Indian Institute of Technology, Kanpur
Department of Mathematics and Statistics
Statistics PhD Admission Test - 2019
Date : May 9, 2019

Name: Time: One hour

Roll/Application Number: Maximum Marks = 60

Category (Tick ✓anyone) : GEN OBC-NCL/EWS SC/ST/PwD

Instruction

1. This question paper consists of 20 questions each carrying 3 marks.

2. The questions are MSQ type, i.e, each question may have more than one correct answer.

3. You will get full 3 marks for full correct answer, 0 for all other cases.

4. This question-cum-answer booklet must be returned to the invigilator before leaving the examination hall.

5. **Please enter your answers only on this page in the space given below.**

Q. No	Answer	Q. No	Answer	Q. No	Answer	Q. No	Answer
1		6		11		16	
2		7		12		17	
3		8		13		18	
4		9		14		19	
5		10		15		20	

Please turn over

Notations : We denote by $\mathbb{R}^n$, $n \geq 1$, the set of n-dimensional real vectors.

1. Let $f : [-1,1] \to \mathbb{R}$ be a continuous function such that it is differentiable on $(-1,1)$.

(a) If $f(-1) = -1$ and $f(1) = 1$, then $f(x) = x$ for all $x \in [-1,1]$.

(b) If $f(-1) = f(1)$, then the equation $f'(x) = 0$ has at least one solution in $(-1,1)$.

(c) If $f'\left(-\frac{1}{2}\right) < f'(\frac{1}{2})$, then for any $y \in \left[f'(-\frac{1}{2}), f'(\frac{1}{2})\right]$ there exists $x \in \left[-\frac{1}{2}, \frac{1}{2}\right]$ such that $f'(x) = y$.

(d) If $f'(0) = 0$, then $\sup\limits_{x \in [-1,1]} f(x) = f(0)$.

2. Let $A = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 : 0 \leq x^2 + y^2 \leq \frac{\pi^2}{4}, 0 \leq z \leq \sqrt{x^2 + y^2} \right\}$. Then the value of the integral

$$\iiint_A \cos(z)\, dx\, dy\, dz$$

is

(a) π (b) 2π (c) 0 (d) 1

3. Consider the matrix $I_n + \mathbf{a}\mathbf{b}^T$, where I_n is the $n \times n$ identity matrix, and $\mathbf{a}$ and $\mathbf{b}$ are non-null vectors. The number of non-zero, distinct eigenvalues of this matrix are:

(a) 1 (b) 2 (c) 3 (d) n

4. Suppose that X is a proper random variable, i.e., $P(-\infty < X < \infty) = 1$, and $\lim\limits_{N \to \infty} N(P(|X| \geq N+1)) = \infty$. Which of the following statements is(are) correct :

(a) $\lim_{N \to \infty} P(X < -N) = 0$.

(b) $\lim_{N \to \infty} P(X \geq N) = 0$.

(c) $E(|X|) < \infty$.

(d) $E(|X|)$ is not finite.

5. Let A be a positive definite matrix, and suppose that $A^{-1} = ((a^{ij}))$ and $e_1 = (1, 0, \ldots, 0)^T$. Then

$$\begin{bmatrix} A & e_1 \\ e_1^T & a^{11} \end{bmatrix} \text{ is}$$

(a) Positive definite (b) Positive semi-definite (c) Negative definite (d) Negative semi-definite.

6. Consider the testing of hypothesis problem $H_0 : X \sim f_0$ against $H_1 : X \sim f_1$, where the probability mass functions f_0 and f_1 are as given below :

x :	-4	-3	0	1	2	5
f_0 :	0.05	0.20	0.30	0.15	0.25	0.05
f_1 :	0.15	0.30	0.05	0.05	0.25	0.20

Then which of the following statements is(are) correct:

(a) The MP level 0.15 test is randomized, but the MP level 0.1 test is non-randomized.
(b) The MP level 0.1 test is randomized, but the MP level 0.15 test is non-randomized.
(c) The MP level 0.15 and level 0.1 tests are both randomized.
(d) The MP level 0.15 and level 0.1 tests are both non-randomized.

7. Suppose that $X_1, \ldots, X_n$ are i.i.d. Bernoulli (p) random variables, i.e., $\mathbf{P}(X_1 = 1) = p$. Let $S_n = \sum_{i=1}^{n} X_i$. Then

(a) $4Var(X_1) \leq 1$

(b) If $p > \frac{1}{2}$, then $\mathbf{P}\Big(S_n = 0\Big) > \mathbf{P}\Big(S_n = n\Big)$ for $n \geq 10$

(c) X_1^2 has the same distribution as X_2X_3 for any value of $p \in [0,1]$

(d) If $np \to \lambda(> 0)$ as $n \to \infty$, then the distribution of S_n can be approximated by Poisson(λ) distribution.

