \right. \\ & \left. \beta^2 L r_0 \left(\pi_0 - \phi\, k\right) - L r_0 \theta_0 \beta_1 \left(\pi_0 - \phi_1\, k\right)\right] B^* + KL\alpha\, r\, \beta_0 \left(\pi_0 - \pi\, k\right) \\ & - sLr_0 \beta_0 \pi_0 - L r_0 \theta_0 \beta \left(\pi_0 - \phi\, k\right) = 0. \end{aligned} \tag{4.24}$$

From (4.24)

$$H_3(0) = -\beta_0 .\pi_0 \left(sr_0 - \alpha\, rK\right) - Kr\alpha\, \beta_0 \pi\, k - r_0 \theta_0 \beta \left(\pi_0 - \phi\, k\right) < 0, \quad \text{due to (3.14).} \tag{4.25}$$

And

$$H_3\left(L^*\right)=\left(\pi_0-\pi_1 k\right)LL^* K\beta_0\alpha_1\left[L^{*2}\alpha_1+L^*\alpha+r\right]+\left(\pi_0-\phi_1 k\right)L\,L^*\beta_1\left[r_0\left(\beta_1 L^{*2}-\theta_0\right)+\beta\pi_0 L^*\right]$$
$$+KL\alpha\,\beta_0\left(\pi_0-\pi\,k\right)\left[L^{*2}\alpha_1+L^*\alpha+r\right]+\left(\pi_0-\phi\,k\right)L\,r_0\,\beta\left[L^{*2}\beta_1-\theta_0+L^*\beta\right]>0.$$

(4.26)

with condition (4.16).

Thus there exists a $B^*,\ 0<B^*<L^*,$ such that $H_3(B^*)=0.$

Now, the sufficient condition for the uniqueness of E_5 is $H_3'\left(B^*\right)>0$ .From (4.24) we can find and $H_3'\left(B^*\right)$ as follows,

$$H_3'\left(B^*\right)=KL\alpha_1\beta_0\left(\pi_0-\pi_1 k\right)\left[3B^{*2}\alpha_1+2B^*\alpha+r\right]+L\beta_1\left(\pi_0-\phi_1 k\right)\left[\left(3B^{*2}\beta_1-\theta_0\right)r_0+\beta\pi_0\right]$$
$$+LK\beta_0\alpha\left(\pi_0-\pi\,k\right)\left[2B^*\alpha_1+\alpha\right]+\left(\pi_0-\phi\,k\right)r_0\left[\beta\,\beta_1 L+L\beta^2+s_0\,\beta_0\,\pi_0\right]>0. \qquad (4.27)$$

By using (4.22) in (4.25) we can check that $H_3'\left(B^*\right)$ is positive.

Hence the existence of E_5.

5. Stability Analysis of the System

Local Stability

To discuss the local stability of system (2.1) as follows,

$$V(E)=\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & 0 & 0 \\ a_{31} & 0 & a_{33} & 0 \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}, \qquad (5.1)$$

where,

$$a_{11} = s - \frac{2s_0 B}{L} - \alpha N - 2\alpha_1 BN - \beta\, I - 2\beta_1 B\, I,\; a_{12} = -\alpha\, B - \alpha_1\, B^2,\; a_{13} = -\beta\, B - \beta_1\, B^2,\; a_{14} = k, \quad (5.2)$$

$$a_{21} = \alpha N + 2\alpha_1 BN,\; a_{22} = r - \frac{2r_0 N}{K} + \alpha B + \alpha_1 B^2,\; a_{31} = \beta I + 2\beta_1 BI,\; a_{33} = \beta B + \beta_1 B^2 - 2\beta_0 I - \theta_0, \; (5.3)$$

$$a_{41} = \pi\alpha N + 2\pi_1\alpha_1 BN + \phi\beta\, I + 2\phi_1\beta_1 BI,\; a_{42} = \pi\alpha\, B + \pi_1\alpha_1 B^2,\; a_{43} = \phi\beta\, B + \phi_1\beta_1 B^2,\; a_{44} = -\pi_0. \; (5.4)$$

We state the following results for $E_i, i = 0,1,2,3,4,5$.

1. E_0 is a saddle point with unstable manifold in $B-N$ plane and stable manifold in $I-F$ plane.
2. E_1 is unstable manifold in $N-I$ plane with condition (4.16) and stable manifold in $B-F$ plane.
3. E_2 is unstable manifold in $B-F$ plane if $s > \pi_0 + \frac{\alpha r K}{r_0}$ otherwise it becomes stable manifold in $B-N-I-F$ plane.
4. E_3 is unstable in $I-$direction and equilibrium point E_4 is unstable in $N-$direction.

The local stability behaviour of equilibrium point E_5 is not clear from (5.1). Hence we use Lyapunov method to prove its stability.

Theorem (5.1) Let the following inequality holds

$$\left[k + M_{\min}\left(\pi\alpha\, N^* + 2\pi_1\alpha_1 B^* N^* + \phi\beta\, I^* + 2\phi_1\beta_1 B^* I^*\right)\right]^2 < \frac{4\pi_0 M_0}{3}\left(\frac{kF^*}{B^*} + \frac{s_0 B^*}{L} + \alpha_1 B^* N^* + \beta_1 B^* I^*\right). \quad (5.5)$$

Then equilibrium point E_5 is locally asymptotically stable. For proof see Appendix A.

Global Stability

The following theorem characterizes the global stability behavior of equilibrium point E_5.

Theorem (5.2) Let the following inequalities hold:

$$\alpha_1^2 \frac{A^2}{\delta^2} < \frac{2r_0 s_0}{3KL}, \tag{5.6}$$

$$\beta_1^2 \frac{A^2}{\delta^2} < \frac{2\beta_0 s_0}{3L}, \tag{5.7}$$

$$\left[\frac{k}{B^*} + C_{\min}\left\{\pi\alpha A_1 + \phi\beta\, A_2 + \left(\pi_1\alpha_1 A_1 + \phi_1\beta_1 A_2\right)\left(\frac{A}{\delta} + B^*\right)\right\}\right]^2 < \frac{4C_0\pi_0 s_0}{9L}. \tag{5.8}$$

Then equilibrium point E_5 is globally stable in the region Ω. For proof, see Appendix B.

6. Persistence of the System

Theorem (6.1) Assume that $s > \alpha\, A_1 + \beta\, A_2$ and $\beta\, B_{\min} + \beta_1 B_{\min}^2 > \theta_0$. Here A, A_1 and A_2 are upper bounds of the populations B, F, N, I respectively and are always positive. Then system (2.1) persists.

Proof: The system (2.1) gives

$$\frac{dB}{dt} = sB - \frac{s_0 B^2}{L} - \alpha BN - \alpha_1 B^2 N - \beta BI - \beta_1 B^2 I + kF, \tag{6.1}$$

$$\frac{dB}{dt} \geq \left(s - \alpha\, A_1 - \beta\, A_2\right)B - \left(\frac{s_0}{L} + \alpha_1 A_1 + \beta_1 A_2\right)B^2. \tag{6.2}$$

According to lemma (3.1), it follows that

$$\lim_{t\to\infty}\inf\ B \geq \frac{L(s-\alpha\ A_1-\beta\ A_2)}{s_0+(\alpha_1 A_1+\beta_1 A_2)L}. \tag{6.3}$$

$$B_{\min} = \frac{L(s-\alpha\ A_1-\beta\ A_2)}{s_0+(\alpha_1 A_1+\beta_1 A_2)L}. \tag{6.4}$$

For $s > \alpha\ A_1 + \beta\ A_2$ and $B_{\min}$ remains always positive.

Again from system (2.1) and equation (6.4), we have

$$\frac{dN}{dt} \geq (r+\alpha B_{\min}+\alpha_1 B_{\min}^2)N - \frac{r}{K}N^2. \tag{6.5}$$

According to lemma (3.1) and comparison principle, it follows that

$$\lim_{t\to\infty}\inf\ N \geq \frac{(r+\alpha B_{\min}+\alpha_1 B_{\min}^2)K}{r}. \tag{6.6}$$

$$N_{\min} = \frac{(r+\alpha B_{\min}+\alpha_1 B_{\min}^2)K}{r}. \tag{6.7}$$

From the system (2.1) and equation (6.4), get

$$\lim_{t\to\infty}\inf\ I \geq \frac{(\beta B_{\min}+\beta_1 B_{\min}^2-\theta_0)}{\beta_0}. \tag{6.8}$$

$$I_{\min} = \frac{(\beta\ B_{\min}+\beta_1 B_{\min}^2-\theta_0)}{\beta_0}. \tag{6.9}$$

For $\beta B_{\min} + \beta_1 B_{\min}^2 > \theta_0$, $I_{\min}$ remains always positive.

From the last equation of the system (2.1) and equations (6.4), (6.7), (6.9), we get

$$\frac{dF}{dt} \geq S_1 - \pi_0 F. \tag{6.10}$$

Where

$$S_1 = \frac{B_{\min}}{r_0\beta_0}\left[\left(\pi\alpha_1^2 K\beta_0 + \phi_1\beta_1^2 r_0\right)B_{\min}^3 + \left\{\left(\pi+\pi_1\right)\alpha\alpha_1 K\beta_0 + \left(\phi+\phi_1\right)\beta\beta_1 r_0\right\}B_{\min}^2 + \right.$$
$$\left.\left\{\left(\pi\alpha^2 + \pi_1\alpha_1 r\right)K\beta_0 + \left(\phi\beta^2 - \theta_0\phi_1\beta_1\right)r_0\right\}B_{\min} + r\pi\alpha\, K\beta_0 - \phi\beta^2\theta_0 r_0\right].$$

Using the lemma (3.1) we get

$$\liminf_{t\to\infty} F \geq \frac{S_1}{\pi_0}.$$

$$F_{\min} = \frac{S_1}{\pi_0}.$$

Hence the theorem.

7. Numerical Simulation

The stability of the equilibrium E_5 is studied by using the following set of parameters.

$$s = 20, s_0 = 4, L = 200, \alpha = (0, 0.3), \alpha_1 = (0, 0.8), \beta = (0, 0.3), \beta_1 = (0,1), k = 30, r = 0.2, r_0 = 4,$$
$$K = 100, \beta_0 = 0.8, \pi = 0.8, \pi_1 = 0.8, \phi = 0.8, \phi_1 = 0.8, \pi_0 = 50, \theta_0 = 2, \delta = 1. \tag{7.1}$$

The equilibrium E_5 corresponding to these parameters values $\alpha = 0.000002, \alpha_1 = 0.0003,$ $\beta = 0.00003, \beta_1 = 0.004$ and others are same as (7.1), $E_5(116.27,\ 106.397,\ \ 65.0986,\ 63.2318)$.

The characteristic polynomial and characteristic roots of the model system corresponding to E_5 is found as follows.

$$\lambda^4 + 158.963\lambda^3 + 10307.9\lambda^2 + 180508\lambda + 624750 = 0. \tag{7.2}$$

$$\lambda_1 = -19.6376, \lambda_{2,3} = -67.3755 \pm 49.1539\ i, \lambda_4 = -4.57389. \tag{7.3}$$

This shows stable local behavior of E_5 with time numerically. Figure (1) shows the global stability of E_5. Figure (2) and Figure (3) show the change in absence and presence of conservation. It is noted that as the values of α and α_1 increase the density of forestry biomass decreases in the absence of conservation but in presence of conservation the forestry biomass can be maintained to its original level. Figure (4) and Figure (5) show the change in B with t . From these Figures it is seen that as β and β_1 increase B decreases but it increases as k increases. From this discussion, it can be concluded that B can be maintained at its original level before depletion by conservation.

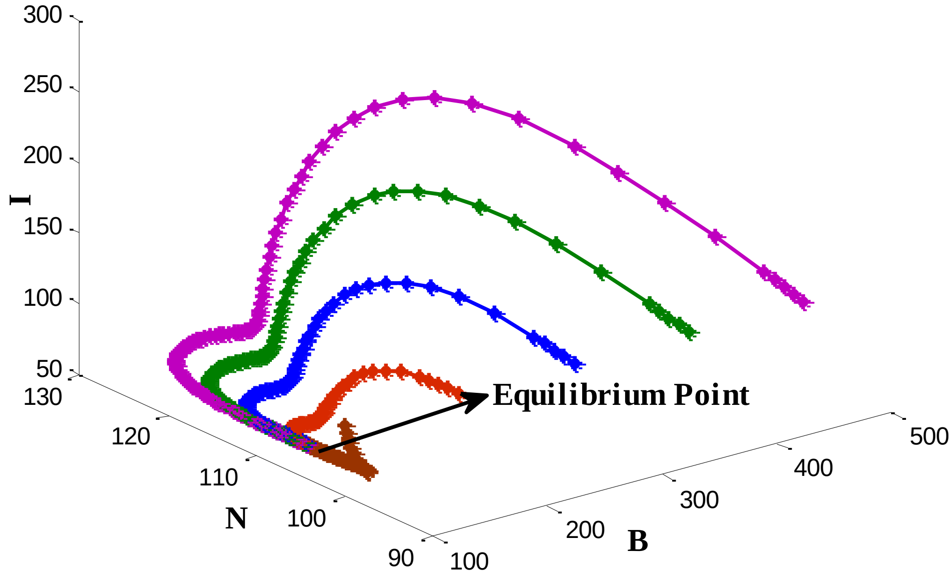

Figure 1. Phase space trajectories of forestry biomass, population and industrialization (E_5) .

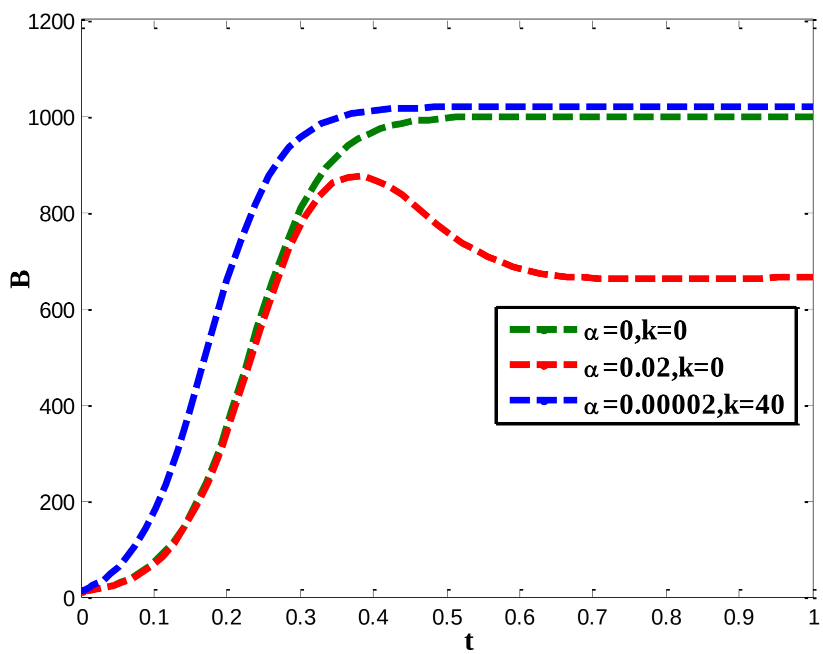

Figure 2. Variation of B with t for different values of α and k.

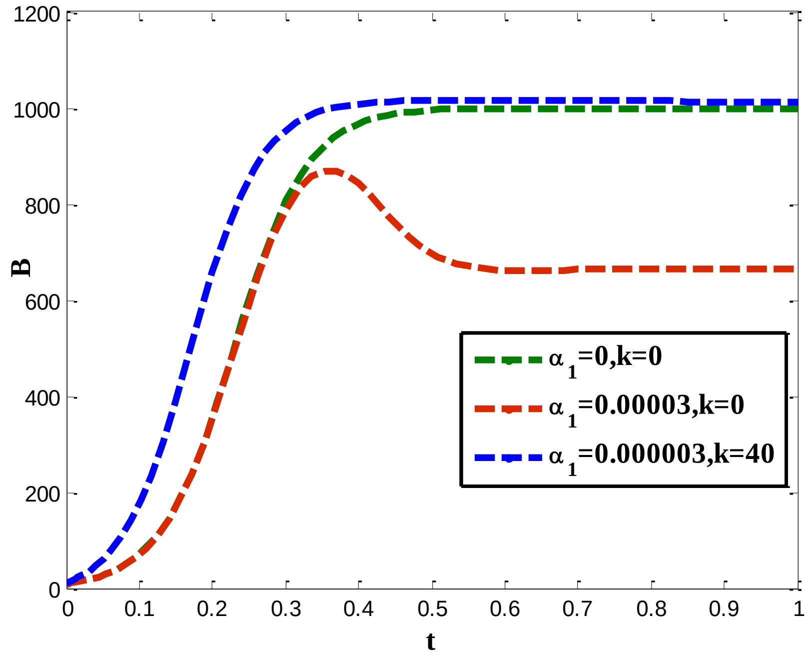

Figure 3. Variation of B with t for different values of α_1 and k.

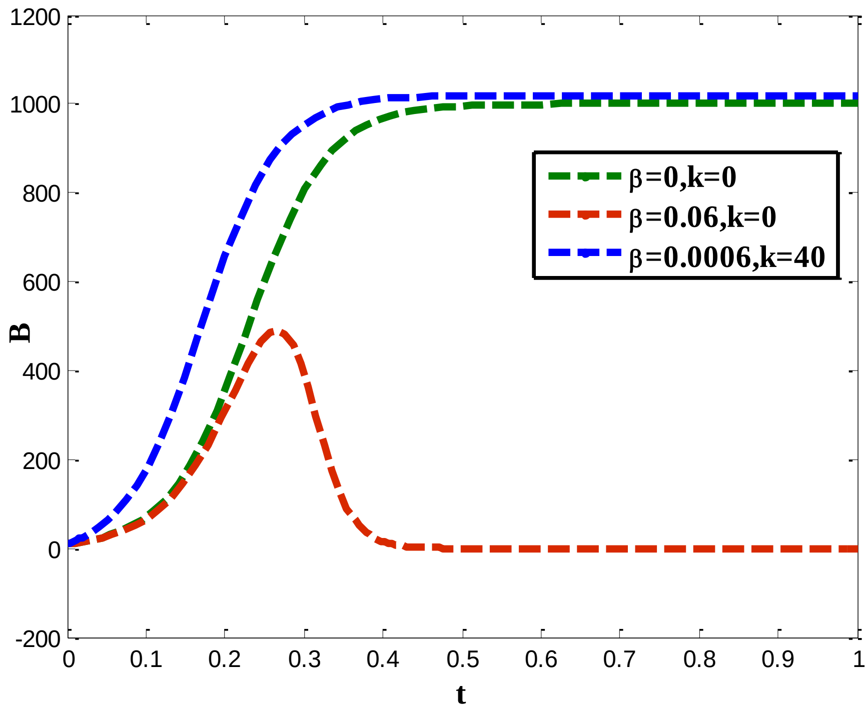

Figure 4. Variation of B with t for different values of β and k.

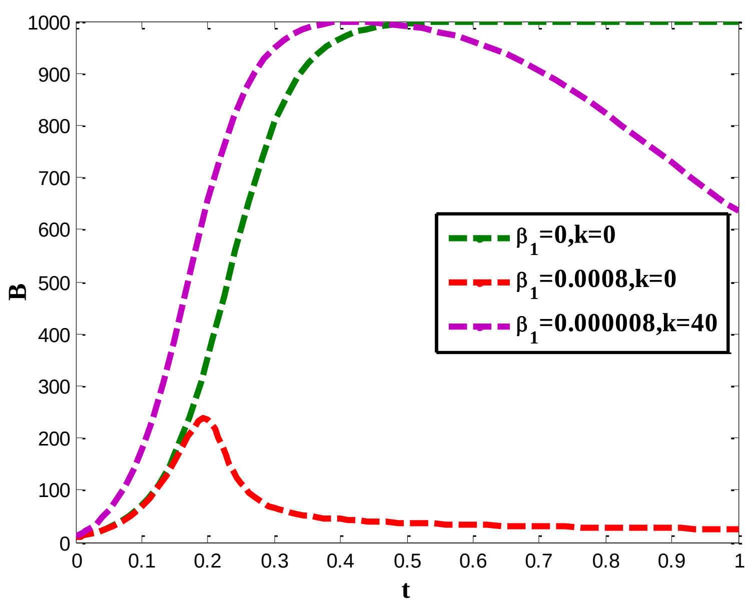

Figure 5. Variation of B with t for different values of β_1 and k.

8. Conclusions

Here a nonlinear mathematical model for depletion of forestry resources caused by population and industrialization has been modelled and analyzed. The effect of conservation on forestry resources has also been modeled and studied. The assumptions regarding the depletion of forestry resources by human population and industrialization are similar to by Shukla, et al. (2011). The conservation model is new and original as the growth rate of commutative biomass density of resources in green belts has been assumed to be proportional to the rate of depletion of commutative biomass density of forest resources.

The system has been assumed to be governed by the following variables

1. The cumulative biomass density of forest resources.
2. The population density of human population.
3. The density of industrialization.
4. The cumulative biomass density of green belts plantation.

The model has been derived by considering the interactions of all the above variables. By using the theorems of nonlinear ordinary differential equations and numerical analysis the model has been analysed.

The analysis of model shows that the forest resources are doomed to extinction in the absence of conservation. In the presence of conservation, however, the forest resources can be maintained to it original level.

