NUMERICAL METHODS FOR NONLINEAR ELLIPTIC PROBLEMS

LINEARIZATION, ERROR ESTIMATION, AND ADAPTIVITY

K. Mitra[1]

joint work with

A. Harnist, A. Javed, K.T.W. Menting, I.S. Pop, F.A. Radu, A. Rappaport, J.S. Stokke, E. Storvik, J.H.M. ten Thije Boonkkamp[1], M. Vohralík

[1] Mathematics & Computer Science Department, TU Eindhoven, The Netherlands

We consider a general formulation of nonlinear elliptic equations satisfying the monotonicity and continuity conditions that define them. For an elliptic operator $\mathcal{R} : H \to (H)^*$ (with H being a Hilbert space), the solution of the corresponding elliptic problem $u \in H$ satisfies $\mathcal{R}(u) = 0$. Then a general iterative linearization scheme for obtaining u constructs an approximating sequence $\{u^i\}_{i\in\mathbb{N}} \subset H$ by introducing a bilinear operator $\mathcal{B}_{\langle i\rangle} : H \times H \to \mathbb{R}$ depending upon $u^i \in H$, such that $u^{i+1} \in H$ solves the linear problem

$$\mathcal{B}_{\langle i\rangle}(u^{i+1} - u^i, v) = -\langle \mathcal{R}(u^i), v\rangle, \quad \forall\, v \in H. \tag{1}$$

We show that for a large class of problems (e.g., porous media flow, mean curvature flow, biological flows, mixed dimensional equations, optimal transport problems, and design of optical systems) and for almost all standard linearization schemes (Newton scheme, Picard scheme, L/M-schemes) this structure holds [1]. Moreover, by considering an operator $\mathcal{B}$ which does not depend on iteration index i and is an inner product in H, we get schemes that are more robust in terms of discretization, nonlinearities, and degeneracies (the 'so called' Zarantonello/L-scheme) [4, 5].

In this setting, we derive a posteriori error estimates that are robust with respect to the ratio of the continuity over monotonicity constants in the dual energy norm invoked by the linearization iterations [1, 3]. This is linked to an orthogonal decomposition of the total error into a linearization error component and a discretization error component [1]. It can be further used to adaptively refine/coarsen the mesh [3], adaptively stop the linearization iterations for efficient error balancing [1], and to adaptively choose the optimal parameters for the convergence of the iterative scheme [2, 5]. Numerical experiments spanning diverse applications illustrate the theoretical results.

REFERENCES

[1] K. Mitra, & M. Vohralík. Guaranteed, locally efficient, and robust a posteriori estimates for nonlinear elliptic problems in iteration-dependent norms. An orthogonal decomposition result based on iterative linearization, HAL Preprint, hal-04156711, v.1., 2023.

[2] J.S. Stokke, K. Mitra, E. Storvik, J.W. Both, & F.A. Radu. An adaptive solution strategy for Richards' equation, Computers & Mathematics with Applications, 152, 155-167, 2023.

[3] A. Harnist, K. Mitra, A. Rappaport, & M. Vohralík. Robust augmented energy a posteriori estimates for Lipschitz and strongly monotone elliptic problems, hal-04033438, 2024.

[4] K.T.W. Menting. *Fixed point methods for the Monge-Ampère equation using Green's functions*, Masters thesis, TU Eindhoven, 2024.

[5] A. Javed, K. Mitra, & I.S. Pop. *Splitting-Based Linearization Schemes for Doubly Degenerate Diffusion Problems*, 2024.