8. Consider three events A, B and C with $\mathbf{P}(A) > 0$, where A and B are mutually independent. Furthermore, B and C are mutually exclusive, and D^c denotes the complement of any event D. Then

(a) A^c and B^c are also mutually independent

(b) B^c and C^c are also mutually exclusive

(c) If A and B are mutually exclusive, then $\mathbf{P}(A \cup B \cup C) = \mathbf{P}(A) - \mathbf{P}(A \cap C) + \mathbf{P}(C)$

(d) If $\mathbf{P}(B) > 0$, then $\mathbf{P}(C|A \cap B) = 0$.

9. Let (X, Y) has the joint p.d.f. $f(x, y) = 2$ if $0 \leq x \leq y \leq 1$, and $= 0$, otherwise.
Let $a = E(Y|X = \frac{1}{2})$ and $b = Var(Y|X = \frac{1}{2})$. Then $(a, b) =$

(a) $(\frac{3}{4}, \frac{7}{12})$ (b) $(\frac{1}{4}, \frac{1}{48})$ (c) $(\frac{1}{4}, \frac{7}{12})$ (d) $(\frac{3}{4}, \frac{1}{48})$

10. $\frac{1}{2\pi^2} \int_{-\infty}^{\infty} \frac{\left(\pi + 2\tan^{-1}(x)\right)}{1+x^2}$ equals

(a) $\frac{1}{2}$ (b) 1 (c) $\frac{1}{4}$ (d) None of (a), (b) and (c)

11. Consider the model $E(Y_1) = 2\beta_1 + \beta_2, E(Y_2) = 2\beta_1 - \beta_2, E(Y_3) = \beta_1 + \alpha\beta_2$, with uncorrelated errors having zero mean and a constant variance σ^2. Let $\hat{\beta}_1$ and $\hat{\beta}_2$ be the best linear unbiased estimators of β_1 and β_2, respectively. The value of α for which $\hat{\beta}_1$ and $\hat{\beta}_2$ are uncorrelated and the corresponding variances of $\hat{\beta}_1$ and $\hat{\beta}_2$ are

(A) $\alpha = 0, \quad Var(\hat{\beta}_1) = 6\sigma^2, \quad Var(\hat{\beta}_2) = 2\sigma^2$
(B) $\alpha = -1, \quad Var(\hat{\beta}_1) = 6\sigma^2, \quad Var(\hat{\beta}_2) = 3\sigma^2$
(C) $\alpha = -1, \quad Var(\hat{\beta}_1) = \frac{\sigma^2}{6}, \quad Var(\hat{\beta}_2) = \frac{\sigma^2}{3}$
(D) $\alpha = 0, \quad Var(\hat{\beta}_1) = \frac{\sigma^2}{6}, \quad Var(\hat{\beta}_2) = \frac{\sigma^2}{2}$

12. Let $X_1, \ldots, X_n$ be i.i.d. random variables from a Gamma distribution $G(\alpha, \lambda)$ with probability density function (p.d.f.)

$$f(x; \alpha, \lambda) = \frac{1}{\Gamma(\alpha)\lambda^\alpha} e^{-\frac{x}{\lambda}} x^{\alpha-1}, \quad x > 0; \quad \alpha, \lambda > 0.$$

If $m_k^/$ and m_k are, respectively, the k^{th} raw and central moments of the sample, then which of the following statements is (are) correct?
(a) $m_1^/$ converges in probability to $\alpha\lambda$, (b) $m_2^/$ does not converge in probability to $\alpha(\alpha+1)\lambda^2$,
(c) m_2 converges in probability to $\alpha\lambda^2$, (d) m_2 does not converge in probability to $\alpha\lambda^2$.

13. A 2^5 factorial experiment is conducted in a randomized block design in which the blocks are constructed by confounding AB, BCD and $ABCDE$. The total number of effects getting confounded and the degree of freedom carried by the treatment sum of squares are

(a) 3 and 24, respectively
(b) 3 and 31, respectively
(c) 7 and 24, respectively
(d) 7 and 31, respectively.