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Appendix A. Proof of Theorem (5.1)

We first linearize model (2.1) by substituting

$$B = B^* + b, N = N^* + n, I = I^* + i, F = F^* + f,$$

where b, n, i, f are small perturbations around equilibrium point E_5. Thus we find the following linearized system

$$\frac{db}{dt} = -\left(\frac{kF^*}{B^*} + \alpha_1 B^* N^* + \beta_1 B^* I^*\right)b - \left(\alpha B^* + \alpha_1 B^{*2}\right)n - \left(\beta B^* + \beta_1 B^{*2}\right)i + kf,$$

$$\frac{dn}{dt} = \left(\alpha N^* + 2\alpha_1 B^* N^*\right)b - \frac{r_0 N^*}{K}n, \tag{A.1}$$

$$\frac{di}{dt} = \left(\beta I^* + 2\beta_1 B^* I^*\right)b - \beta_0 I^* i,$$

$$\frac{df}{dt} = \left(\pi\alpha N^* + 2\pi_1\alpha_1 B^* N^* + \phi\beta I^* + 2\phi_1\beta_1 B^* I^*\right)b + \left(\pi\alpha B^* + \pi_1\alpha_1 B^{*2}\right)n + \left(\phi\beta B^* + \phi_1\beta_1 B^{*2}\right)i - \pi_0 f.$$

Then we consider the following positive definite function:

$$W = \frac{p_1}{2}b^2 + \frac{p_2}{2}n^2 + \frac{p_3}{2}i^2 + \frac{p_4}{2}f^2, \tag{A.2}$$

where p_1, p_2, p_3 and p_4 are positive constants.

From equations (A.1) and (A.2) we get

$$\frac{dW}{dt} = -w_{11}b^2 + w_{12}bn + w_{13}bi + w_{14}bf + w_{24}nf + w_{34}fi - w_{22}n^2 - w_{33}i^2 - w_{44}f^2. \tag{A.3}$$

where

$$w_{11} = \left\{\frac{kF^*}{B^*} + \frac{s_0 B^*}{L} + \alpha_1 B^* N^* + \beta_1 B^* I^*\right\} p_1,\; w_{12} = \left\{\left(\alpha N^* + 2\alpha_1 B^* N^*\right) p_2 - \left(\alpha B^* + \alpha_1 B^{*2}\right) p_1\right\},$$

$$w_{13} = \left\{\left(\beta I^* + 2\beta_1 B^* I^*\right) p_3 - \left(\beta B^* + \beta_1 B^{*2}\right) p_1\right\},$$
$$w_{14} = p_1 k + p_4\left(\pi\alpha N^* + 2\pi_1\alpha_1 B^* N^* + \phi\beta\, I^* + 2\phi_1\beta_1 B^* I^*\right),\; w_{22} = \frac{p_2 r_0 N^*}{K},\; w_{33} = \beta_0 p_3 I^*,$$
$$w_{44} = p_4\pi_0,\; w_{24} = p_4\left(\pi\alpha B^* + \pi_1\alpha_1 B^{*2}\right),\; w_{34} = p_4\left(\phi\beta\, B^* + \phi_1\beta_1 B^{*2}\right).$$

From equation (4.16) it is noted that w_{11} is positive.

First we choose $p_2 = \dfrac{\left(\alpha B^* + \alpha_1 B^{*2}\right) p_1}{\alpha N^* + 2\alpha_1 B^* N^*}$, $p_3 = \dfrac{\left(\beta B^* + \beta_1 B^{*2}\right) p_1}{\beta I^* + 2\beta_1 B^* I^*}$.

Then $\dfrac{dW}{dt}$ to be negative definite, the following inequalities must hold

$$\left[p_1 k + p_4\left(\pi\alpha\, N^* + 2\pi_1\alpha_1 B^* N^* + \phi\beta\, I^* + 2\phi_1\beta_1 B^* I^*\right)\right]^2 < \frac{4\pi_0 p_1 p_4}{3}\left(\frac{kF^*}{B^*} + \frac{s_0 B^*}{L} + \alpha_1 B^* N^* + \beta_1 B^* I^*\right), \tag{A.4}$$

From (A.4), we get following inequalities

$$p_4 > \frac{3p_1k^2}{2\pi_0\left(\frac{kF^*}{B^*}+\frac{s_0B^*}{L}+\alpha_1B^*N^*+\beta_1B^*I^*\right)} = p_1M_0, \tag{A.5}$$

$$p_4 < \frac{2\pi_0 p_1}{\left(\pi\alpha\ N^* + 2\pi_1\alpha_1B^*N^* + \phi\beta\ I^* + 2\phi_1\beta_1B^*I^*\right)}\left(\frac{kF^*}{B^*}+\frac{s_0B^*}{L}+\alpha_1B^*N^*+\beta_1B^*I^*\right) = p_1M_1, \tag{A.6}$$

$$p_4 < \frac{4\pi_0r_0N^*\left(\alpha B^* + \alpha_1B^{*2}\right)p_1}{3K\left(\pi\alpha\ B^* + \pi_1\alpha_1B^{*2}\right)^2\left(\alpha N^* + 2\alpha_1B^*N^*\right)} = p_1M_2, \tag{A.7}$$

$$p_4 < \frac{4\beta_0I^*\left(\beta B^* + \beta_1B^{*2}\right)p_1}{3\left(\phi\beta\ B^* + \phi_1\beta_1B^{*2}\right)^2\left(\beta I^* + 2\beta_1B^*I^*\right)} = p_1M_3, \tag{A.8}$$

Then p_4 can be chosen as

$$p_4 < p_1\ Min\left(M_1, M_2, M_3\right) = p_1M_{\min}. \tag{A.9}$$

From (A.4), the condition $\frac{dW}{dt}$ would be negative definite for all arbitrary values of p_1 if the following condition is satisfied

$$\left[k + M_{\min}\left(\pi\alpha\ N^* + 2\pi_1\alpha_1B^*N^* + \phi\beta\ I^* + 2\phi_1\beta_1B^*I^*\right)\right]^2 <$$

$$\frac{4\pi_0M_0}{3}\left(\frac{kF^*}{B^*}+\frac{s_0B^*}{L}+\alpha_1B^*N^*+\beta_1B^*I^*\right).$$

As stated in Theorem (5.1).

Appendix B. Proof of Theorem (5.2)

For finding the condition of global stability at E_5 we construct the Lyapunov function

$$H = \frac{m_1}{2}\left(B - B^* - B^* \ln\frac{B}{B^*}\right) + m_2\left(N - N^* - N^* \ln\frac{N}{N^*}\right) + m_3\left(I - I^* - I^* \ln\frac{I}{I^*}\right) + \frac{m_4}{2}\left(F - F^*\right)^2. \tag{B.1}$$

Where m_1, m_2, m_3 and m_4 are positive constants to be chosen suitably.

Differentiating H with respect to time t along the solutions of model (2.1), we get

$$\frac{dH}{dt} = m_1 \frac{\left(B - B^*\right)}{B}\frac{dB}{dt} + \frac{m_2\left(N - N^*\right)}{N}\frac{dN}{dt} + \frac{m_3\left(I - I^*\right)}{I}\frac{dI}{dt} + m_4\left(F - F^*\right)\frac{dF}{dt}. \tag{B.2}$$

Using system of equation (2.1) in (B.2), we get after simplification,

$$\begin{aligned}\frac{dH}{dt} = & -m_1\left\{\frac{s_0}{L} + \alpha_1 N + \beta_1 I + \frac{kF}{BB^*}\right\}\left(B - B^*\right)^2 - \frac{m_2 r_0}{K}\left(N - N^*\right)^2 - m_3\beta_0\left(I - I^*\right)^2 - m_4\pi_0\left(F - F^*\right)^2 + \\ & \left\{m_2\left(\alpha + \alpha_1\left(B + B^*\right)\right) - m_1\left(\alpha + \alpha_1 B^*\right)\right\}\left(B - B^*\right)\left(N - N^*\right) + \left\{m_3\left(\beta + \beta_1\left(B + B^*\right)\right) - m_1\left(\beta + \beta_1 B^*\right)\right\} \\ & \left(B - B^*\right)\left(I - I^*\right) + \left\{\frac{m_1 k}{B^*} + m_4\left(\pi\alpha\, N + \left(\pi_1\alpha_1 N + \phi_1\beta_1 I\right)\left(B + B^*\right) + \phi\beta\, \mathrm{I}\right)\right\}\left(B - B^*\right)\left(F - F^*\right) + \\ & m_4\left(\pi\alpha\, B^* + \pi_1\alpha_1 B^{*2}\right)\left(N - N^*\right)\left(F - F^*\right) + m_4\left(\phi\beta\, B^* + \phi_1\beta_1 B^{*2}\right)\left(I - I^*\right)\left(F - F^*\right).\end{aligned}$$

Here choosing $m_1 = m_2 = m_3 = 1$.

The $\frac{dH}{dt}$ to be negative definite, the following inequalities hold

$$\alpha_1^2 \frac{A^2}{\delta^2} < \frac{2r_0 s_0}{3KL}, \tag{B.3}$$

$$\beta_1^2 \frac{A^2}{\delta^2} < \frac{2\beta_0 s_0}{3L}, \tag{B.4}$$

$$\left[\frac{k}{B^*} + m_4\left\{\pi\alpha A_1 + \phi\beta\ A_2 + \left(\pi_1\alpha_1 A_1 + \phi_1\beta_1 A_2\right)\left(\frac{A}{\delta} + B^*\right)\right\}\right]^2 < \frac{4m_4\pi_0 s_0}{9L}, \tag{B.5}$$

From (B.5), we have following results

$$m_4 > \frac{9L}{2\pi_0 s_0}\left(\frac{k}{B^*}\right)^2 = C_0, \tag{B.6}$$

$$m_4 < \frac{2\pi_0 s_0}{9L\left\{\pi\alpha A_1 + \phi\beta\ A_2 + \left(\pi_1\alpha_1 A_1 + \phi_1\beta_1 A_2\right)\left(\frac{A}{\delta} + B^*\right)\right\}^2} = C_1, \tag{B.7}$$

$$m_4 < \frac{2\pi_0 r_0}{3K\left(\pi\alpha\ B^* + \pi_1\alpha_1 B^{*2}\right)^2} = C_2, \tag{B.8}$$

$$m_4 < \frac{2\pi_0\beta_0}{3\left(\phi\beta\ B^* + \phi_1\beta_1 B^{*2}\right)^2} = C_3, \tag{B.9}$$

Now we choose m_4 as

$$m_4 = Min\left(C_1, C_2, C_3\right) = C_{\min}. \tag{B.10}$$

From (B.5), we get

$$\left[\frac{k}{B^*} + C_{\min}\left\{\pi\alpha A_1 + \phi\beta\ A_2 + \left(\pi_1\alpha_1 A_1 + \phi_1\beta_1 A_2\right)\left(\frac{A}{\delta} + B^*\right)\right\}\right]^2 < \frac{4C_0\pi_0 s_0}{9L}. \tag{B.11}$$

Then, $\frac{dH}{dt}$ is negative definite under conditions (B.3), (B.4) and (B.11) in region Ω as stated in the Theorem (5.2).

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Modeling the Depletion of Forest Resources due to Population and Industrialization and Its Effect on Survival of Wildlife Species

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Abstract

In this paper, by using a mathematical model, the depletion of forest resources by human population and industrialization and its effect on survival of wildlife species wholly depends upon various forest resources are studied. In the modeling process, we assume that the growth rate of the commutative density of wildlife species wholly depend on the commutative density of forest resources and is governed by a logistic equation, the carrying capacity of which is depleted by population and industrialization. The local and global stability of the interior equilibrium point as well as the persistence of the system are analyzed using the theory of nonlinear ordinary differential equations. The model analysis shows that due to deforestation caused by population and industrialization, the forest resources are driven to extinction and due to this the survival of wildlife species is very much threatened. These analytical results are confirmed by using numerical simulation.

Keywords: **Forestry Biomass, Population, Industrialization, Wildlife, Stability, Persistence.**

1. Introduction

Humans have exploited contaminated or somehow touched every part of the planet. The total impact of people on the planet, sometimes called the human footprint, has greatly intensified over the past couple of centuries, largely because of explosive human population growth.Forests are extensively affected by human activities. For example, slash-and-burn shifting agriculture has been a common practice within Amazonian rainforests for the past 3,500 years. The forests cover about one-third of Earth's land area and are home to most of the terrestrial plants and animal species. It is important to recognize that forests are not just a habitat for biodiversity; they are also a major source of fuel and building materials, furniture etc. Today, many developing countries are logging or burning their forests to make way for agriculture or for industrialization. It is not surprising, then, that deforestation of forest resources can have serious effect on biodiversity and threat to wildlife species. Repeatedly, scientists have seen that deforestation can spell doom for the species that rely upon forest habitats.For example, the Eurasian red squirrel (Sciurus vulgaris) requires forest habitat. Historically, Britain was dominated by forests, and the red squirrel was once common and widespread throughout the region. However, because much of the land has been converted to agriculture, the remaining patches of native forest are the remaining patches of native forest are severely fragmented, and the red squirrel is now rare (Trivedi, 2001). Another example of Singapore, 95% of the forest area has been lost since the British colonization in 1819.A comparison of historical and modern checklists of species indicates that 26% of plant species, 34%

of bird species, and 42% of mammal species that were originally recorded on this island are now extinct because its forests were replaces by human land uses (Brook et al.2003).

The depletion of forest biomass by human population and industrialization has been investigated both theoretically as well as experimentally by many researchers (Rawat and Ginwal 2009; Dubey and Narayanan 2010; Agarwal et al. 2010; Agarwal and Devi 2012; Chaudhary et al. 2013; Misra et al. 2014) .In particular, Shukla et al. (1988, 1989) have proposed a mathematical model for the depletion of forest resources where they have shown that the forest resources may be doomed to extinction under growing population living in the forest or by industrialization which is wholly depends on forest resources. Shukla's results have also been found by other researchers showing their concern about biodiversity caused by deforestation. Agarwal and Pathak (2015) have proposed and analyzed a mathematical model for conservation of forestry biomass and wildlife population. In above paper, we assumed that the growth rate of wildlife conservation is proportional to the depletion of forestry biomass due to wildlife population. We have also studied the affect of illegal trade in forestry biomass and wildlife population. We have not shown the effect of population and industrialization on the forestry biomass and wildlife population in the previous study. Keeping these things are in the mind. In this paper, therefore, we propose a nonlinear mathematical model to study the effect of deforestation by human population and industrialization on wildlife species living in the forest habitat.

2. Mathematical Model

Consider a forest habitat, the resource of which is being exploited by human population and industrialization for its resources. Let B be the cumulative biomass density of forest resources, N be the human population density, I be density of Industrialization (i.e. the number of industries in a unit area) and W be the density of wildlife species in the forest. In the modelling process, the growth rate of commutative density of wildlife species which is wholly dependent on the forest resources is governed by logistic model, the carrying capacity of which is depleted by human population and industrialization. The dynamics of the problem is assumed to be governed by the following ordinary differential equations:

$$
\begin{aligned}
\frac{dB}{dt} &= sB - \frac{s_0 B^2}{L} - \alpha BN - \alpha_1 B^2 N - \beta BI - \beta_1 B^2 I - s_1 BW, \\
\frac{dN}{dt} &= rN - \frac{r_0 N^2}{K} + \alpha BN + \alpha_1 B^2 N, \\
\frac{dI}{dt} &= \beta BI + \beta_1 B^2 I - \beta_0 I^2 - \theta_0 I, \\
\frac{dW}{dt} &= \pi s_1 BW - \phi_0 W^2 - (\phi_1 N + \phi_2 I) W^2 - \psi_0 W.
\end{aligned}
\tag{2.1}
$$

with initial conditions $B(0) = B_0 \geq 0, N(0) = N_0 \geq 0, I(0) = I_0 \geq 0, W(0) = W_0 \geq 0.$

Here $\alpha, \alpha_1, \beta, \beta_1, s_1, \pi, \beta_0, \theta_{0,} \phi_0, \phi_1, \phi_2$ and ψ_0 are positive constants. The depletion rates of forestry biomass by population and industrialization is represented by α and β respectively whereas, α_1 and β_1 are the depletion rate coefficients of forestry biomass due to crowding by these factors. The constants s and r are growth rate coefficients of B and N respectively and the constant s_0, L, r_0 and K are interaction coefficients related to carrying capacity of B and N respectively. The constant s_1 is depletion of biomass by wildlife population. The growth rate of wildlife wholly depends on the commutative density of biomass; it is represented by $\pi\, s_1 BW$, where π is growth rate coefficient for wildlife population. The constants ϕ_0, ϕ_1 and ϕ_2 are depletion coefficient of wildlife population due to intra specific interaction, population and industrialization. The constant θ_0 and ψ_0 are government policies which decreases the growth rate of industrialization and decreasing in growth rate of wildlife by smugglering.

3. Boundedness of the System

We state the boundedness of the system in the following lemma without proof.

Lemma 3.1: The set $\Omega = \{(B, N, I, W) : 0 \leq B \leq A_1, 0 \leq N \leq A_2, 0 \leq I \leq A_3$ and $0 \leq W \leq A_4 \}$, Where $A_1 = \frac{sL}{s_0}$, $A_2 = \frac{\left(r + \alpha A_1 + \alpha_1 A_1^2\right)K}{r_0}$, $A_3 = \frac{\beta A_1 + \beta_1 A_1^2 - \theta_0}{\beta_0}$ and $A_4 = \frac{\pi\, s_1 A_1 - \psi_0}{\phi_0}$ is the region of attraction for all solutions initiating in the interior of the positive octant with conditions $\beta A_1 + \beta_1 A_1^2 > \theta_0$ and $\pi\, s_1 A_1 > \psi_0$.

From the above lemma and the model (2.1) it is noted that

$$s > \left\{\left(\alpha + \alpha_1 A_1\right)A_2 + \left(\beta + \beta_1 A_1\right)A_3 + s_1 A_4\right\}. \tag{3.1}$$

The condition (3.1) will be used in later of the analysis of this paper.

4. Equilibrium Analysis of the System

The system (2.1) has ten nonnegative equilibria namely, $E_0(0,0,0,0)$, $E_1\left(\frac{sL}{s_0},0,0,0\right)$, $E_2\left(0,\frac{rK}{r_0},0,0\right)$, $E_3\left(\hat{B},\hat{N},0,0\right)$, $E_4\left(\widehat{B},0,\widehat{I},0\right)$, $E_5\left(\overline{B},0,0,\overline{W}\right)$, $E_6\left(\tilde{B},\tilde{N},\tilde{I},0\right)$, $E_7\left(\breve{B},0,\breve{I},\breve{W}\right)$, $E_8\left(\vec{B},\vec{N},0,\vec{W}\right)$, and $E_9\left(B^*,N^*,I^*,W^*\right)$. The existence of equilibrium points $E_0(0,0,0,0)$, $E_1\left(\frac{sL}{s_0},0,0,0\right)$ or $E_2\left(0,\frac{rK}{r_0},0,0\right)$, is obvious. The existences of other equilibria are shown in Appendix A.

5. Stability Analysis of the System

Local Stability

To discuss the local stability of system (2.1), we compute the variational matrix $V(E)$ of system (2.1). The entries of general variational matrix are given by differentiating the right side of system (2.1) with respect to B, N, I and W, i.e.

$$V(E) = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & 0 & 0 \\ a_{31} & 0 & a_{33} & 0 \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}, \tag{5.1}$$

where

$$a_{11} = s - \frac{2s_0 B}{L} - \alpha N - 2\alpha_1 BN - \beta I - 2\beta_1 BI - s_1 W,\ a_{12} = -\alpha B - \alpha_1 B^2,\ a_{13} = -\beta B - \beta_1 B^2,\ a_{14} = -s_1 B,$$
$$a_{21} = \alpha N + 2\alpha_1 BN,\ a_{22} = r - \frac{2r_0 N}{K} + \alpha B + \alpha_1 B^2,\ a_{31} = \beta I + 2\beta_1 BI,\ a_{33} = \beta B + \beta_1 B^2 - 2\beta_0 I - \theta_0,$$
$$a_{41} = \pi s_1 W,\ a_{42} = -\phi_1 W^2,\ a_{43} = -\phi_2 W^2,\ a_{44} = \pi s_1 B - 2\phi_0 W - 2(\phi_1 N + \phi_2 I)W - \psi_0.$$

Accordingly, the linear stability analysis about the equilibrium points $E_i, i = 0\ \text{to}\ 9$ gives the following results:

5. The equilibrium point E_0 is unstable manifold in $B - N$ plane and equilibrium point E_1 is unstable manifold in $N - I - W$ plane and similarly the equilibrium point E_2 is unstable manifold in $B -$ direction respectively.
6. The equilibrium point E_3 is unstable manifold in $I - W$ plane and equilibrium point E_4 is stable manifold in $B - W$ plane but this is unstable manifold in $N - W$ plane.
7. The equilibrium point E_5 is unstable manifold in $N - I$ plane and equilibrium point E_6 is unstable manifold in $W -$ direction.
8. Similar way equilibrium point E_7 is unstable in $N -$ direction and equilibrium point E_8 is unstable in $I -$ direction.

The stability behaviour of equilibrium point E_9 is not obvious from equation (5.1).Hence we use Lyapunov method to prove its stability. Following statement of the theorem shows the stability of the equilibrium point E_9. For proof see Appendix B.

Theorem (5.1) Let the following inequality holds

$$m_2 \phi_1^2 W^{*4} < \frac{2m_0 r_0 N^*}{K}\left(\phi_0 W^* + \phi_1 N^* W^* + \phi_2 N^* I^*\right), \tag{5.2}$$
$$m_2 \phi_2^2 W^{*4} < 2m_1 \beta_0 I^*\left(\phi_0 W^* + \phi_1 N^* W^* + \phi_2 N^* I^*\right). \tag{5.3}$$

where

$$m_0 = \frac{\alpha B^* + \alpha_1 B^{*2}}{\alpha N^* + 2\alpha_1 B^* N^*}, m_1 = \frac{\beta B^* + \beta_1 B^{*2}}{\beta I^* + 2\beta_1 B^* I^*}, m_2 = \frac{B^*}{\pi W^*}. \tag{5.4}$$

Then equilibrium point E_9 is locally asymptotically stable.

Global Stability

The following theorem characterizes the global stability behavior of equilibrium point E_9.For proof, see Appendix C.

Theorem (5.2) Let the following inequalities hold:

$$\alpha_1^4 A_1^2 B^{*2} < \frac{r_0 \alpha}{K}\left(\alpha + \alpha_1 B^*\right)\left(\frac{s_0}{L} + \alpha_1 N^* + \beta_1 I^*\right), \tag{5.5}$$

$$\beta_1^4 A_1^2 B^{*2} < \beta_0 \beta\left(\beta + \beta_1 B^*\right)\left(\frac{s_0}{L} + \alpha_1 N^* + \beta_1 I^*\right), \tag{5.6}$$

$$\frac{\phi_1^2}{\pi} A_4^2 < \frac{r_0 \alpha}{K\left(\alpha + \alpha_1 B^*\right)}\left(\phi_0 + \phi_1 N^* + \phi_2 I^*\right), \tag{5.7}$$

$$\frac{\phi_2^2}{\pi} A_4^2 < \frac{\beta_0 \beta}{\left(\beta + \beta_1 B^*\right)}\left(\phi_0 + \phi_1 N^* + \phi_2 I^*\right). \tag{5.8}$$

Then equilibrium point E_9 is globally stable in the region Ω.

The above theorems imply that under certain conditions the system (2.1) exists but the cumulative biomass density of forestry resources decreases due to increase the densities of population and industrialization (see figures 3-6).

6. Persistence of the System

Theorem (6.1) The system (2.1) persists with conditions $s > \alpha A_{2\backslash} + \beta A_3 + s_1 A_4$, $\beta B_{\min} + \beta_1 B_{\min}^2 > \theta_0$ and $\pi s_1 B_{\min} > \psi_0$. These conditions are always satisfied from Lemma (3.1) and equation (3.1). For proof, see Appendix D.

7. Numerical Results

The stability of the non-linear model system (2.1), in the positive octant, is investigated numerically by using the following set of parameters.

$$s = 20, s_0 = 20, L = 10{,}000, \alpha = 1\times10^{-9}, \alpha_1 = 2\times10^{-8}, \beta = 2\times10^{-7}, \beta_1 = 2\times10^{-8}, r = 0.1, r_0 = 35,$$
$$K = 8000, \beta_0 = 0.2, s_1 = 0.4, \pi = 0.1, \phi_0 = 10, \phi_1 = 0.000002, \phi_2 = 0.000002, \theta_0 = 0.2, \psi_0 = 0.2. \quad (7.1)$$

The interior equilibrium point of the model system (2.1) corresponding to the above parameters values is:

$$E_9(5550.49,\ 163.69,\ \ 3.08634,\ 22.2012).$$

The characteristic polynomial and characteristic roots of the model system corresponding to the interior equilibrium point are given as:

$$\lambda^4 + 234.473\lambda^3 + 4751.74\lambda^2 + 6029.04\lambda + 1966.04 = 0. \quad (7.2)$$

$$\lambda_1 = -212.215, \lambda_2 = -20.9228, \lambda_3 = -0.717307, \lambda_4 = -0.617291. \quad (7.3)$$

From equation (7.3), it is clear that all characteristic roots of the characteristic polynomial (7.2) are negative. So, the interior equilibrium of the model system (2.1) is locally asymptotically stable.

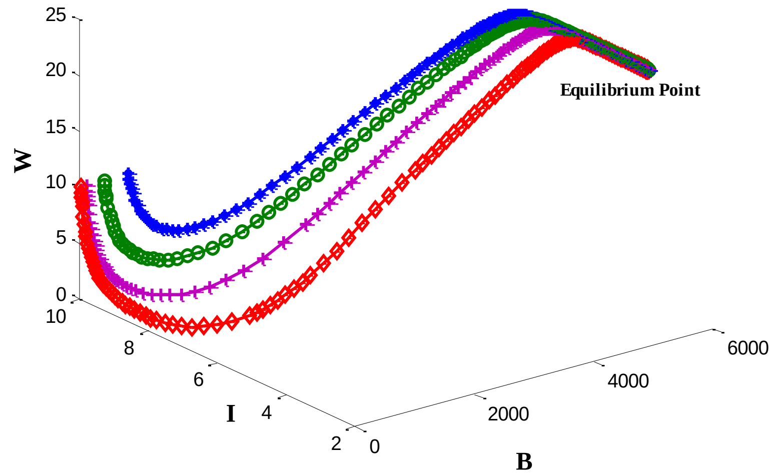

Figure 1. Phase space of W verses B .

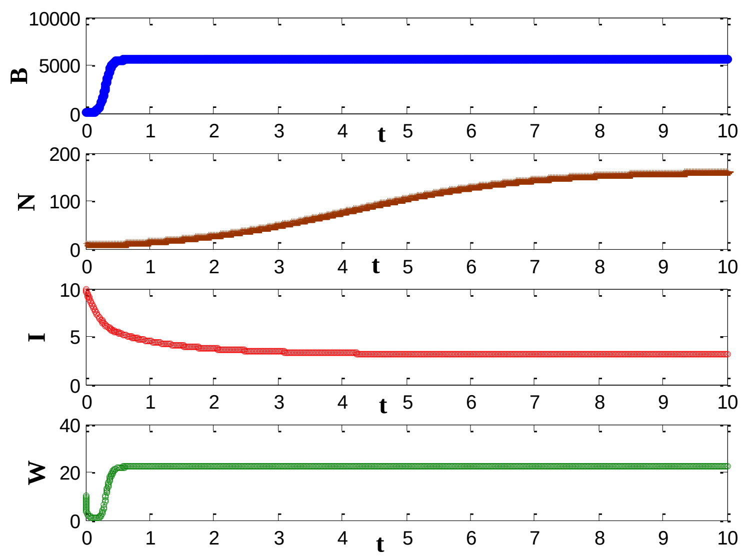

Figure 2. Stable behavior of B, N, I and W with time and other parameter values are same as (7.1).

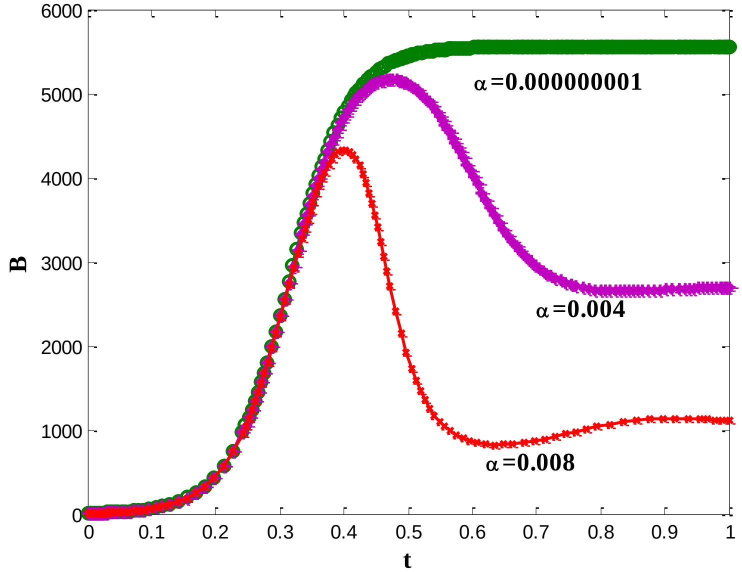

Figure 3. Variation of B with time for different values of α and other parameter values are same as (7.1).

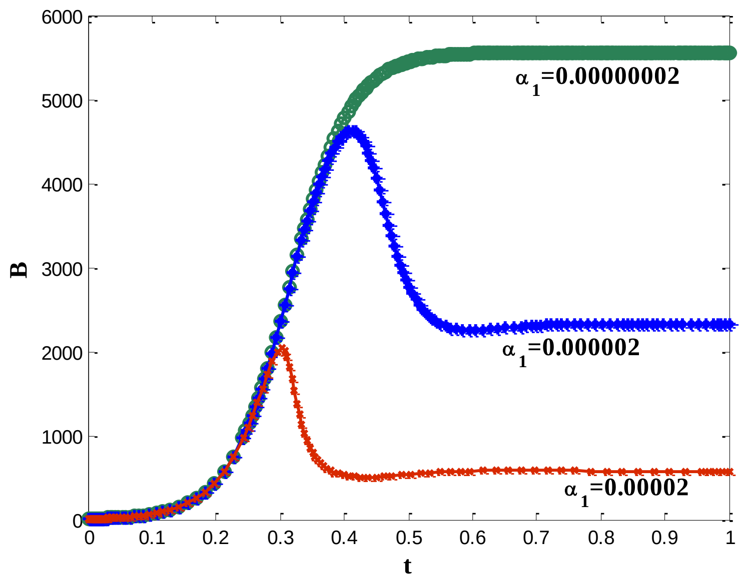

Figure 4. Variation of B with time for different values of α_1 and other parameter values are same as (7.1).

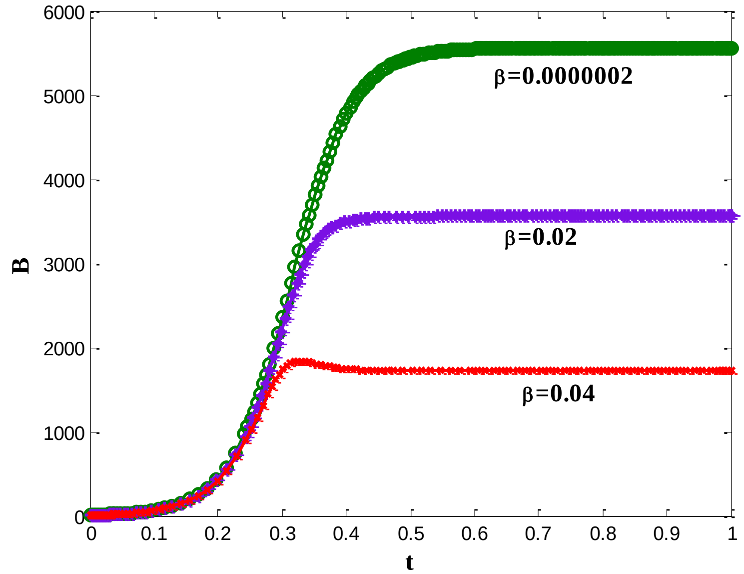

Figure 5. Variation of B with time for different values of β and other parameter values are same as (7.1).

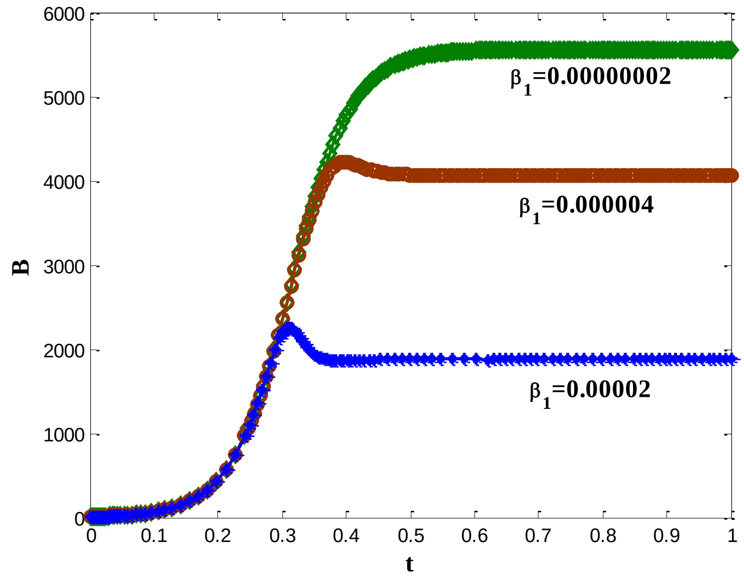

Figure 6. Variation of B with time for different values of β_1 and other parameter values are same as (7.1).

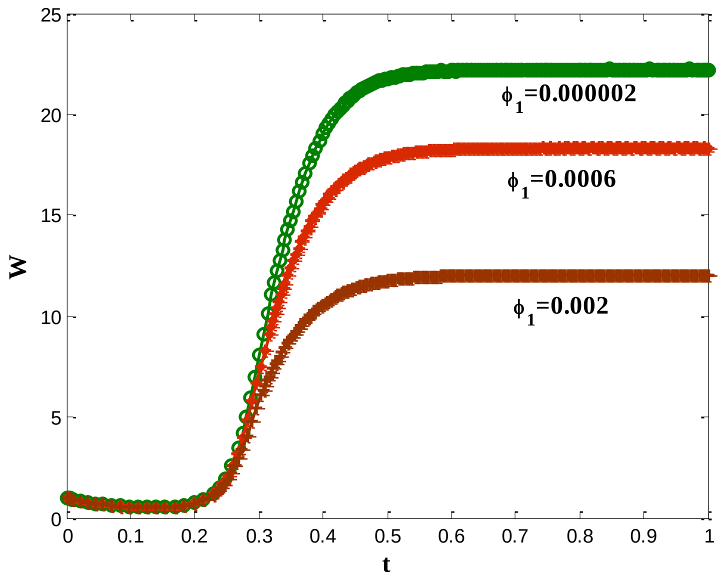

Figure 7. Variation of W with time for different values of ϕ_1 and other parameter values are same as (7.1).

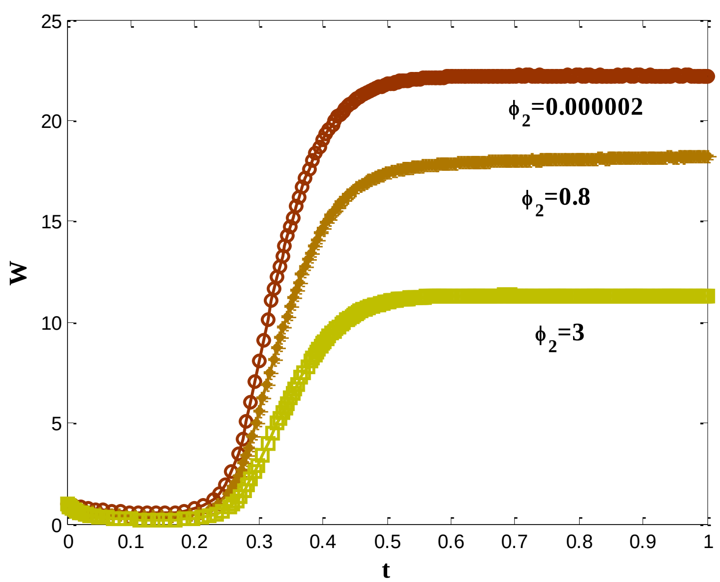

Figure 8. Variation of W with time for different values of ϕ_2 and other parameter values are same as (7.1).

Using the above set of parameters the model (2.1) has been solved numerically and various results are displayed graphically in Figures 1-8.Figure 1 shows the global stability of the system (2.1). The variation of the populations with time is shown in Figure 2 which shows the stable behavior of the system. Figure 3 to Figure 6 show the effects of human population and industrialization on the forestry biomass density B. These figures show that as the population and industrialization increase the growth rate of forestry biomass decreases and it may become extinct after a long time. The effects of population and industrialization on wildlife species are shown in Figure 7 and Figure 8,which show that wildlife species decreases due to population and industrialization.

8. Conclusions

In this paper, we have proposed a nonlinear mathematical model to study the depletion of forest resources by human population and industrialization and its effect on wildlife species living in the forest. The following variables have been considered.

1. The commutative biomass density of forest resources.
2. The density of human population living in the forest habitat.
3. The density of industrialization wholly dependent upon forest resources.
4. The commutative density of wildlife population in the forest habitat.

The proposed model has been analyzed by the stability theory of differential equations. The conditions of existence of equilibrium points and their stability in both local and global cases have been obtained. The condition, under which the system persists, by using differential inequality, has been found. By using numerical simulation, the effects of various parameters on the depletion of forestry biomass by population and industrialization have been shown graphically. The effect of depletion of forestry biomass on the survival of wildlife species has also been shown graphically. The following results have been found form the study:

1. Due to deforestation caused by population and industrialization, the forest resources are doomed to extinction.

2. The survival of wildlife species wholly dependent upon cumulative biomass density of forestry resources is very much threatened and the species are not likely to survive for long.

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Appendix A.

Existence of $E_3\left(\hat{B},\hat{N},0,0\right)$,

Here $\hat{B}$ and $\hat{N}$ are the positive solutions of the following algebraic equations

$$s-\frac{s_0\hat{B}}{L}-\alpha\hat{N}-\alpha_1\hat{B}\hat{N}=0, \quad \text{(A.1)}$$

$$r-\frac{r_0\hat{N}}{K}+\alpha\hat{B}+\alpha_1\hat{B}^2=0. \quad \text{(A.2)}$$

From equation (A.2), we get

$$\hat{N}=\frac{K\left(r+\alpha\hat{B}+\alpha_1\hat{B}^2\right)}{r_0}. \quad \text{(A.3)}$$

Using the value of $\hat{N}$ in the equation (A.1), we define the function $H_1\left(\hat{B}\right)$ as follows,

$$H_1\left(\hat{B}\right)=\alpha_1^2KL\hat{B}^3+2\alpha_1\alpha KL\hat{B}^2+\left(r_0s_0+L\alpha_1Kr+KL\alpha^2\right)\hat{B}+L\left(\alpha Kr-sr_0\right)=0. \quad \text{(A.4)}$$

Now from (A.4) we get

$$H_1(0)=L\left(\alpha Kr-sr_0\right)<0, \text{ due to (3.9).} \quad \text{(A.5)}$$

and

$$H_1\left(L^*\right)=\alpha_1^2KLL^{*3}+2\alpha_1\alpha KLL^{*2}+\left(L\alpha_1Kr+KL\alpha^2\right)L^*+L\alpha Kr>0. \quad \text{(A.6)}$$

Where $L^*=\dfrac{sL}{s_0}$.

Thus there exists a $\hat{B}$, $0<\hat{B}<L^*$, such that $H_1(\hat{B})=0$.

Now, the sufficient condition for the uniqueness of E_3 is $H_1'\left(\hat{B}\right)>0$. For this we find $H_1'\left(\hat{B}\right)$ from (A.4) as follows.

$$H_1'(\hat{B}) = 3\alpha_1^2 KL\hat{B}^2 + 4\alpha_1\alpha KL\hat{B} + r_0 s_0 + L\alpha_1 Kr + KL\alpha^2 > 0. \tag{A.7}$$

This completes the existence of E_3.

Existence of $E_4(\widehat{B}, 0, \widehat{I}, 0)$,

Here $\widehat{B}$ and $\widehat{I}$ are the positive solutions of the following algebraic equations

$$s - \frac{s_0\widehat{B}}{L} - \beta\,\widehat{I} - \beta_1\widehat{B}\widehat{I} = 0, \tag{A.8}$$

$$\beta\widehat{B} + \beta_1\widehat{B}^2 - \beta_0\widehat{I} - \theta_0 = 0. \tag{A.9}$$

From equation (A.9), we get

$$\widehat{I} = \frac{\beta\widehat{B} + \beta_1\widehat{B}^2 - \theta_0}{\beta_0}. \tag{A.10}$$

The value of $\widehat{I}$ remains positive if $\beta\widehat{B} + \beta_1\widehat{B}^2 > \theta_0$. Using the value of $\widehat{I}$ in the equation (A.8), we define the function $H_2(\hat{B})$ as follows,

$$H_2(\widehat{B}) = \beta_1^2 L\widehat{B}^3 + 2L\beta\,\beta_1\,\widehat{B}^2 + (\beta_0 s_0 + L\beta^2 - \beta_1\theta_0 L)\widehat{B} - L\beta_0\,s - \beta\,L\theta_0 = 0. \tag{A.11}$$

From (A.11), we have

$$H_2(0) = -L\beta_0 s - \beta\,L\theta_0 < 0, \tag{A.12}$$

$$H_2(L^*) = \beta_1\,L\,L^*(\beta_1\,L^{*2} - \theta_0) + \beta\,L\{(\beta_1 L^{*2} - \theta_0) + \beta_1 L^{*2} + \beta\,L^*\} > 0. \tag{A.13}$$

With condition $\beta_1 L^{*2} > \theta_0$.

Thus, there exists a $\widehat{B}$, $0 < \widehat{B} < L^*$, such that $H_2(\widehat{B}) = 0$.

Now, the sufficient condition for the uniqueness of E_4 is $H_2'(\widehat{B}) > 0$. For this, we find $H_2'(\widehat{B})$ as follows.

$$H_2'(\hat{B}) = 3\beta_1^2 L\hat{B}^2 + 4L\beta\beta_1\hat{B} + \beta_0 s_0 L(\beta^2 - \beta_1\theta_0) > 0. \tag{A.14}$$

With condition $\beta^2 > \beta_1\,\theta_0$.

This completes the existence of E_4.

Existence of $E_5(\overline{B}, 0, 0, \overline{W})$,

Here $\overline{B}$ and $\overline{W}$ are the positive solutions of the following algebraic equations

$$s - \frac{s_0\overline{B}}{L} - s_1\overline{W} = 0, \tag{A.15}$$

$$\pi\, s_1\overline{B} - \phi_0\overline{W} - \psi_0 = 0. \tag{A.16}$$

Solving the above equations, we get

$$\overline{B} = \frac{(s\phi_0 + s_1\psi_0)L}{s_0\phi_0 + s_1^2\pi\, L}, \overline{W} = \frac{\pi\, s\, s_1 L - \psi_0 s_0}{s_0\phi_0 + s_1^2\pi\, L}. \tag{A.17}$$

With condition $\pi\, s_1 s L > \psi_0 s_0$.

This completes the existence of E_5.

Existence of $E_6(\tilde{B}, \tilde{N}, \tilde{I}, 0)$,

Here $\tilde{B}, \tilde{N}$ and $\tilde{I}$ are the positive solutions of the following algebraic equations

$$s - \frac{s_0\tilde{B}}{L} - \alpha\tilde{N} - \alpha_1\tilde{B}\tilde{N} - \beta\,\tilde{I} - \beta_1\tilde{B}\tilde{I} = 0, \tag{A.18}$$

$$r - \frac{r_0\tilde{N}}{K} + \alpha\tilde{B} + \alpha_1\tilde{B}^2 = 0, \tag{A.19}$$

$$\beta\,\tilde{B} + \beta_1\tilde{B}^2 - \beta_0\tilde{I} - \theta_0 = 0. \tag{A.20}$$

From equations (A.19) and (A.20), we get

$$\tilde{N} = \frac{K\left(r + \alpha\tilde{B} + \alpha_1\tilde{B}^2\right)}{r_0}, \hat{I} = \frac{\beta\tilde{B} + \beta_1\tilde{B}^2 - \theta_0}{\beta_0}. \tag{A.21}$$

Using the values of $\tilde{N}$ and $\tilde{I}$ in the equation (A.18), we define the function $H_3\left(\hat{B}\right)$ as follows,

$$\begin{aligned} H_3\left(\tilde{B}\right) &= \left(Lr_0\beta_1^2 + \beta_0 LK\alpha_1^2\right)\tilde{B}^3 + 2\left(\alpha\alpha_1\beta_0 LK + L\beta\,\beta_1\, r_0\right)\hat{B}^2 + \\ &\left(\beta_0 r_0 s_0 + \beta_0 LK\alpha_1 r + \alpha^2\beta_0 LK + Lr_0\beta^2 - \beta_1 r_0\theta_0 L\right)\tilde{B} - L\beta_0\left(sr_0 - Kr\alpha\right) - \beta\, r_0\theta_0 L = 0. \end{aligned} \tag{A.22}$$

From (A.22), we have

$$H_3(0) = -L\beta_0\left(s\, r_0 - K\, r\alpha\right) - \beta\, r_0\theta_0 L < 0, \quad \text{due to (3.1).} \tag{A.23}$$

$$\begin{aligned} H_3\left(L^*\right) &= L\, r_0\left(\beta_1^2 L^{*3} - \beta\theta_0\right) + \alpha_1^2\, K\, L\beta_0 L^{*3} + 2\alpha\alpha_1\beta_0 K\, L\, L^{*2} \\ &+ \left(\beta_0 LK\alpha_1 r + \alpha^2 KL\beta_0 + \beta^2 r_0 L\right)L^* + \beta_1 r_0 LL^*\left(\beta\, L^* - \theta_0\right) > 0. \end{aligned} \tag{A.24}$$

With conditions $\beta_1^2 L^{*3} > \beta\theta_0$ and $\beta\, L^* > \theta_0$.

Thus, there exists a $\tilde{B},\ 0 < \tilde{B} < L^*$, such that $H_3(\tilde{B}) = 0$.

Now, the sufficient condition for the uniqueness of E_6 is $H_3'\left(\tilde{B}\right) > 0$. For this, we find $H_3'\left(\tilde{B}\right)$ as follows.

$$\begin{aligned} H_3'\left(\tilde{B}\right) &= 3\left(Lr_0\beta_1^2 + \beta_0 LK\alpha_1^2\right)\tilde{B}^2 + 4\,\alpha\,\alpha_1\,\beta_0 LK\hat{B} + \beta_0 r_0 s_0 + \beta_0 LK\alpha_1 r + \\ &\alpha^2\beta_0 LK + Lr_0\beta^2 + \beta_1 r_0 L\left(\beta\,\hat{B} - \theta_0\right) > 0. \end{aligned} \tag{A.25}$$

With condition $\beta\,\hat{B} > \theta_0$.

This completes the existence of E_6.

Existence of $E_7\left(\breve{B}, 0, \breve{I}, \breve{W}\right)$,

Here $\breve{B}, \breve{I}$ and $\breve{W}$ are the positive solutions of the following algebraic equations

$$s-\frac{s_0\breve{B}}{L}-\beta\,\breve{I}-\beta_1\breve{B}\breve{I}-s_1\breve{W}=0, \tag{A.26}$$

$$\beta\,\breve{B}+\beta_1\breve{B}^2-\beta_0\breve{I}-\theta_0=0, \tag{A.27}$$

$$\pi\,\mathrm{s}_1\,\breve{B}-\phi_0\breve{W}-\phi_2\breve{I}\breve{W}-\psi_0=0. \tag{A.28}$$

From equation (A.27), we get

$$\breve{I}=\frac{\beta\breve{B}+\beta_1\breve{B}^2-\theta_0}{\beta_0}. \tag{A.29}$$

With condition $\beta\breve{B}+\beta_1\breve{B}^2>\theta_0$

Using the value of $\breve{I}$ in the equation (A.28), we get

$$\breve{W}=\frac{\beta_0\left(\pi\,s_1\breve{B}-\psi_0\right)}{\beta_0\phi_0+\phi_2\left(\beta\breve{B}+\beta_1\breve{B}^2-\theta_0\right)}. \tag{A.30}$$

With condition $\pi\,s_1\breve{B}>\psi_0$.

Using the value of $\breve{W}$ and $\breve{I}$ in the equation (A.26), we get we define the function $H_4\left(\breve{B}\right)$ as follows,

$$\begin{aligned}H_4\left(\breve{B}\right)=\,&\beta_1^2L\phi_2\breve{B}^5+3L\beta\beta_1^2\phi_2\breve{B}^4+\left(\beta_0s_0\beta_1\phi_2+3L\beta^2\beta_1\phi_2+\beta_1^2\beta_0L\phi_0-2L\beta_1^2\theta_0\phi_2\right)\breve{B}^3+\left(\beta^3\phi_2L+s_0\beta_0\beta\,\phi_2\right.\\&\left.-sL\beta_0\beta_1\phi_2+2L\beta_1\beta\beta_0\phi_0-Ls\beta_0\beta_1\phi_2-4L\beta\beta_1\theta_0\phi_2\right)\breve{B}^2+\left(s_0\beta_0^2\phi_0-sL\phi_2\beta\beta_0+\beta_0^2L\pi\,s_1^2\right.\\&\left.+L\beta^2\beta_0\phi_0-L\beta_0\beta_1\theta_0\phi_0\right)\breve{B}-sL\beta_0^2\phi_0-L\beta\beta_0\theta_0\phi_0+Ls\theta_0\beta_0\phi_2+L\beta\theta_0^2\phi_2-Ls_1\beta_0^2\psi_0=0.\end{aligned} \tag{A.31}$$

From (A.31), we have

$$H_4(0)=-sL\beta_0\left(\beta_0\phi_0-\theta_0\phi_2\right)-L\beta\theta_0\left(\beta_0\phi_0-\theta_0\phi_2\right)-Ls_1\beta_0^2\psi_0<0, \tag{A.32}$$

With $\beta_0\phi_0>\theta_0\phi_2$.

$$H_4\left(L^*\right)=\beta_1^2 L\phi_2 L^{*5}+3L\beta\,\beta_1^2\,\phi_2\,L^{*4}+L\,L^{*2}\beta_1\phi_2\left(3\beta\,\beta_1\,L^{*2}-2\beta_1\,\theta_0 L^*-4\beta\,\theta_0\right)+L\,\beta_0\,\phi_0\left(2\beta\,\beta_1 L^{*2}-\beta_1\,\theta_0 L^*-\beta\,\theta_0\right)$$
$$L\,L^{*2}\beta^3\,\phi_2+L\,s\,\theta_0\,\beta_0\,\phi_2+L\,\beta\,\theta_0^2\,\phi_2+L\,s_1\,\beta_0^2\left(L^*\,\pi-\psi_0\right)+LL^*\beta^2\beta_0\phi_0>0. \tag{A.33}$$

With conditions $3\beta\,\beta_1\,L^{*2}>2\beta_1\,\theta_0 L^*+4\beta\,\theta_0, 2\beta\,\beta_1\,L^{*2}>\beta_1\,\theta_0 L^*+\beta\,\theta_0, L^*\,\pi>\psi_0.$

Thus, there exists a $\breve{B},\ \ 0<\breve{B}<L^*,$ such that $H_4(\breve{B})=0.$

Now, the sufficient condition for the uniqueness of E_7 is $H_4'\left(\breve{B}\right)>0$. For this, we find $H_4'\left(\breve{B}\right)$ as follows.

$$H_4'\left(\breve{B}\right)=5\beta_1^2 L\phi_2\breve{B}^4+\beta_1\,\phi_2\,L\left(9\beta\,\beta_1\breve{B}^2-2\,s\,\beta_0-4\,\beta\,\theta_0-6\beta_1\,\theta_0\,\breve{B}\right)\breve{B}+L\,s_1^2\,\beta_0^2\,\pi+L\,\beta^2\,\beta_0\,\phi_0+s_0\,\beta_0^2\,\phi_0$$
$$\beta_0\,\phi_2\,L\left(3\breve{B}^2\,\beta_1^2-s\,\beta\right)+3\,\breve{B}^2\left(3\,L\,\beta^2\,\beta_1\,\phi_2+s_0\,\beta_0\,\beta_1\,\phi_2\right)+\beta_0\,\phi_0\,\beta_1\,L\left(4\,\beta\,\breve{B}-\theta_0\right)$$
$$+2\,\breve{B}\,\left(L\,\beta^3\,\phi_2+s_0\,\beta\,\beta_0\,\phi_2\right)>0. \tag{A.34}$$

With conditions $9\beta\,\beta_1\,\breve{B}^2>2\,s\,\beta_0+4\beta\,\theta_0+6\,\beta_1\,\theta_0\,\breve{B}, 3\beta_1^2\breve{B}^2>s\beta, 4\,\beta\,\breve{B}>\theta_0.$

After multiplying (A.31) by 2 and (A.34) by $\breve{B}$ and comparing we note that $H_4'\left(\breve{B}\right)$ is positive.

This completes the existence of E_7.

Existence of $E_8\left(\vec{B},\vec{N},0,\vec{W}\right)$,

Here $\vec{B},\vec{N}$ and $\vec{W}$ are the positive solutions of the following algebraic equations

$$s-\frac{s_0\vec{B}}{L}-\alpha\,\vec{N}-\alpha_1\vec{B}\vec{N}-s_1\vec{W}=0, \tag{A.35}$$

$$r-\frac{r_0\vec{N}}{K}+\alpha\,\vec{B}-\alpha_1\vec{B}^2=0, \tag{A.36}$$

$$\pi\,s_1\vec{B}-\phi_0\vec{W}-\phi_1\vec{N}\,\vec{W}-\psi_0=0. \tag{A.37}$$

From equations (A.36) and (A.37), we get

$$\bar{N} = \frac{K\left(r + \alpha\bar{B} + \alpha_1\bar{B}^2\right)}{r_0}, \bar{W} = \frac{\left(\pi\, s_1\bar{B} - \psi_0\right) r_0}{\phi_0 r_0 + \phi_1 K\left(r + \alpha\bar{B} + \alpha_1\bar{B}^2\right)}. \tag{A.38}$$

With condition $\pi\, s_1\bar{B} > \psi_0$.

Using the value of $\bar{N}$ and $\bar{W}$ in the equation (A.35), we define the function $H_5\left(\hat{B}\right)$ as follows,

$$H_5\left(\bar{B}\right) = p_1\,\bar{B}^5 + p_2\,\bar{B}^4 + p_3\,\bar{B}^3 + p_4\bar{B}^2 + p_5\bar{B} + p_6 = 0. \tag{A.39}$$

Where

$p_1 = \alpha_1^2 L\, K^2\,\phi_1,\ p_2 = 3K^2 L\alpha\,\alpha_1^2\,\phi_1,\ p_3 = K\,L\,r_0\,\alpha_1^2\,\phi_0 + K\,r_0\,s_0\,\alpha_1\,\phi_1 + 3\,K^2\,L\,\alpha^2\,\alpha_1\,\phi_1 + 2K^2\,L\,r\,\alpha_1^2\,\phi_1,$

$p_4 = 2\,K\,L\,r_0\,\alpha\,\alpha_1\,\phi_0 + K\,r_0\,s_0\,\alpha\,\phi_1 + K^2\,L\,\alpha^2\,\phi_1 - K\,L\,r_0\,s\,\alpha_1\,\phi_1 + 4K^2\,L\,r\,\alpha\,\alpha_1\,\phi_1,$

$p_5 = L\,r_0^2\,s_1^2\pi + r_0^2 s_0\phi_0 + K\,L\,r_0\alpha^2\,\phi_0 + K\,L\,r r_0\,\alpha_1\,\phi_0 + K r r_0 s_0\phi_1 - K L r_0 s\alpha\phi_1 + 2K^2 L r\alpha^2\phi_1,$

$p_6 = -K\,L\,r\,\phi_1\left(s\,r_0 - K\,r\,\alpha\right) - L\,r_0\,\phi_0\left(s\,r_0 - K\,r\,\alpha - r_0\,s_1\,\psi_0\right).$

From (A.39), we have

$$H_5(0) = -K\,L\,r\,\phi_1\left(s\,r_0 - K\,r\,\alpha\right) - L\,r_0\,\phi_0\left(s\,r_0 - K\,r\,\alpha - r_0\,s_1\,\psi_0\right) < 0, \text{ due to (3.9).} \tag{A.40}$$

$$H_5\left(L^*\right) = p_1\,L^{*5} + p_2\,L^{*4} + p_3\,L^{*3} + p_4 L^{*2} + p_5 L^* + p_6 > 0. \tag{A.41}$$

Thus, there exists a $\bar{B},\ 0 < \bar{B} < L^*,$ such that $H_5(\bar{B}) = 0.$

Now, the sufficient condition for the uniqueness of E_8 is $H_5'\left(\bar{B}\right) > 0$. For this, we find $H_5'\left(\bar{B}\right)$ as follows.

$$H_5'\left(\bar{B}\right) = 5p_1\,\bar{B}^4 + 4p_2\,\bar{B}^3 + 3p_3\,\bar{B}^2 + 2p_4\bar{B} + p_5 > 0. \tag{A.42}$$

After multiplying (A.40) by 2 and (A.42) by $\bar{B}$ and comparing we note that $H_5'\left(\bar{B}\right)$ is positive.

This completes the existence of E_8.

Existence of $E_9\left(B^*, N^*, I^*, W^*\right)$

To determine B^*, N^*, I^*, W^* we get the following equations from (2.1)

$$s-\frac{s_0 B^*}{L}-\alpha N^*-\alpha_1 B^* N^*-\beta I^*-\beta_1 B^* I^*-s_1 W^*=0, \tag{A.43}$$

$$r-\frac{r_0 N^*}{K}+\alpha B^*+\alpha_1 B^* N^*=0, \tag{A.44}$$

$$\beta B^*+\beta_1 B^{*2}-\beta_0 I^*-\theta_0=0, \tag{A.45}$$

$$\pi s_1 B^*-\phi_0 W^*-\left(\phi_1 N^*+\phi_2 I^*\right) W^*-\psi_0=0. \tag{A.46}$$

From equations (A.44), (A.45) and (A.46), we get

$$N^*=\frac{K\left(r+\alpha B^*+\alpha_1 B^{*2}\right)}{r_0},\ I^*=\frac{\left(\beta B^*+\beta_1 B^{*2}-\theta_0\right)}{\beta_0}, \tag{A.47}$$

The value of I^* is always positive with condition $\beta B^*+\beta_1 B^{*2}>\theta_0$.

$$W^*=\frac{r_0\,\beta_0\left(\pi s_1 B^*-\psi_0\right)}{r_0\left(\phi_0\,\beta_0-\phi_2\,\theta_0\right)+r\phi_1 K\beta_0+B^*\left(\alpha\phi_1 K\beta_0+\phi_2 r_0\beta\right)+B^{*2}\left(\phi_1 K\beta_0\alpha_1+\phi_2 r_0\beta_1\right)}. \tag{A.48}$$

If $\pi s_1 B^*>\psi_0$ and $\phi_0\,\beta_0>\phi_2\,\theta_0$ then W^* remains positive.

Using the value of N^*, I^* and W^* in the equation (A.43), we get

$$H_6\left(B^*\right)=q_1 B^{*5}+q_2 B^{*4}+q_3 B^{*3}+q_4 B^{*2}+q_5 B^*+q_6=0. \tag{A.49}$$

Where

$q_1=M_2 M_3, q_2=M_1 M_3+M_2 M_4, q_3=M M_3+M_4 M_1+M_2 M_5, q_4=M M_4+M_1 M_5+M_2 M_6,$
$q_5=M M_5+M_1 M_6+s_1^2\,L\,r_0\,\beta_0\,\pi, q_6=M_6 M-s_1\,L\,r_0\,\beta_0\,\psi_0,\ M=r_0\left(\phi_0\,\beta_0-\phi_2\,\theta_0\right)+K\,\phi_1\,\beta_0\,r,$
$M_1=K\,\phi_1\,\beta_0\,\alpha+r_0\,\phi_2\,\beta,\ M_2=K\,\phi_1\,\beta_0\,\alpha_1+r_0\,\phi_2\,\beta_1,\ M_3=\alpha_1^2\,L\,K\,\beta_0+\beta_1^2\,L,$
$M_4=\alpha\,\alpha_1\,L\,K\,\beta_0+\alpha\,\alpha_1\,L\,K\,\beta_0\,r+2\beta\,\beta_1\,L,$
$M_5=r\,\alpha_1\,L\,K\,\beta_0+s_0\,r_0\,\beta_0+\alpha^2 L\,K\,\beta_0\,r+\beta^2\,L-\beta_1\,L\,\theta_0,\ M_6=r^2\,\alpha\,L\,K\,\beta_0-s\,L\,r_0\,\beta_0-\beta\,L\,\theta_0.$

From equation (A.49), we get

$$H_6(0) = M_6\, M - s_1\, L\, r_0\, \beta_0\, \psi_0 < 0, \tag{A.50}$$

With condition $M_6\, M < s_1\, L\, r_0\, \beta_0\, \psi_0$.

and

$$H_6(L^*) = q_1 L^{*5} + q_2 L^{*4} + q_3 L^{*3} + q_4 L^{*2} + q_5 L^* + q_6. \tag{A.51}$$

which is positive due to (3.1).

Thus there exists a B^*, $0 < B^* < L^*$, such that $H_6(B^*) = 0$. Now, the sufficient condition for the uniqueness of E_9 is $H_6'(B^*) > 0$ .From (4.49) we can find and $H_6'(B^*)$ as follows,

$$H_6'(B^*) = 5\, q_1\, B^{*4} + 4\, q_2\, B^{*3} + 3\, q_3\, B^{*2} + 2\, q_4\, B^* + q_5 > 0. \tag{A.52}$$

After multiplying (A.49) by 2 and (A.52) by B^* and comparing it is noted that $H_6'(B^*)$ is positive.

This completes the existence of E_9.

Appendix B. Proof of Theorem (5.1)

We first linearize model (2.1) by substituting

$B = B^* + b, N = N^* + n, I = I^* + i, W = W^* + w.$ Where b, n, i, w are small perturbations around equilibrium point E_9. We get following linearize system

$$\begin{aligned}
\frac{db}{dt} &= \left(-\frac{s_0 B^*}{L} - \alpha_1 B^* N^* - B^* I^* \beta_1\right) b - \left(\alpha\, B^* + \alpha_1 B^{*2}\right) n - \left(\beta\, B^* + \beta_1 B^{*2}\right) i - s_1 B^* w, \\
\frac{dn}{dt} &= \left(\alpha N^* + 2\alpha_1 B^* N^*\right) b - \frac{r_0 N^*}{K} n, \\
\frac{di}{dt} &= \left(\beta\, I^* + 2\beta_1 B^* I^*\right) b - \beta_0 I^* i,\ \frac{dw}{dt} = \pi\, s_1 W^* b - \phi_1 W^{*2} n - \phi_2 W^{*2}\, i - \left(\phi_0 W^* + \phi_1 N^* W^* + \phi_2 N^* I^*\right) w.
\end{aligned} \tag{B.1}$$

Then we consider the following positive definite function:

$$V = \frac{1}{2} b^2 + \frac{m_0}{2} n^2 + \frac{m_1}{2} i^2 + \frac{m_2}{2} w^2. \tag{B.2}$$

Differentiating V with respect to time t along the linearize system (B.1), we get

$$\frac{dV}{dt} = -v_{11}b^2 + v_{12}bn + v_{13}bi + v_{14}bw + v_{24}nw + v_{34}iw - v_{22}n^2 - v_{33}i^2 - v_{44}w^2, \tag{B.3}$$

where

$$v_{11} = \frac{s_0 B^*}{L} + \alpha_1 B^* N^* + B^* I^* \beta_1,\ v_{12} = -\alpha B^* - \alpha_1 B^{*2} + m_0\left(\alpha N^* + 2\alpha_1 B^* N^*\right),$$
$$v_{13} = -\beta B^* - \beta_1 B^{*2} + m_1\left(\beta I^* + 2\beta_1 B^* I^*\right),\ v_{14} = m_2 \pi\, s_1 W^* - s_1 B^*,\ v_{22} = \frac{m_0 r_0 N^*}{K},\ v_{33} = m_1 \beta_0 I^*,$$
$$v_{44} = m_2\left(\phi_0 W^* + \phi_1 N^* W^* + \phi_2 N^* I^*\right),\ v_{24} = m_2 \phi_1 W^{*2},\ v_{34} = m_2 \phi_2 W^{*2}. \tag{B.4}$$

Now choosing $m_0 = \dfrac{\alpha B^* + \alpha_1 B^{*2}}{\alpha N^* + 2\alpha_1 B^* N^*}, m_1 = \dfrac{\beta B^* + \beta_1 B^{*2}}{\beta I^* + 2\beta_1 B^* I^*}, m_2 = \dfrac{B^*}{\pi W^*}.$ (B.5)

we have,

$$\frac{dV}{dt} = -v_{11}b^2 - v_{24}nw - v_{34}iw - v_{22}n^2 - v_{33}i^2 - v_{44}w^2. \tag{B.6}$$

From equation (B.6), we concluded that $\dfrac{dV}{dt}$ is negative definite and equilibrium point E_9 is locally asymptotically stable under the following conditions

$$v_{24}^2 < 2v_{22}v_{44} \quad \text{and} \quad v_{34}^2 < 2v_{33}v_{44}. \tag{B.7}$$

Which are given in (5.2) and (5.3).

Appendix C. Proof of Theorem (5.2)

For finding the condition of global stability at E_9 we construct the Lyapunov function

$$H = B - B^* - B^* \ln\frac{B}{B^*} + c_0\left(N - N^* - N^* \ln\frac{N}{N^*}\right) + c_1\left(I - I^* - I^* \ln\frac{I}{I^*}\right) + \\ c_2\left(W - W^* - W^* \ln\frac{W}{W^*}\right). \tag{C.1}$$

Differentiating H with respect to time t along the solutions of model (2.1), we get

$$\frac{dH}{dt} = \frac{(B - B^*)}{B}\frac{dB}{dt} + \frac{c_0(N - N^*)}{N}\frac{dN}{dt} + \frac{c_1(I - I^*)}{I}\frac{dI}{dt} + \frac{c_2(W - W^*)}{W}\frac{dW}{dt}. \tag{C.2}$$

Using system of equation (2.1), we get after some algebraic manipulations as

$$\begin{aligned}\frac{dH}{dt} = &-\left\{\frac{s_0}{L} + \alpha_1 N^* + \beta_1 I^*\right\}(B - B^*)^2 - \frac{c_0 r_0}{K}(N - N^*)^2 - c_1\beta_0(I - I^*)^2 \\ &- c_2(\phi_0 + \phi_1 N^* + \phi_2 I^*)(W - W^*)^2 + \left[-(\alpha + \alpha_1 B) + c_0(\alpha + \alpha_1 B^*) + c_0\alpha_1 B\right] \\ &(B - B^*)(N - N^*) + \left[-(\beta + \beta_1 B) + c_1(\beta + \beta_1 B^*) + c_1\beta_1 B\right](B - B^*)(I - I^*) \\ &+ s_1(\pi c_2 - 1)(B - B^*)(W - W^*) + (-\phi_1 c_2 W)(N - N^*)(W - W^*) \\ &+ (-\phi_2 c_2 W)(I - I^*)(W - W^*).\end{aligned} \tag{C.3}$$

Here choosing $c_0 = \dfrac{\alpha}{\alpha + \alpha_1 B^*}, c_1 = \dfrac{\beta}{\beta + \beta_1 B^*}, c_2 = \dfrac{1}{\pi}.$ (C.4)

Now $\dfrac{dH}{dt}$ to be negative definite if the following inequalities hold,

$$\alpha_1^4 A_1^2 B^{*2} < \frac{r_0\alpha}{K}(\alpha + \alpha_1 B^*)\left(\frac{s_0}{L} + \alpha_1 N^* + \beta_1 I^*\right), \tag{C.5}$$

$$\beta_1^4 A_1^2 B^{*2} < \beta_0\beta(\beta + \beta_1 B^*)\left(\frac{s_0}{L} + \alpha_1 N^* + \beta_1 I^*\right), \tag{C.6}$$

$$\frac{\phi_1^2}{\pi} A_4^2 < \frac{r_0\alpha}{K(\alpha + \alpha_1 B^*)}(\phi_0 + \phi_1 N^* + \phi_2 I^*), \tag{C.7}$$

$$\frac{\phi_2^2}{\pi} A_4^2 < \frac{\beta_0\beta}{(\beta + \beta_1 B^*)}(\phi_0 + \phi_1 N^* + \phi_2 I^*). \tag{C.8}$$

Thus the model (2.1) is globally stable in the region Ω .

Appendix D. Proof of Theorem (6.1)

From the first equation of the system (2.1), we have

$$\frac{dB}{dt} = sB - \frac{s_0 B^2}{L} - \alpha BN - \alpha_1 B^2 N - \beta BI - \beta_1 B^2 I - s_1 BW, \tag{D.1}$$

$$\frac{dB}{dt} \geq \left(s - \alpha\, A_2 - \beta\, A_3 - s_1 A_4\right)B - \left(\frac{s_0}{L} + \alpha_1 A_2 + \beta_1 A_3\right)B^2. \tag{D.2}$$

According to lemma (3.1) and comparison principle, it follows that

$$\liminf_{t\to\infty} B \geq \frac{L\left(s - \alpha\, A_2 - \beta\, A_3 - s_1 A_4\right)}{s_0 + \left(\alpha_1 A_2 + \beta_1 A_3\right)L}. \tag{D.3}$$

$$B_{\min} = \frac{L\left(s - \alpha\, A_2 - \beta\, A_3 - s_1 A_4\right)}{s_0 + \left(\alpha_1 A_2 + \beta_1 A_3\right)L}. \tag{D.4}$$

Since $s > \alpha\, A_{2\backslash} + \beta\, A_3 + s_1 A_4$, $B_{\min}$ remains always positive (see 3.1).

From the second equation of the system (2.1) and equation (D.4), we have

$$\frac{dN}{dt} \geq \left(r + \alpha B_{\min} + \alpha_1 B_{\min}^2\right)N - \frac{r_0}{K} N^2. \tag{D.5}$$

According to lemma (3.1) and comparison principle, it follows that

$$\liminf_{t\to\infty} N \geq \frac{\left(r + \alpha B_{\min} + \alpha_1 B_{\min}^2\right)K}{r_0}. \tag{D.6}$$

$$N_{\min} = \frac{\left(r + \alpha B_{\min} + \alpha_1 B_{\min}^2\right)K}{r_0}. \tag{D.7}$$

From the third equation of the system (2.1) equation (D.4), we have

$$\liminf_{t\to\infty} I \geq \frac{\left(\beta B_{\min} + \beta_1 B_{\min}^2 - \theta_0\right)}{\beta_0}. \tag{D.8}$$

$$I_{\min} = \frac{\left(\beta\, B_{\min} + \beta_1 B_{\min}^2 - \theta_0\right)}{\beta_0}. \tag{D.9}$$

The value of $I_{\min}$ is always positive with condition $\beta\, B_{\min} + \beta_1 B_{\min}^2 > \theta_0$.

From the last equation of the system (2.1) and equations (D.4), (D.7), (D.9), we get

$$\frac{dW}{dt} \geq \left(\pi\, s_1 B_{\min} - \psi_0\right) W - \pi_0 W^2. \tag{D.10}$$

Where $\pi_0 = \dfrac{1}{r_0 \beta_0}\left[\left(\phi_1 \alpha_1 K \beta_0 + \phi_2 \beta_1 r_0\right) B_{\min}^2 + \left(\phi_1 \alpha K \beta_0 + \phi_2 \beta\, r_0\right) B_{\min} + r_0\left(\phi_0\, \beta_0 - \phi_2\, \theta_0\right) + \phi_1 r K \beta_0\right].$

According to lemma (3.1) and comparison principle, it follows that

$$\liminf_{t\to\infty} W \geq \frac{\pi\, s_1 B_{\min} - \psi_0}{\pi_0}. \tag{D.11}$$

$$W_{\min} = \frac{\pi\, s_1 B_{\min} - \psi_0}{\pi_0}. \tag{D.12}$$

If $\pi\, s_1 B_{\min} > \psi_0$ then $W_{\min}$ remains positive.

This completes the proof of the theorem.

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Modeling and Analysis of the Impact of Awareness Programs on Rural Population for Conservation of Forest Resources

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Abstract

The rural populations, living around and in the peripheral region of forest are highly dependent on its resources for fuel, food and other livelihoods. It is, therefore, very desirable to educate the rural population to use the forest resources by keeping in view the concepts of sustainable development and conservation. In this paper, therefore a mathematical model is proposed and analyzed consisting of four variables, namely, the cumulative biomass density of resources, the population density of rural population living around and in the peripheral regions of forest, the population density of the rural population which are made aware (educated) about the need of conservation of forest resources and the cumulative density of awareness programs by various sources including media. The analysis shows that as the density of awared population increases, the cumulative biomass density of forest resources increases leading to sustainable conservation of resources. The result is confirmed by simulation study of the proposed model.

Keywords: Forestry Biomass, Population, Tribal Population Awareness Programs, Stability, Persistence.

1. Introduction

The rural population living around and in the peripheral regions of the forest use its resources for fuel, food, fodder etc. These people cause the degradation of forest land for agriculture, use timber for housing and other household needs. They use the forest grass land for grazing and in the process destroy the younger trees, newly planted trees, bushes etc; (Gibbs,2010; Liu et al. 2010,Subrit et al.,2010). It is therefore important to make the rural population aware about the need for conservation of forest resources so that they can be sustained for future. Our main aim of this paper therefore, is to model and analyse the impact of awareness programs to rural population by various sources including media on conservation of forest resources.

There have been many studies investigating the impact of industrialization and increasing population pressure on forest resources. Many researchers have investigated the depletion of resource biomass by overgrowing population, toxicants and industrialization, Dubey and Dass (1999), Shukla et al. (1989, 1996, and 1997), Agarwal et al. (2010). Shukla et al. (1989) have studied the effects of population on the depletion of forestry resources. Dubey and Dass (1999) have proposed and analyzed a mathematical model to study the survival of species dependent on a resource, which is depleted due to industrialization. Agarwal et al. (2010) proposed a ratio dependent mathematical model on the depletion of forestry resources due to industrialization pressure. Shukla et al. (2011) have also studied the effect of technology on the conservation of forestry resources.

It has been observed that in the tropics, where deforestation has occurred, due to agricultural expansion, has given way to large-scale, enterprise-driven forest conversion (Rudel, 2007; Asner et al., 2013). Sayer and Cassman (2013) have argued that future agricultural demand for land could be met without significant forest loss if agricultural innovations are applied. Sloan and Sayer (2015) presented a review on Forest Resources Assessment of 2015 showing positive global trends but still they have been observed that forest loss and degradation persist in poor tropical countries. In view of the above literature, it is observed that none of them considered the effect of awareness programs on rural population for sustainable management of forest resources. Here our aim is to study the impact of awareness programs on rural population for conservation of forest resources by media.

2. Mathematical Model

Let B be the cumulative biomass density of forest resources, P be the density of rural population, P_A be the density of aware rural population by media etc and A be the cumulative density of awareness programs to educate the rural population. It is assumed that cumulative biomass density is growing at intrinsic growth rate s with carrying capacity L.It is assumed further that the rural population density is also growing logistically with growth rate r and carrying capacity K.The rural population with density P is depleting the biomass density B with a rate α but it is conserved by educated population P_A by the rate s_0. The rate at which rural population is educated about sustainable consumption of forest resources is γ and ν is the fractional part of γ. The constants r_0 and γ_0 are the migration rates of the rural and aware rural population. The constant η is the rate at which awareness programs are launched, which is assumed to the proportional to the rural population. The constants η_0 and θ are the failure rate of awareness programs by different ways. Keeping in view of these considerations, the non-linear model is proposed as follows

$$\begin{aligned}
\frac{dB}{dt} &= sB\left(1-\frac{B}{L}\right)-\alpha BP + s_0 P_A, \\
\frac{dP}{dt} &= rP\left(1-\frac{P}{K}\right)+\alpha BP-\gamma\, P\, A - r_0 P, \\
\frac{dP_A}{dt} &= \nu\,\gamma\, P\, A-\gamma_0 P_A, \\
\frac{dA}{dt} &= \eta PA-\eta_0 A^2-\theta\, A.
\end{aligned} \tag{2.1}$$

where $B(0)>0, P(0)\geq 0, P_A(0)\geq 0, A(0)\geq 0.$

The last equation governing A is generalized version of the model considered by Misra et al. (2011).

3. Boundedness of the System

In the following lemma, we state the bounds of the various variables which would be needed in our study.

Lemma 3.1: The set $\Omega=\{(B,P,P_A,A): 0\leq B+P+P_A\leq \mathrm{W_m}$ and $0\leq A\leq A_m\}$, is the region of attraction for all solutions initiating in the interior of the positive octant,

where $W_m=\dfrac{(s+\delta)^2 L}{4s\delta}+\dfrac{(r-r_0+\delta)^2 K}{4r\delta}$, $A_m=\dfrac{\eta W_m-\theta}{\eta_0}$, $\gamma_0>s_0$, $\nu\leq 1$, $r>r_0$ and $\eta\, W_m>\theta.$

Proof: Let (B,P,P_A,A) be solution with positive initial values (B_0,P_0,P_{A0},A_0). Let assume function W as

$$W=B+P+P_A, \tag{3.1}$$

$$\frac{dW}{dt}=\frac{dB}{dt}+\frac{dP}{dt}+\frac{dP_A}{dt}. \tag{3.2}$$

From system (2.1), we get

$$\frac{dW}{dt} \le sB - \frac{sB^2}{L} - (\gamma_0 - s_0)P_A - (1-\nu)\gamma\, PA + (r - r_0)P - \frac{rP^2}{K}. \tag{3.3}$$

Assuming that $\nu < 1, \gamma_0 > s_0$, so from equation (3.3), we get

$$\frac{dW}{dt} + \delta W \le (s+\delta)B - \frac{sB^2}{L} + (r - r_0 + \delta)P - \frac{rP^2}{K}. \tag{3.4}$$

$$\frac{dW}{dt} \le \frac{(s+\delta)^2 L}{4s} + \frac{(r - r_0 + \delta)^2 K}{4r} - \delta W.$$

with condition $r > r_0$. (3.5)

According to comparison principle, it follows that

$$W_m = \frac{(s+\delta)^2 L}{4s\delta} + \frac{(r - r_0 + \delta)^2 K}{4r\delta}. \tag{3.6}$$

Thus $0 \le B \le W_m, 0 \le P \le W_m$ and $0 \le P_A \le W_m$. (3.7)

Now from the system (2.1), we get

$$\frac{dA}{dt} \le \eta\, W_m\, A - \eta_0\, A^2 - \theta\, A. \tag{3.8}$$

According to comparison principle again, we get

$$A_m = \frac{\eta\, W_m - \theta}{\eta_0}. \tag{3.9}$$

The value of A_m is always positive if $\eta\, W_m > \theta$.

This completes the proof of lemma.

From the above and the model (2.1) it is noted that (see article 6)

$$s > \alpha W_m\ , r + \alpha\ B_{\min} > \gamma\ A_m + r_0 \text{ and } \eta\ P_{\min} > \theta. \tag{3.10}$$

4. Equilibrium Analysis:

The system (2.1) has five nonnegative equilibrium points namely; trivial equilibrium $E_0(0,0,0,0)$, boundary equilibrium points $E_1(L,0,0,0)$, $E_2\left(0, \frac{(r-r_0)K}{r}, 0,0\right)$, $E_3(\overline{B}, \overline{P}, 0,0)$ and interior equilibrium $E_4(B^*, P^*, P_A{}^*, A^*)$. The existence of equilibrium points $E_0(0,0,0,0)$, $E_1(L,0,0,0)$ or $E_2\left(0, \frac{(r-r_0)K}{r}, 0,0\right)$ is obvious with condition (3.5). We show the existence of $E_3(\overline{B}, \overline{P}, 0,0)$ and interior equilibrium point $E_4(B^*, P^*, P_A{}^*, A^*)$ as follows:

Existence of $E_3(\overline{B}, \overline{P}, 0,0)$

Here $\overline{B}$ and $\overline{P}$ are the positive solutions of the following equations:

$$s\left(1-\frac{B}{L}\right) - \alpha P = 0, \tag{4.1}$$

$$r\left(1-\frac{P}{K}\right) + \alpha B - r_0 = 0. \tag{4.2}$$

From equations (4.1) and (4.2) we get

$$\overline{B} = \frac{[sr-(r-r_0)\alpha K]L}{LK\alpha^2 + sr},\ \ \overline{P} = (L\alpha + r - r_0)\alpha K\,. \tag{4.3}$$

The values of $\overline{B}$ and $\overline{P}$ are always positive if following condition holds with equation (3.5):

$$sr > (r-r_0)\alpha K\,. \tag{4.4}$$

Existence of $E_2(B^*, P^*, P_A{}^*, A^*)$

Here $B^*, P^*, P_A{}^*$ and A^* are the positive solutions of the system of algebraic equations given below

$$sB^*\left(1-\frac{B^*}{L}\right) - \alpha B^* P^* + s_0 P_A{}^* = 0, \tag{4.5}$$

$$r\left(1-\frac{P*}{K}\right)+\alpha B*-\gamma\, A*-r_0\ =0, \tag{4.6}$$

$$\nu\,\gamma\, P*A*-\gamma_0 P_A*=0, \tag{4.7}$$

$$\eta\, P*-\eta_0\ A*-\theta=0. \tag{4.8}$$

From equation (4.8), we get

$$A^* = \frac{\eta\, P^* - \theta}{\eta_0}. \tag{4.9}$$

The value of A^* remains positive if $P^* > \dfrac{\theta}{\eta}$.

Using equation (4.9) in equation (4.7), we have

$$P_A^* = \frac{\nu\,\gamma\, P^*\left(\eta P^* - \theta\right)}{\gamma_0\eta_0}. \tag{4.10}$$

Using the equation (4.9) in the equation (4.6), we get P^* as follows,

$$P^* = d_1 + d_2 B^*. \tag{4.11}$$

where

$$d_1 = \frac{\eta_0\left(r-r_0\right)+\theta\,\gamma}{\eta_0\, D}, d_2 = \frac{\alpha}{D} \text{ and } D = \frac{r}{K}+\frac{\gamma\,\eta}{\eta_0}. \tag{4.12}$$

Now using (4.10) and (4.11) in (4.5), we get the following quadratic equation for determining B^*

$$A_0 B^{*2} - B_0 B^* - C_0\ = 0. \tag{4.13}$$

where

$$A_0 = \frac{s}{L}+\alpha\, d_2 - \frac{s_0\gamma\,\nu\,\eta\, d_2^2}{\gamma_0\eta_0},\ \ B_0 = s - \alpha\, d_1 + \frac{s_0\gamma\,\nu\, d_2\left(2\eta d_1 - \theta\right)}{\gamma_0\eta_0},\quad C_0 = \frac{s_0\gamma\,\nu\, d_1\left(d_1\eta - \theta\right)}{\gamma_0\eta_0}. \tag{4.14}$$

It is assumed that even if there is no the conservation i.e. $s_0 = 0$, the equilibrium B^* exists. Then when $s_0 = 0, C_0 = 0$. From (4.13) and (2.1) in this case it is seen that $A_0 > 0, B_0 > 0$. Thus for $s_0 \neq 0$, also we assume that

$$A_0 > 0 \text{ and } B_0 > 0. \tag{4.15}$$

Now solving (4.13), we get a positive root

$$B^* = \frac{B_0 + \sqrt{B_0^2 + 4A_0C_0}}{2A_0}, \tag{4.16}$$

under condition (4.15).

After knowing B^* from (4.16) we can determine P^*, P_A^* and A^* from (4.11),(4.10) and (4.9) respectively.

To see the effect of conservation parameter s_0, we can find $\frac{dB^*}{ds_0}$ from (4.13) as

$$\frac{dB^*}{ds_0} = \frac{NB^*}{A_0B^{*2} + C_0} > 0. \tag{4.17}$$

where $N = \frac{\gamma\, \nu\, d_1(d_1\eta - \theta)}{\gamma_0\eta_0} + \frac{2\gamma\, \eta\, d_1 d_2}{\gamma_0\eta_0} B^* + \frac{\gamma\, \eta\, d_2^2}{\gamma_0\eta_0} B^{*2}$. (4.18)

It is therefore concluded that B^* increases s_0 increases i.e. the biomass density increases as conservation effort by rural population increases.

5. Stability Analysis

5.1 Local Stability

To discuss the local stability of system (2.1) as follows,

$$V(E)=\begin{bmatrix} e_{11} & e_{12} & e_{13} & 0 \\ e_{21} & e_{22} & 0 & e_{24} \\ 0 & e_{32} & e_{33} & e_{34} \\ 0 & e_{42} & 0 & e_{44} \end{bmatrix}.$$

Where the entries in the matrix are

$$e_{11}=s-\frac{2sB}{L}-\alpha P,\quad e_{12}=-\alpha B,\quad e_{13}=s_0,\quad e_{21}=\alpha P,\quad e_{22}=r-\frac{2rP}{K}+\alpha B-\gamma A-r_0,\quad e_{24}=-\gamma P,$$

$$e_{32}=\nu\gamma A,\ e_{33}=-\gamma_0,\ e_{34}=\nu\gamma P,\ e_{42}=\eta A,\ e_{44}=\eta P-2\eta_0 A-\theta. \tag{5.1}$$

Accordingly, the linear stability analysis about the equilibrium points $E_i, i=0,1,2,3,4$ gives the following results:

9. The equilibrium point E_0 is unstable manifold in $B-P$ plane and stable manifold in P_A-A plane.
10. The equilibrium point E_1 is unstable manifold in $P-$ direction and stable manifold in $B-P_A-A$ plane.
11. The equilibrium point E_2 is unstable manifold in $B-P-A$ plane and stable manifold in P_A- direction.
12. The equilibrium point E_3 is unstable manifold in $A-$ direction and stable manifold in P_A- direction.

The stability behaviour of equilibrium point E_4 is not obvious. However, in the following theorem we give sufficient conditions for equilibrium point E_4 to be locally asymptotically stable.

Theorem (5.1) Let the following inequality holds:

$$s_0^2 < \frac{1}{3}\gamma_0\left(\frac{2sB^*}{L} + \alpha P^* - s\right)Min(M_{21}, M_{22}). \tag{5.2}$$

Then the equilibrium point E_4 is locally asymptotically stable. For proof, see Appendix A.

5.2 Global Stability

The following theorem characterizes the global stability behavior of equilibrium point E_4.

Theorem (5.2) Let the following inequality holds:

$$\alpha^2\left(W_m - B^*\right)^2 < \frac{2rB^*}{3K}\left(\frac{sB^*}{L} + \alpha P^* - s\right), \tag{5.3}$$

$$s_0^2 < \frac{2}{3}\gamma_0\left(\frac{sB^*}{L} + \alpha P^* - s\right)Min(C_{21}, C_{22}). \tag{5.4}$$

Then equilibrium point E_4 is globally stable in the region Ω. For proof, see Appendix B.

6. Persistence

Theorem (6.1) Assume that $s > \alpha W_m$, $r + \alpha\, B_{\min} > \gamma\, A_m + r_0$ and $\eta\, P_{\min} > \theta.$. Here W_m and A_m are upper bounds of the populations B, P, P_A, A respectively and always positive. Then system (2.1) persists.

Proof: From the system (2.1), we have

$$\frac{dB}{dt} \geq (s - \alpha\, W_m)B - \frac{sB^2}{L}. \tag{6.1}$$

According to lemma (3.1) and comparison principle, it follows that

$$B_{\min} = \frac{(s - \alpha\, W_m)L}{s}. \tag{6.2}$$

With condition $s > \alpha\, W_m$, $B_{\min}$ remains always positive.

From the system (2.1) and equation (6.2), we have

$$\frac{dP}{dt} \geq \left(r + \alpha\, B_{\min} - \gamma\, A_m - r_0\right)P - \frac{r}{K}P^2. \tag{6.3}$$

According to lemma (3.1) and comparison principle, it follows that

$$P_{\min} = \frac{\left(r + \alpha\, B_{\min} - \gamma\, A_m - r_0\right)K}{r}. \tag{6.4}$$

With condition $r + \alpha B_{\min} > \gamma A_m + r_0$, $P_{\min}$ remains always positive.

From the last equation of the system (2.1) and equation (6.4), we have

$$\frac{dA}{dt} \geq \eta P_{\min} A - \eta_0 A^2 - \theta\, A. \tag{6.5}$$

Using comparison principle, it follows that

$$A_{\min} = \frac{\eta\, P_{\min} - \theta}{\eta_0}. \tag{6.6}$$

With condition $\eta\, P_{\min} > \theta$, $A_{\min}$ remains always positive.

From the system (2.1) and equations (6.2), (6.4), (6.6), we get

$$\frac{dP_A}{dt} \geq \nu\, \gamma\, P_{\min} A_{\min} - \gamma_0 P_A. \tag{6.7}$$

According to lemma (3.1) and comparison principle, it follows that

$$P_{A_{\min}} = \frac{\nu\, \gamma\, P_{\min} A_{\min}}{\gamma_0}. \tag{6.8}$$

This completes the proof of the theorem. Thus, system (2.1) persists if $s > \alpha W_m$, $r + \alpha\, B_{\min} > \gamma\, A_m + r_0$ and $\eta\, P_{\min} > \theta$..

7. Numerical Simulation

The stability of the non-linear model system (2.1), in the positive octant, is investigated numerically by using the following set of parameters.

$$s=20,\ \ L=25000,\ \ s_0=4,\ \ r=40,\ \ K=20000,\ \ \gamma=10,\ \ r_0=10,\ \ \gamma_0=8,\ \ \eta=2\ \ \alpha=0.000008,$$
$$\eta_0=1, \nu=1, \theta=0.02. \tag{7.1}$$

The interior equilibrium point of the model system (2.1) corresponding to the feasible parameters values as (7.1) is: $E_4(25001,\ 1.50984,\ \ 5.6991,\ 3.019698)$.

Figures 1, 2 and 3 are drawn to study the effect of important parameters on the system. Figure 1 shows the variation of resource biomass with cumulative density of awareness programs.

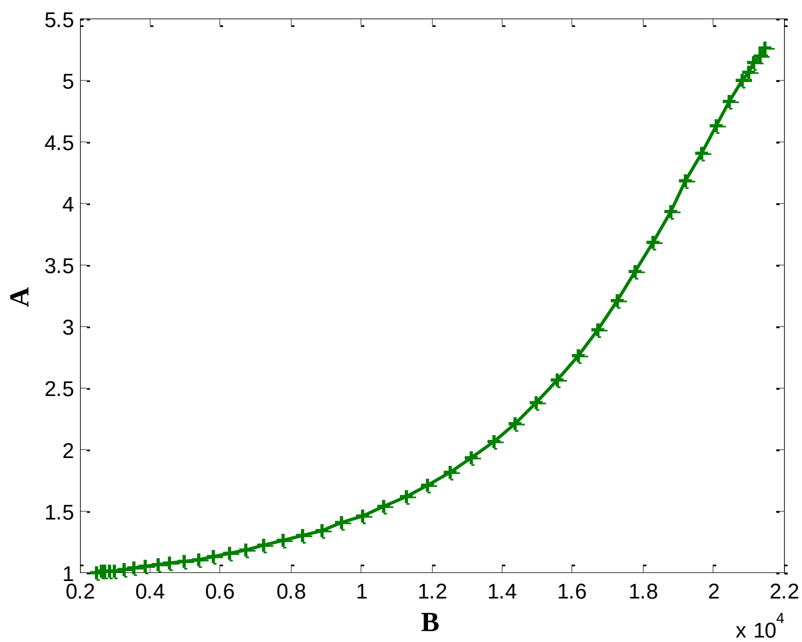

Figure 1. Graph B versus A .

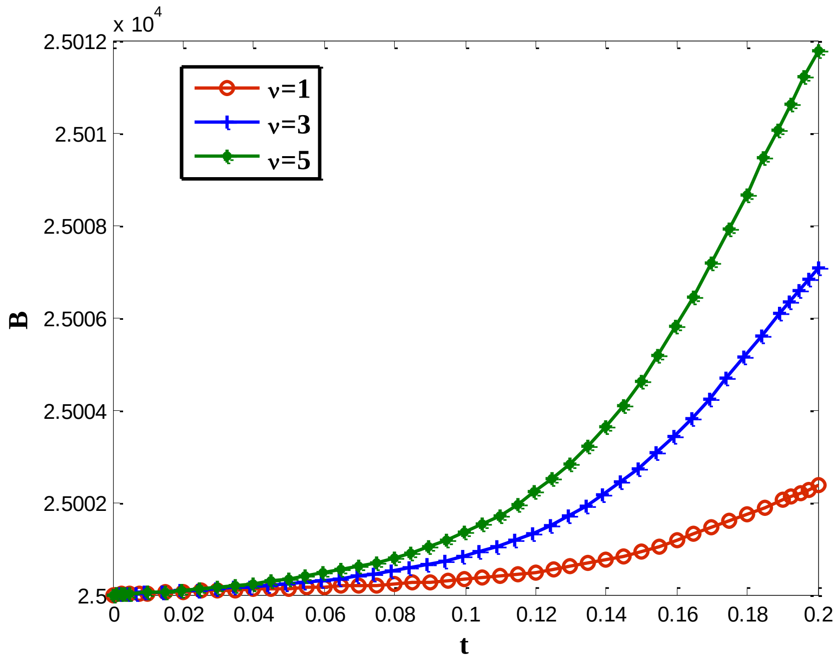

Figure 2. Variation of resource biomass with time for different intensity of ν .

In Figure 2 the variation of resource biomass with time for different intensity of ν in the system is studied. It is observed that as rate of educated rural population in the system increases equilibrium value of resource biomass increases. As the rural people become aware about sustainable management of resources, the equilibrium level of resource biomass increases. Figure 3 and Figure 4 show that density of aware rural population and cumulative density of awareness programs increase with increasing the values of ν and η respectively.

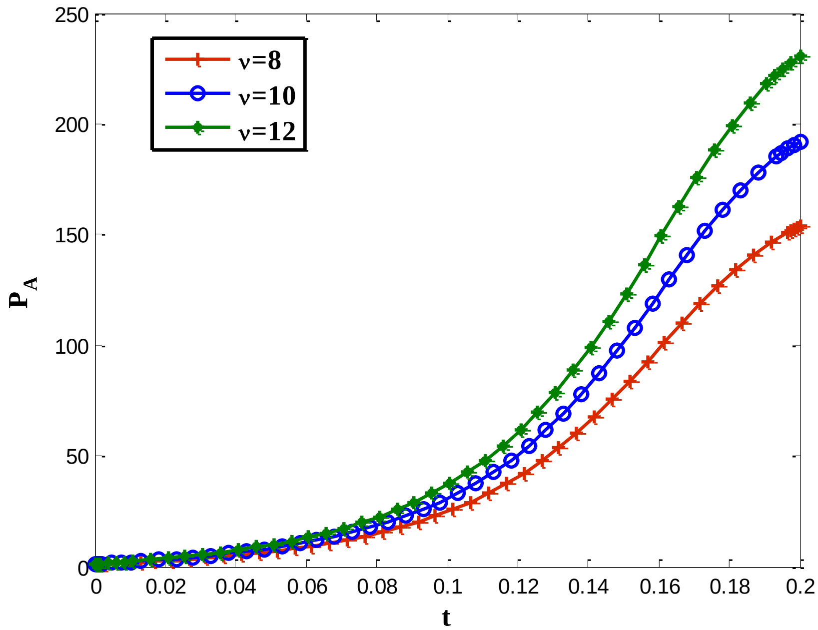

Figure 3. Variation of aware rural population with time for different intensity of ν .

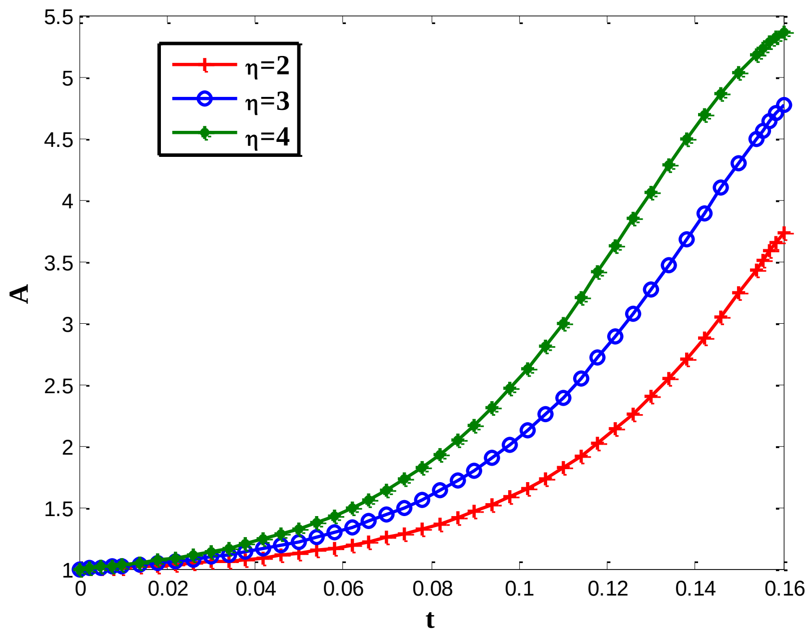

Figure 4. Variation of awareness programs with time for different intensity of η .

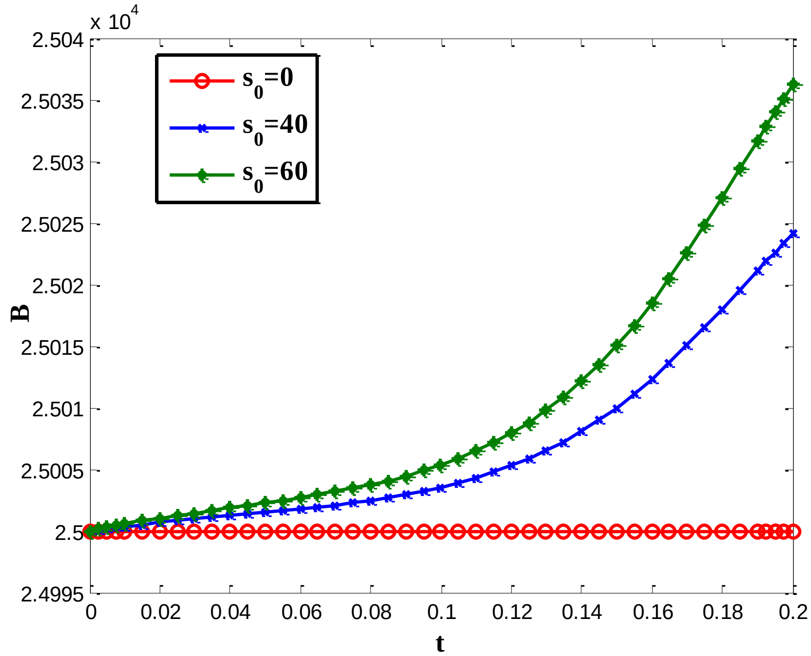

Figure 5. Variation of resource biomass with time for different rates of s_0 **.**

In Figure 5, the variation of resource biomass with time for different rate of s_0 is shown where s_0 is the rate at which resource biomass is benefitted by the awareness program launched. It is observed from the figure that density of resource biomass decreases gradually by continuous use of resource biomass without any awareness program. However when awareness program is launched then resource biomass retains it equilibrium value and approaches the same equilibrium level as if it is unused by the rural population.

8. Conclusions

The rural population is highly dependent on forest resources for its livelihood. The degradation and depletion of forest resources increase due to various activities related to rural population. Therefore it is imperative to educate rural population to use the forest products judiciously and make efforts for sustainable management of forest resources. Keeping this in view we have proposed and analyzed a non linear mathematical model to study the impact of awareness program among rural population. In the model the following four variables have been used.

I. The cumulative density of forest resources.

II. The density of rural population is living around the forest.

III. The density of aware population.

IV. The density of rural aware programs.

The analysis of the model includes equilibrium analysis, local and global stability analysis and persistence of the system. Numerical simulation has been performed to justify the analytical findings and graphs are plotted to study the variations of important variables of the system with time for different parameters. The conditions for the local and global stability of the system have been determined and these conditions are further justified numerically by considering a given set of parameter values. The model analysis has shown that as the rate of awareness programs increases the equilibrium value of the cumulative biomass density increases. This implies that as rural people become aware about the sustainable management of resources, the original equilibrium level of the biomass density can be maintained.

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Appendix A. Proof of Theorem (5.1)

To prove Theorem, we first linearize model (2.1) by substituting

$$B = B^* + b, P = P^* + p, P_A = P_A^* + p_A, A = A^* + a.$$

Where b, p, p_A, a are small perturbations around equilibrium point E_4. We get following linearize system

$$\frac{db}{dt} = sb - \frac{2sbB^*}{L} - \alpha bP^* - \alpha B^* p + s_0 p_A,$$

$$\frac{dp}{dt} = -\frac{rP^* p}{K} + \alpha P^* b - \gamma P^* a, \tag{A.1}$$

$$\frac{dp_A}{dt} = \nu \gamma A^* p + \nu \gamma P^* a - \gamma_0 p_A,$$

$$\frac{da}{dt} = \eta A^* p - \eta_0 A^* a.$$

Then we consider the following positive definite function:

$$U = \frac{1}{2} m_0 b^2 + \frac{1}{2} m_1 p^2 + \frac{1}{2} m_2 p_A^2 + \frac{1}{2} m_3 a^2. \tag{A.2}$$

Differentiating U with respect to time t along the linearized system (A.1), we get

$$\frac{dU}{dt} = -u_{11} b^2 + u_{12} bp + u_{13} bp_A + u_{24} ap + u_{23} pp_A + u_{34} ap_A - u_{22} p^2 - u_{33} p_A^2 - u_{44} a^2. \tag{A.3}$$

Where

$$u_{11} = m_0\left(\frac{2sB^*}{L} + \alpha P^* - s\right),\ u_{12} = \alpha\left(m_1 P^* - m_0 B^*\right),\ u_{13} = m_0 s_0,\ u_{22} = \frac{r\, m_1 P^*}{K},\ u_{33} = m_2 \gamma_0,$$

$$u_{44} = m_3 \eta_0 A^*,\ u_{24} = m_3 \eta\, A^* - m_1 \gamma\, P^*,\ u_{34} = m_2 \nu\, \gamma\, P^*,\ u_{23} = m_2 \nu\, \gamma\, A^*.$$

It is noted from (4.5) that u_{11} is positive.

Choosing $m_1 P^* = m_0 B^*$ and $m_3 \eta\, A^* = m_1 \gamma\, P^*$, $\frac{dU}{dt}$ simplified as

$$\frac{dU}{dt} = -u_{11} b^2 + u_{13} b p_A + u_{23} p p_A + u_{34} a p_A - u_{22} p^2 - u_{33} p_A^2 - u_{44} a^2. \tag{A.4}$$

Then for $\frac{dU}{dt}$ to be negative definite, the following inequalities must hold:

$$m_0 s_0^2 < \frac{2 m_2 \gamma_0}{3}\left(\frac{2sB^*}{L} + \alpha P^* - s\right), \tag{A.5}$$

$$m_2 < \frac{4 m_0 \gamma_0 r B^*}{3K \nu\, \gamma^2 A^{*2}} = m_0\, M_{21}, \tag{A.7}$$

$$m_2 < \frac{2 m_0 \gamma_0 \eta_0 B^*}{3\eta\, \nu\, \gamma\, P^{*2}} = m_0\, M_{22}. \tag{A.8}$$

Now we choose m_2 as

$$m_2 = \frac{1}{2} m_0 Min\left(M_{21}, M_{22}\right). \tag{A.9}$$

Then from (A.5), the condition $\frac{dU}{dt}$ would be negative definite if

$s_0^2 < \frac{1}{3}\gamma_0\left(\frac{2sB^*}{L} + \alpha P^* - s\right) Min\left(M_{21}, M_{22}\right)$ as stated in Theorem (5.1).

Appendix B. Proof of Theorem (5.2)

For finding the condition of global stability at E_4 in region Ω we construct the Lyapunov function

$$H=\frac{1}{2}c_0\left(B-B^*\right)^2+c_1\left(P-P^*-P^*\ln\frac{P}{P^*}\right)+\frac{1}{2}c_2\left(P_A-P_A^*\right)^2+\frac{1}{2}c_3\left(A-A^*-A^*\ln\frac{A}{A^*}\right). \quad \text{(B.1)}$$

Differentiating H with respect to time t along the solutions of model (2.1), we get

$$\frac{dH}{dt}=c_0\left(B-B^*\right)\frac{dB}{dt}+\frac{c_1\left(P-P^*\right)}{P}\frac{dP}{dt}+c_2\left(P_A-P_A^*\right)\frac{dP_A}{dt}+c_3\frac{\left(A-A^*\right)}{A}\frac{dA}{dt}. \quad \text{(B.2)}$$

Using system of equation (2.1), we get after some algebraic manipulations as

$$\frac{dH}{dt}=-c_0\left[\frac{s\left(B+B^*\right)}{L}+\alpha P^*-s\right]\left(B-B^*\right)^2-\frac{rc_1}{K}\left(P-P^*\right)^2-\gamma_0c_2\left(P_A-P_A^*\right)^2-c_3\eta_0\left(A-A^*\right)^2$$
$$-\alpha\left(\mathrm{c}_0B-c_1\right)\left(B-B^*\right)\left(P-P^*\right)+c_0s_0\left(B-B^*\right)\left(P_A-P_A^*\right)+c_2\nu\,\gamma\,P^*\left(P_A-P_A^*\right)\left(A-A^*\right)+$$
$$\left(c_3\eta-c_1\gamma\right)\left(A-A^*\right)\left(P-P^*\right)+c_2\nu\,\gamma\,A\left(P_A-P_A^*\right)\left(P-P^*\right). \quad \text{(B.3)}$$

First we choose $c_1=c_0B^*, c_3=c_1\dfrac{\gamma}{\eta}=c_0\dfrac{\gamma}{\eta}B^*$. (B.4)

Then $\dfrac{dH}{dt}$ to be negative definite, the following inequalities must hold.

$$\alpha^2\left(W_m-B^*\right)^2<\frac{2rB^*}{3K}\left(\frac{sB^*}{L}+\alpha P^*-s\right), \quad \text{(B.5)}$$

$$c_0s_0^2<\frac{2c_2\gamma_0}{3}\left(\frac{sB^*}{L}+\alpha P^*-s\right), \quad \text{(B.6)}$$

$$c_2<\frac{4r\,c_0\gamma_0B^*}{3K\nu\,\gamma^2A_m^2}=c_0C_{21}, \quad \text{(B.7)}$$

$$c_2<\frac{2c_0\eta_0\,\gamma_0B^*}{3\nu\,\gamma\,\eta\,P^{*2}}=c_0C_{22}. \quad \text{(B.8)}$$

Then c_2 can be chosen as

$$c_2 < c_0 \; Min(C_{21}, C_{22}). \tag{B.9}$$

Then from (B.6), we have

$$s_0^2 < \frac{2}{3}\gamma_0\left(\frac{sB^*}{L} + \alpha P^* - s\right) Min(C_{21}, C_{22}). \tag{B.10}$$

Then $\frac{dH}{dt}$ is negative definite under condition (B.5) and (B.10) as stated in the Theorem (5.2).

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Modeling the control of CO_2 from the atmosphere by using greenbelts plantation and seaweeds cultivation

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Abstract

The most important greenhouse gases in the atmosphere are carbon dioxide, methane, nitrous oxide, ozone, CFCs. Among these, carbon dioxide is the main contributor to global warming and therefore its control is very desirable. In this paper, the control of a CO_2 from the atmosphere by using greenbelts plantation and seaweeds cultivation has been modeled and analyzed. The proposed model consists of four nonlinearly interacting variables namely; the cumulative biomass densities of greenbelts plantation and seaweeds cultivation, the concentration of carbon dioxide in the atmosphere and the atmospheric temperature. The model is analyzed using the stability theory of differential equations. The analysis shows that the concentration of carbon dioxide in the atmosphere decreases as the biomass densities of greenbelts plantation and seaweeds cultivation

increases. Thus, it can be considerably controlled by using large area of plantation. The numerical simulation of the model has also been performed to confirm these analytical results.

Keywords: Model, global warming, seaweeds, greenbelts, stability

1. Introduction

One of the most serious problems that our society faces today is global warming due to increase of greenhouse gases in the atmosphere caused by various human population density related factors. The main global warming contributors in the atmosphere are carbon dioxide (CO_2) and methane (CH_4). The CO_2 gas is the largest contributor to global warming with a share of 60 percent, UNFC (1992). Since global warming due to carbon dioxide affects human life and the environment, its control from the atmosphere is absolutely necessary, Houghton et al. (2015).

Greenbelt plays a significant role to regulate the concentration of carbon dioxide in the atmosphere as it acts like a natural sink, Malhi and Grace (2000). Similarly, seaweeds cultivation also helps in reducing thus reducing CO_2 from the atmosphere, Mathien et al. (2016).

A simple chemical equation for photosynthesis in green leaves to transform CO_2 gas into glucose is given below

Carbon dioxide + water + light energy = glucose + oxygen

$$6CO_2 + 6H_2O + \text{light energy} \rightarrow C_6H_{12}O_6 + 6O_2$$

It is now well documented that rising atmospheric CO_2 levels and the consequent changes in ocean carbon chemistry (ocean acidification) have profound implications for the physiology of many marine organisms, Robinson (2007). The effect of climate change on life and livelihood has been studied, Rautela and Karki (2015).

Some modeling studies have been conducted to study the abatement of carbon dioxide in the atmosphere using nonlinear mathematical models, Shukla et al (2015), Sundar et al (2014), Misra and Verma (2013), Tennakone (1990). Apart from a greenbelt plantation, the idea of removing carbon dioxide from the atmosphere by using seaweed cultivation is also important. Therefore, in the this paper, we propose a mathematical model to study the abatement of carbon dioxide from the atmosphere using greenbelts plantation and seaweeds cultivation, Mathien et al. (2016), Duarte et al. (2005).

2. Mathematical model

To model the problem of control of carbon dioxide concentration from the atmosphere by using greenbelts plantation and seaweeds cultivation, let B_1 be the biomass density of greenbelts plantation, B_2 be the biomass density of seaweeds cultivation, C be the concentration of carbon dioxide in the atmosphere and C_0 be the equilibrium concentration of carbon dioxide which corresponds to the equilibrium temperature T_0 , T be the average atmospheric temperature and T_0 be the equilibrium temperature. We have made the following assumptions in making the mathematical model,

1. The cumulative emission rate of carbon dioxide from different sources is assumed to be a constant.

2. Carbon dioxide is used by leafy trees in greenbelts during the process of photosynthesis in the presence of sunlight thereby lowering the concentration of carbon dioxide in the atmosphere.
3. Carbon dioxide is also used in the growth of seaweeds by photosynthesis process. Due to this, its concentration decreases in the atmosphere.
4. The growth of biomass density B_1 in a greenbelt is assumed to be governed by a logistic equation where s_1 and L_1 are the intrinsic growth rate and the carrying capacity of B_1 respectively.
5. The abatement in carbon dioxide concentration in the atmosphere is assumed to be directly proportional to the biomass density of B_1 as well as the concentration of carbon dioxide (i.e. $\pi_1 B_1 C$), π_1 being the growth rate coefficient of biomass due to uptake of carbon dioxide.
6. The growth rate of biomass density B_2, seaweeds cultivation, is also assumed to be governed by a logistic equation where s_2 and L_2 are the intrinsic growth rate and the carrying capacity of B_2 respectively.
7. Due to seaweeds cultivation, the abatement in carbon dioxide concentration in the atmosphere is assumed to be directly proportional to the biomass density of seaweeds as well as the concentration of carbon dioxide (i.e. $\pi_2 B_2 C$), π_2 being the rate at which carbon dioxide.
8. The cumulative rate of discharge of CO_2 from different sources such as chimneys of power plants, industries, etc. is Q_0 with its natural depletion rate $\alpha_0 C$.

9. The depletion of carbon dioxide due to interaction with greenbelts is assumed to be in direct proportion to the cumulative concentration of carbon dioxide and the biomass density of greenbelts plantation (i.e. $\alpha_1 B_1 C$), α_1 being the rate of depletion of carbon dioxide due to natural factors.

10. The depletion of carbon dioxide due to interaction with seaweeds is assumed to be in direct proportion to the cumulative concentration of carbon dioxide and the biomass density of seaweeds cultivation (i.e. $\alpha_2 B_2 C$), α_2 being the rate of depletion of carbon dioxide due to seaweeds.

11. The atmospheric temperature is directly proportional to the difference of its cumulative concentration C of CO_2 and equilibrium concentration C_0 i.e. $\theta(C - C_0)$, where θ is its growth rate coefficient and constant θ_0 is the natural depletion rate coefficient of atmospheric temperature and T_0 be the average atmospheric temperature..

In view of the above assumptions, a four dimensional mathematical model governing the phenomenon of control of carbon dioxide from the atmosphere by using greenbelts plantation and seaweeds cultivation is proposed as follows:

$$\frac{dB_1}{dt} = s_1 B_1 - \frac{s_1 B_1{}^2}{L_1} + \pi_1 B_1 C \tag{1}$$

$$\frac{dB_2}{dt} = s_2 B_2 - \frac{s_2 B_2{}^2}{L_2} + \pi_2 B_2 C \tag{2}$$

$$\frac{dC}{dt} = Q_0 - \alpha_0 C - \alpha_1 B_1 C - \alpha_2 B_2 C \tag{3}$$

$$\frac{dT}{dt} = \theta(C - C_0) - \theta_0 (T - T_0) \tag{4}$$

with $B_1(0) \geq 0, B_2(0) \geq 0, C(0) \geq 0, T(0) \geq 0$. Here all the parameters are assumed to be positive.

To analyze the model system (1) – (4), we need the bounds of dependent variables. For this, we establish the region of attraction in the following lemma.

Lemma: The set

$\Omega = \{(B_1, B_2, C, T) \in R_+^4 : 0 \leq B_1 \leq B_{1m}, 0 \leq B_2 \leq B_{2m}, 0 \leq C \leq C_m, 0 \leq T \leq T_m\}$, where

$$B_{1m} = \frac{L_1}{s_1}\left(s_1 + \pi_1 \frac{Q_0}{\alpha_0}\right),\ B_{2m} = \frac{L_2}{s_2}\left(s_2 + \pi_2 \frac{Q_0}{\alpha_0}\right),\ C_m = \frac{Q_0}{\alpha_0},\ T_m = T_0 + \frac{\theta}{\theta_0}\left(\frac{Q_0}{\alpha_0} - C_0\right),$$

is the region of attraction for all solutions of the model system (1) – (4) initiating in the interior of positive octant.

As discussed above, $C > C_0$ for all practical purposes and hence it is obvious that $\frac{Q_0}{\alpha_0} > C_0$.

3. Equilibrium Analysis

The model system (1)-(4) has four non-negative equilibria which are listed as follows:

(i) $E_1\left(0, 0, \frac{Q}{\alpha_0}, \hat{T}\right)$, where, $\hat{T} = T_0 + \frac{\theta}{\theta_0}\left(\frac{Q}{\alpha_0} - C_0\right)$

(ii) $E_2(0, \overline{B}_2, \overline{C}, \overline{T})$

(iii) $E_3(\tilde{B}_1, 0, \tilde{C}, \tilde{T})$

(v) $E^*(B_1^{\,*}, B_2^{\,*}, C^*, T^*)$

Here it may be noted that the existence of equilibrium E_1 is obvious.

3.1. Existence of $E_2(0, \overline{B}_2, \overline{C}, \overline{T})$

The solution of $E_2(0, \overline{B}_2, \overline{C}, \overline{T})$ is given by the following set of algebraic equations,

$$s_2 - \frac{s_2 B_2}{L_2} + \pi_2 C = 0 \tag{5}$$

$$Q_0 - \alpha_0 C - \alpha_1 B_1 C - \alpha_2 B_2 C = 0 \tag{6}$$

$$\theta(C - C_0) - \theta_0 (T - T_0) = 0 \tag{7}$$

From (5), we get

$$B_2 = \frac{L_2}{s_2}(s_2 + \pi_2 C) \tag{8}$$

Using equation (8) in equation (6), we get

$$F(C) = Q_0 - (\alpha_0 + \alpha_2 L_2)C - \frac{\pi_2 \alpha_2 L_2}{s_2} C^2 = 0 \tag{9}$$

From which we get

(i) $F(0) = Q_0 > 0$

(ii) $F\left(\dfrac{Q_0}{\alpha_0}\right) = -\alpha_2 L_2 \dfrac{Q_0}{\alpha_0} - \dfrac{\pi_2 \alpha_2 L_2}{s_2}\left(\dfrac{Q_0}{\alpha_0}\right)^2 < 0$

This implies that there exists a root of $F(C) = 0$ (say $C = \overline{C}$) in $0 \le C \le \dfrac{Q_0}{\alpha_0}$.

For the uniqueness of the root, we have

$$F'(C) = -(\alpha_0 + \alpha_2 L_2) - 2\frac{\pi_2 \alpha_2 L_2}{s_2} C < 0$$

Thus $F(C) = 0$ has a unique root (say $C = \overline{C}$) in $0 \le C \le \frac{Q_0}{\alpha_0}$ without any condition.

3.2. Existence of $E_3(\tilde{B}_1, 0, \tilde{C}, \tilde{T})$

The solution of $E_4(\tilde{B}_1, 0, \tilde{C}, \tilde{T})$ is given by the following set of algebraic equations,

$$s_1 - \frac{s_1 B_1}{L_1} + \pi_1 C = 0 \tag{10}$$

$$Q_0 - \alpha_0 C - \alpha_1 B_1 C - \alpha_2 B_2 C = 0 \tag{11}$$

$$\theta(C - C_0) - \theta_0 (T - T_0) = 0 \tag{12}$$

From (10), we get

$$B_1 = \frac{L_1}{s_1}(s_1 + \pi_1 C) \tag{13}$$

Using equation (13) in equation (11), we get

$$G(C) = Q_0 - (\alpha_0 + \alpha_1 L_1)C - \frac{\pi_1 \alpha_1 L_1}{s_1} C^2 = 0 \tag{14}$$

From which we get

(i) $G(0) = Q_0 > 0$

(ii) $G\left(\frac{Q_0}{\alpha_0}\right) = -\alpha_1 L_1 \frac{Q_0}{\alpha_0} - \frac{\pi_1 \alpha_1 L_1}{s_1}\left(\frac{Q_0}{\alpha_0}\right)^2 < 0$

This implies that there exists a root of $G(C)=0$ (say $C=\tilde{C}$) in $0 \le C \le \frac{Q_0}{\alpha_0}$.

For the uniqueness of the root, we have

$$G'(C) = -(\alpha_0 + \alpha_1 L_1) - 2\frac{\pi_1 \alpha_1 L_1}{s_1} C < 0$$

Thus $G(C)=0$ has a unique root (say $C=\tilde{C}$) in $0 \le C \le \frac{Q_0}{\alpha_0}$ without any condition.

3.3. Existence of $E^*(B_1^*, B_2^*, C^*, T^*)$

The solution of $E^*(B_1^*, B_2^*, C^*, T^*)$ is given by the following set of algebraic equations,

$$s_1 - \frac{s_1 B_1}{L_1} + \pi_1 C = 0 \qquad (15)$$

$$s_2 - \frac{s_2 B_2}{L_2} + \pi_2 C = 0 \qquad (16)$$

$$Q_0 - \alpha_0 C - \alpha_1 B_1 C - \alpha_2 B_2 C = 0 \qquad (17)$$

$$\theta(C - C_0) - \theta_0 (T - T_0) = 0 \qquad (18)$$

From (15), we get

$$B_1 = \frac{L_1}{s_1}(s_1 + \pi_1 C) \qquad (19)$$

From (16), we get

$$B_2 = \frac{L_2}{s_2}(s_2 + \pi_2 C) \tag{20}$$

Using equations (19) and (20) in equation (17), we get

$$H(C) = Q_0 - (\alpha_0 + \alpha_1 L_1 + \alpha_2 L_2)C - \left(\frac{\pi_1 \alpha_1 L_1}{s_1} + \frac{\pi_2 \alpha_2 L_2}{s_2}\right)C^2 = 0 \tag{21}$$

From which we get

(i) $H(0) = Q_0 > 0$

(ii) $H\left(\frac{Q_0}{\alpha_0}\right) = -(\alpha_1 L_1 + \alpha_2 L_2)\frac{Q_0}{\alpha_0} - \left(\frac{\pi_1 \alpha_1 L_1}{s_1} + \frac{\pi_2 \alpha_2 L_2}{s_2}\right)\left(\frac{Q_0}{\alpha_0}\right)^2 < 0$

This implies that there exists a root of $H(C) = 0$ (say C^*) in $0 \le C \le \frac{Q_0}{\alpha_0}$.

For the uniqueness of the root, we have

$$H'(C) = -(\alpha_0 + \alpha_1 L_1 + \alpha_2 L_2) - 2\left(\frac{\pi_1 \alpha_1 L_1}{s_1} + \frac{\pi_2 \alpha_2 L_2}{s_2}\right)C < 0$$

Thus $H(C) = 0$ has a unique root (say $C = C^*$) in $0 \le C \le \frac{Q_0}{\alpha_0}$ without any condition.

3.4. Variations of C and T with different parameters

Variation of C with α_1

Differentiating (21) with respect to α_1, we have

$$\frac{dC}{d\alpha_1} = -\frac{L_1 C\left(1+\frac{\pi_1 C}{s_1}\right)}{(\alpha_0 + \alpha_1 L_1 + \alpha_2 L_2) + 2C\left(\frac{\pi_1 \alpha_1 L_1}{s_1} + \frac{\pi_2 \alpha_2 L_2}{s_2}\right)} < 0,$$

This implies that the concentration of carbon dioxide C decreases as the rate of its absorption by greenbelt (i.e. α_1**)** increases**.** This implies that more greenbelts plantation decrease the cumulative concentration of CO_2 from the atmosphere.

Variation of T with α_1

Differentiating (18) with respect to 'C', we get

$$\frac{dT}{dC} = \frac{\theta}{\theta_0} > 0$$

Since $\frac{dC}{d\alpha_1} < 0$, therefore, from $\frac{dT}{d\alpha_1} = \frac{dT}{dC}\frac{dC}{d\alpha_1}$, we have $\frac{dT}{d\alpha_1} < 0$

This implies that the atmospheric temperature T decreases as the rate of absorption of CO_2 (i.e. α_1**)** by greenbelts increases. Similar results are also found with seaweeds cultivation.

4. Stability Analysis

4.1. Local stability

In order to study the local stability character of the equilibria, we first compute the Jacobian matrices corresponding to each equilibrium and from these we note the following,

(i) E_1 is saddle point with unstable manifold locally in the $B_1 - B_2$ plane and with stable manifold locally in the $C - T$ plane.

(ii) E_2 is also a saddle point with stable manifold locally in the $B_2 - C - T$ space and with unstable manifold locally in the B_1 direction.

(ii) E_3 is also a saddle point with stable manifold locally in the $B_1 - C - T$ space and with unstable manifold locally in the B_2 direction.

To present local stability behaviour of the model system (1) - (4) corresponding to the equilibrium $E^*(B_1{}^*, B_2{}^*, C^*, T^*)$, we compute the following Jacobian matrix,

$$J(E^*) = \begin{bmatrix} -\dfrac{s_1 B_1{}^*}{L_1} & 0 & \pi_1 B_1{}^* & 0 \\ 0 & -\dfrac{s_2 B_2{}^*}{L_2} & \pi_2 B_2{}^* & 0 \\ -\alpha_1 C^* & -\alpha_2 C^* & -(\alpha_0 + \alpha_1 B_1{}^* + \alpha_2 B_2{}^*) & 0 \\ 0 & 0 & \theta & -\theta_0 \end{bmatrix}$$

It is clear that one eigenvalue of the above Jacobian matrix $J(E^*)$ is negative and the others are given by the following characteristic equation,

$$x^3 + A_1 x^2 + A_2 x + A_3 = 0 \tag{22}$$

where,

$$A_1 = \frac{s_1 B_1{}^*}{L_1} + \frac{s_2 B_2{}^*}{L_2} + \alpha_0 + \alpha_1 B_1{}^* + \alpha_2 B_2{}^* > 0$$

$$A_2 = \frac{s_1 s_2 B_1^* B_2^*}{L_1 L_2} + \frac{s_2 B_2^*}{L_2}(\alpha_0 + \alpha_1 B_1^* + \alpha_2 B_2^*) + (\alpha_0 + \alpha_1 B_1^* + \alpha_2 B_2^*)\frac{s_1 B_1^*}{L_1}$$

$$+ \pi_1 \alpha_1 B_1^* C^* + \pi_2 \alpha_2 B_2^* C^* > 0$$

$$A_3 = \frac{s_1 s_2 B_1^* B_2^*}{L_1 L_2}(\alpha_0 + \alpha_1 B_1^* + \alpha_2 B_2^*) + \frac{s_1 B_1^*}{L_1}\pi_2 \alpha_2 B_2^* C^* + \frac{s_2 B_2^*}{L_2}\pi_1 \alpha_1 B_1^* C^* > 0$$

It can be easily checked that $A_1 A_2 - A_3 > 0$ without any condition and thus, by Routh-Hurwitz criteria, all the roots of equation (22) are negative or have negative real part. Hence equilibrium E^* is locally asymptotically stable.

4.2. Nonlinear stability of the equilibrium $E^*(B_1^*, B_2^*, C^*, T^*)$

To establish nonlinear stability, we use the following positive definite function

$$V = m_1\left(B_1 - B_1^* - B_1^* \log\frac{B_1}{B_1^*}\right) + m_2\left(B_2 - B_2^* - B_2^* \log\frac{B_2}{B_2^*}\right) + \frac{1}{2}m_3(C - C^*)^2$$

$$+ \frac{1}{2}m_4(T - T^*)^2 \qquad (23)$$

where $m_i\ (i = 1, 2, 3, 4)$ are positive constants to be chosen appropriately.

Differentiating (23) with respect to t and using the model system (1) - (4), we get

$$\frac{dV}{dt} = -\frac{m_1 s_1}{L_1}(B_1 - B_1^*)^2 - \frac{m_2 s_2}{L_2}(B_2 - B_2^*)^2 - m_3(\alpha_0 + \alpha_1 B_1 + \alpha_2 B_2)(C - C^*)^2$$
$$- m_4 \theta_0 (T - T^*)^2 + (m_1 \pi_1 - m_3 \alpha_1 C^*)(B_1 - B_1^*)(C - C^*)$$

$$+ (m_2 \pi_2 - m_3 \alpha_2 C^*)(B_2 - B_2^*)(C - C^*) + m_4 \theta (C - C^*)(T - T^*)$$

After some algebraic manipulations and by choosing $m_3 = 1$, we get

$$m_1 = \frac{\alpha_1 C^*}{\pi_1}, m_2 = \frac{\alpha_2 C^*}{\pi_2} \text{ and } m_4 < \frac{2\alpha_0 \theta_0}{\theta^2}$$

Thus, $\frac{dV}{dt}$ is negative definite without any condition inside the region of attraction Ω showing that V is a Lyapunov's function and hence $E^*(B_1^*, B_2^*, C^*, T^*)$ is nonlinearly asymptotically stable without any condition.

5. Persistent of the system

From the lemma, it follows that for any $\varepsilon > 0$, there exist $t_0 > 0$ such that for all $t \geq t_0$,

$$B_1(t) < B_{1m} + \varepsilon,\ B_2(t) < B_{2m} + \varepsilon,\ C(t) < C_m + \varepsilon,\ T(t) < T_m + \varepsilon$$

Now from equation (1) of the model system (1) – (4), we have $\frac{dB_1}{dt} \geq s_1 B_1 \left(1 - \frac{B_1}{L_1}\right)$ from which we get $\liminf_{t\to\infty} B_1(t) \geq L_1$. Similarly, $\liminf_{t\to\infty} B_2(t) \geq L_2$

Now, from equation (3), we have,

$$\frac{dC}{dt} \geq Q_0 - \alpha_0 C - \alpha_1 (B_{1m} + \varepsilon) C - \alpha_2 (B_{2m} + \varepsilon) C$$

From this we get, $\liminf_{t\to\infty} C(t) \geq \frac{Q_0}{\alpha_0 + \alpha_1 (B_{1m} + \varepsilon) + \alpha_2 (B_{2m} + \varepsilon)}$

This is true for all $\varepsilon > 0$ thus it follows that $\liminf_{t\to\infty} C(t) \geq C_a$

where $C_a = \dfrac{Q_0}{\alpha_0 + \alpha_1 B_{1m} + \alpha_2 B_{2m}}$

Now from equation (4), we have

$$\frac{dT}{dt} \geq \theta(C_a - C_0) - \theta_0(T - T_0)$$

From this, we get, $\liminf_{t\to\infty} T(t) \geq T_a$

where $T_a = \dfrac{\theta(C_a - C_0) + \theta_0 T_0}{\theta_0}$, $C_a > C_0$

Taking, $M_1 = \min(L_1, L_2, C_a, T_a)$ and $M_2 = \max(B_{1m}, B_{2m}, C_m, T_m)$, it follows that,

$$M_1 \leq \liminf_{t\to\infty} X(t) \leq \limsup_{t\to\infty} X(t) \leq M_2$$

where $X(t) = \{B_1(t), B_2(t), C(t), T(t)\}$

This shows that the model system (1) – (4) is persistent if $C_a > C_0$.

6. Numerical Simulation

In this section, we have performed numerical simulations for the validity of analytical findings using MAPLE 18 by using the following set of parameters.

$$Q = 20, \delta = 0.1, \delta_0 = 0.1, s_1 = 0.5, s_2 = 0.1, s_{11} = 0.00003, s_{22} = 0.00002, C_0 = 10$$

$$\delta_1 = 0.0001, \delta_2 = 0.0001,\ L_1 = 2000, L_2 = 5000, \mu_1 = 0.3, \theta = 0.001, \theta_0 = 0.2, T_0 = 14.5$$

Equilibrium values of E^* as obtained are given follows;

$$B_1^* = 2002.9895, B_2^* = 5024.9131, C^* = 24.9131, T^* = 14.5745$$

The eigenvalues corresponding to E^* so obtained are $-0.2000, -0.1008$, -0.8019 and 0.5012 which are all negative depicting that the equilibrium E^* is locally asymptotically stable. Further, nonlinear stability of equilibrium E^* is shown in fig.1 where trajectories in $B_1 - B_2 - C$ plane with different initial starts approach the equilibrium point E^* revealing the nonlinear stability character of E^*.

The variation of concentration C of carbon dioxide for different values of it's depletion rate coefficients due to greenbelt (i.e. $\alpha_1 = 0.0001, 0.0002, 0.0003$) and seaweeds (i.e. $\alpha_2 = 0.0001, 0.00015, 0.0002$) with time is shown in figures 2 and 3 respectively. From these figures, we note that the concentration of carbon dioxide decreases as the rate of depletion coefficients increases which in turn may decrease the equilibrium level of average atmospheric temperature (figure 4 and 5 respectively). In figures 6 and 7 an attempt has been made to depict the variation of concentration of carbon dioxide and average atmospheric temperature considering three different cases showing the significant role of simultaneous effect of greenbelt and seaweeds on abatement in carbon dioxide concentration in the atmosphere. The simultaneous effect of growth rate of greenbelt and seaweeds due to carbon dioxide on reduction in carbon dioxide concentration and lowering the average atmospheric temperature is shown in figures 8 and 9 respectively. From these figures we depict that the concentration of carbon dioxide and average atmospheric temperature decreases as the growth of greenbelt and seaweeds increases.

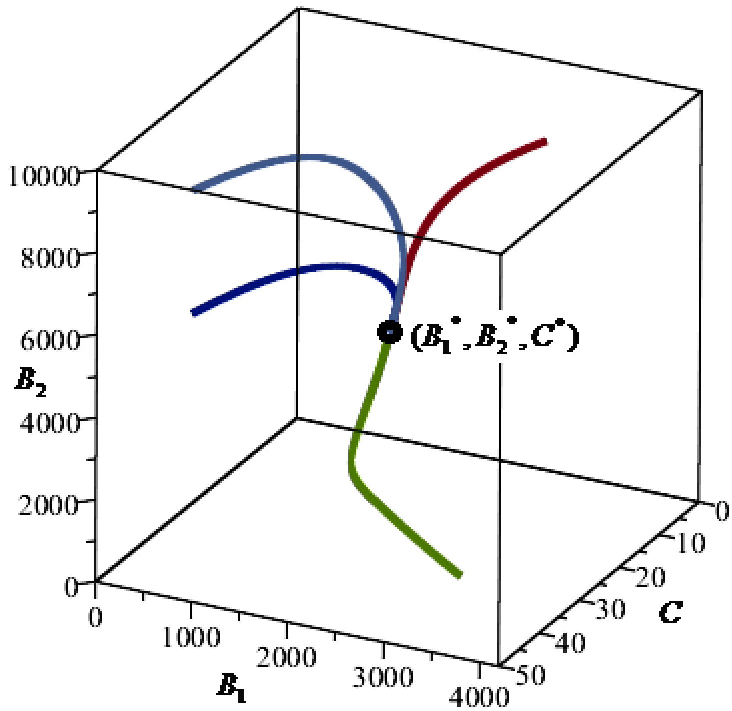

Figure 1 Nonlinear stability in $B_1 - B_2 - C$ plane

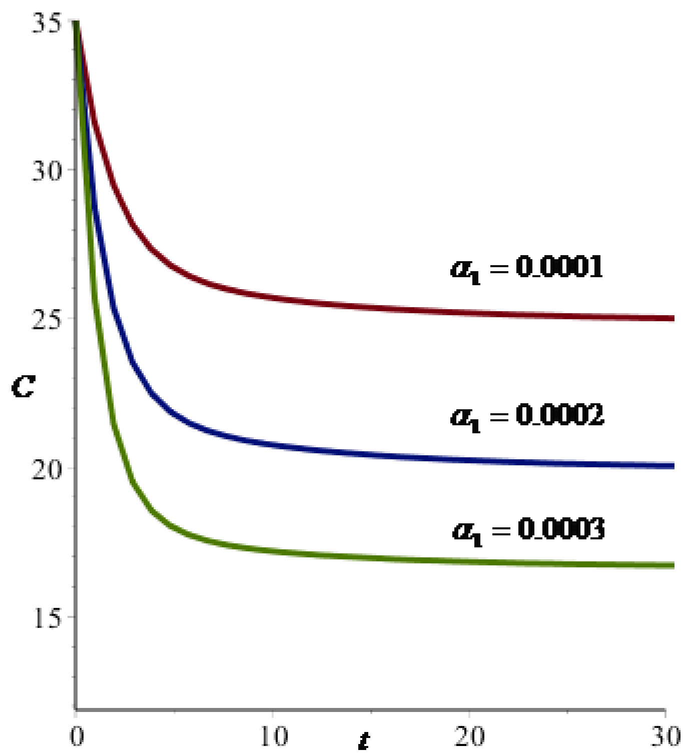

Figure 2 Variation of C with time for different values of α_1 when $\alpha_2 = 0.0001$

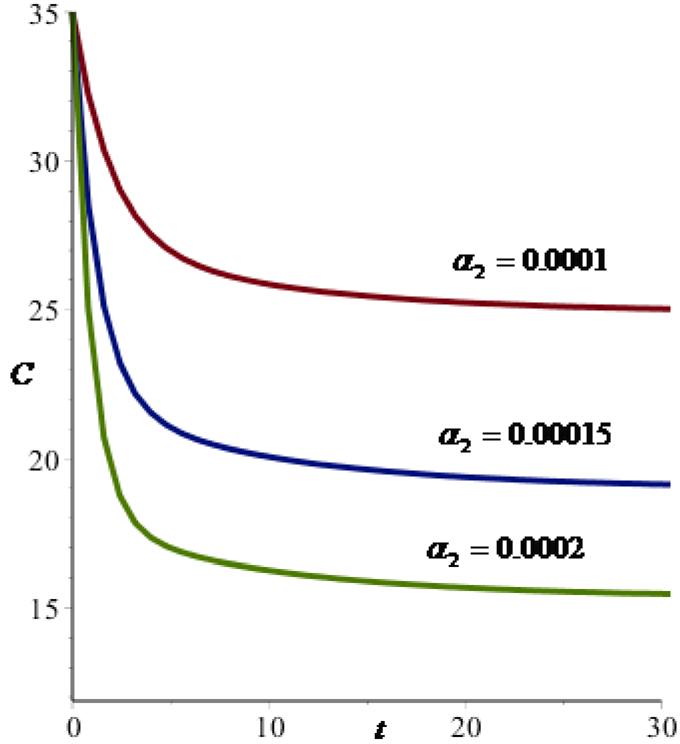

Figure 3 Variation of C with time for different values of α_2 when $\alpha_1 = 0.0001$

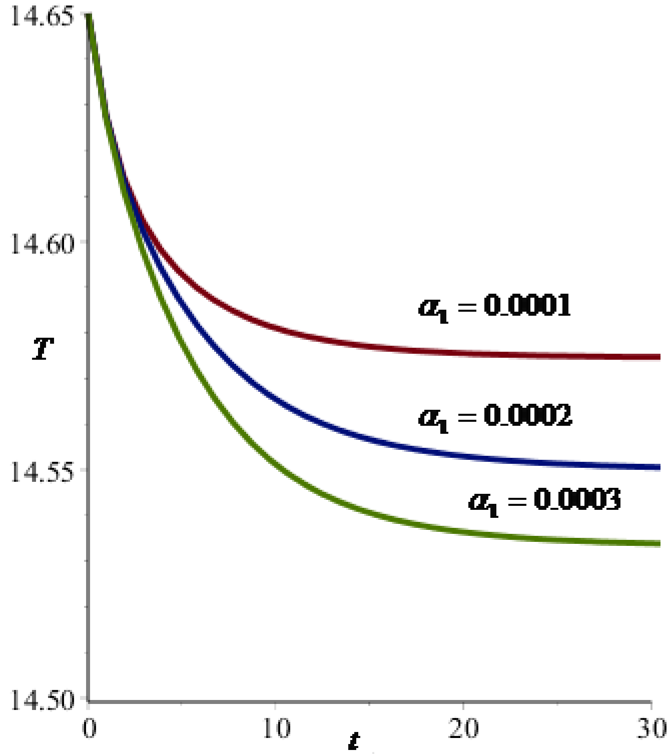

Figure 4 Variation of T with time for different values of α_1 when $\alpha_2 = 0.0001$

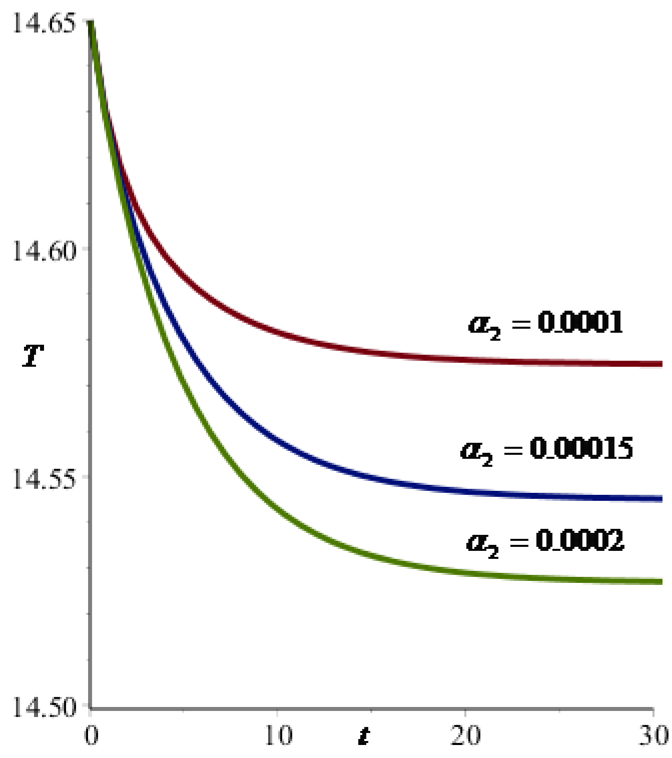

Figure 5 Variation of T with time for different values of α_2 when $\alpha_1 = 0.0001$

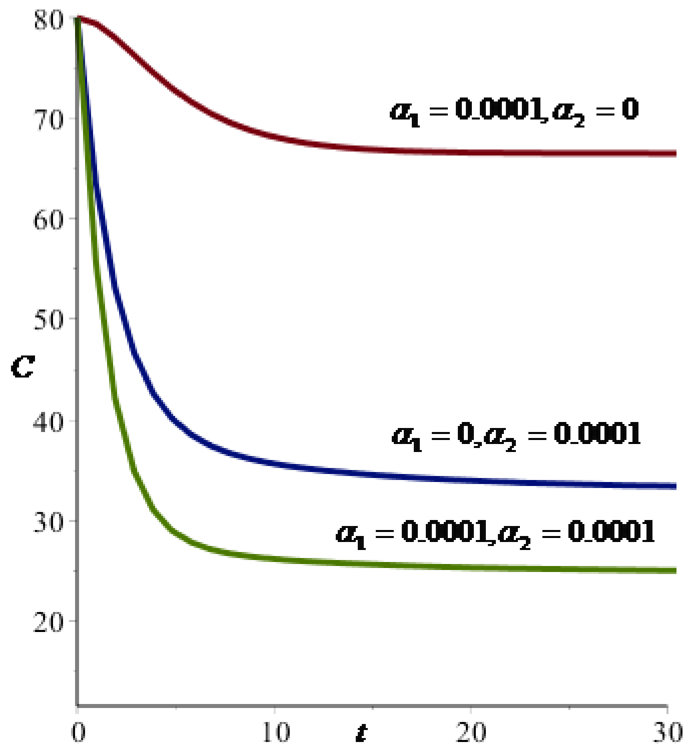

Figure 6 Variation of C with time for different values of α_1 and α_2

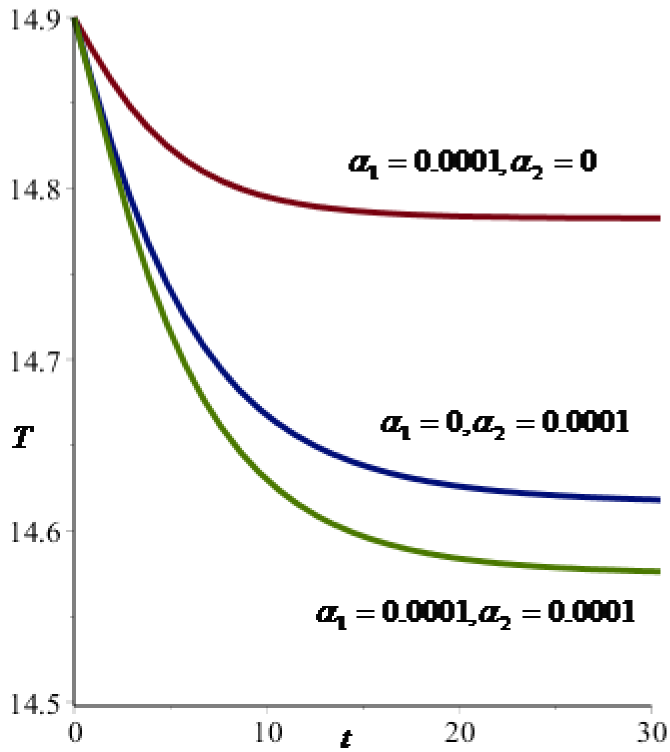

Figure 7 Variation of T with time for different values of α_1 and α_2

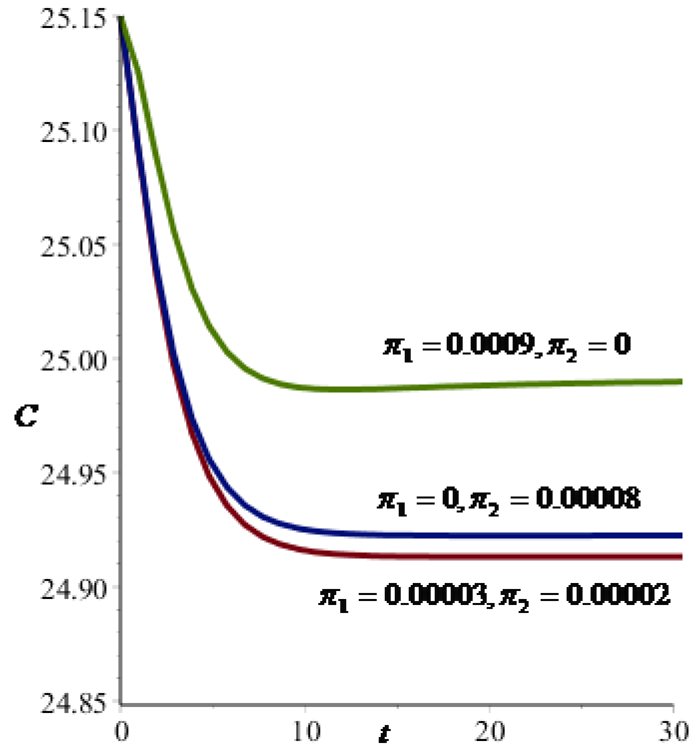

Figure 8 Variation of C with time for different values of π_1 and π_2

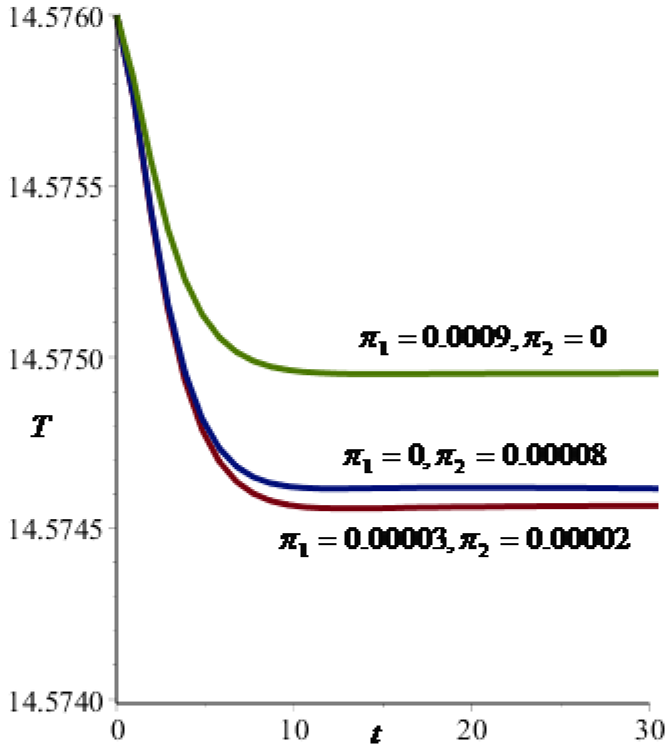

Figure 9 Variation of T with time for different values of π_1 and π_2

7. Conclusions

In the modeling process, the following four variables are considered,

1. Biomass density of greenbelt B_1
2. Biomass density of seaweed cultivation B_2
3. Cumulative concentration of CO_2
4. Atmospheric temperature T

In this paper, we have proposed a nonlinear mathematical model. The nonlinear model is analyzed using the stability theory of differential equations. The effect of greenbelt plantation and seaweeds cultivation on carbon dioxide mitigation is studied analytically and numerically. The analysis of equilibria has been conducted to comprehend the feasibility of the model system. The analysis of the model has shown that as the biomass density of greenbelt plantation and seaweeds cultivation increases, the concentration of CO_2 decreases and in turn it lowers the average atmospheric temperature.

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