14. Consider the multiple linear regression model $y = X\beta + \epsilon$, where y is a $n \times 1$ vector of n observations on the dependent variable, X is a non-stochastic $n \times p$ full column rank matrix of n observations on p explanatory variables, β is a $p \times 1$ vector of fixed regression coefficients, and ϵ is a $n \times 1$ vector of random errors with mean null vector and non-null covariance matrix Ω. The bias vector and covariance matrix of ordinary least squares estimator of β are

(a) β and $(X'X)^{-1}$, respectively
(b) null vector and $(X'\Omega X)^{-1}$, respectively
(c) null vector and $(X'X)^{-1}X'\Omega X(X'X)^{-1}$, respectively
(d) β and $(X'\Omega X)^{-1}X'X(X'\Omega X)^{-1}$, respectively

15. $X_1, \ldots, X_n$ be a random sample from the uniform distribution $U[\theta - 1, \theta + 1]$, where $\theta \in (-\infty, \infty)$ is unknown. Let $X_{(1)} = \min(X_1, \ldots, X_n)$ and $X_{(n)} = \max(X_1, \ldots, X_n)$. Which of the following statements is(are) true?

(a) Minimal sufficient statistic is complete.

(b) $X_{(1)}$ is an MLE of θ.

(c) MLE of θ is not unique.

(d) $X_{(n)} + X_{(1)}$ is an ancillary statistic.

16. $\{X_t\}$, $\{Y_t\}$ and $\{\epsilon_t\}$ are three mutually independent sequence of random variables. Here $\{X_t\}$ and $\{\epsilon_t\}$ are both i.i.d. sequence of $N(0,1)$ random variables, and $\{Y_t\}$ is an i.i.d. sequence of $N(1,1)$ random variables. Further, $Z_t = \epsilon_t + \epsilon_{t-1}$. Which of the following statements is(are) true?

(a) $P_t = (1 - Y_t)\, Z_t + X_t$ is a covariance stationary process that is NOT a white noise

(b) $Q_t = Z_{2t} + Z_{t-1}$ is a covariance stationary process

(c) $R_t = Z_{2t} - Z_{2t-1}$ is a white noise process

(d) $S_t = Y_t + (1 - Z_t)$ is a strict stationary process

17. Let $f_\rho(x, y)$ denote the joint probability density function of bivariate normal distribution with mean vector $\underline{O} = (0,0)^t$ and the variance-covariance matrix

$$\Sigma = \begin{bmatrix} 1 & \rho \\ \rho & 4 \end{bmatrix},$$

where $|\rho| < 2$. Let $(X, Y)^t$ be a random vector having the joint probability density function

$$g(x, y) = \frac{1}{4} f_{\frac{1}{2}}(x, y) + \frac{3}{4} f_1(x, y), \quad -\infty < x, y < \infty.$$

Then

(A) $\text{Cov}(X, Y) = \frac{7}{8}$	(B) $\text{Var}(X - Y) = \frac{13}{4}$
(C) $\text{Var}(X + Y) = \frac{17}{4}$	(D) $\text{Cov}(Y + X, Y - X) = 2$

18. Customers are arriving in a Super Market according to the Poisson Process $\{N(t) : t \geq 0\}$ with intensity $\lambda = 2$ arrivals per hour. Let S_n $(n = 1, 2, \ldots)$ denote the waiting time for the arrival of the n^{th} customer. Define

$$p = \Pr(S_3 \geq 2), \quad q = E(N(4) | N(1) = 2), \quad r = \text{Cov}(N(4), N(2)) \quad \text{and} \quad s = \text{Var}(S_1 | N(2) = 1).$$

Then

(A) $p = 4e^{-2}$ $\qquad$ (B) $q = 8$ $\qquad$ (C) $r = 4$ $\qquad$ (D) $s = \frac{1}{3}$

19. Let $X_1, \ldots, X_n$ be a random sample from the following probability density function:

$$f(x|\lambda) = \lambda e^{-\lambda x}; \qquad x > 0.$$

Here $\lambda > 0$. The prior on λ is Gamma(a, b), where the probability density function of a Gamma(a, b) is

$$\pi(x|a, b) = \frac{b^a}{\Gamma(a)} x^{a-1} e^{-bx}; \qquad x > 0, \quad a > 0, \quad b > 0.$$

Which of the following statements is(are) correct :

(a) The Bayes estimate of λ with respect to the squares error loss function exists for any $n > 1$.

(b) The Bayes estimate of λ with respect to the absolute error loss function exists for any $n > 1$.

(c) $100(1-\alpha)\%$ credible interval of λ, for $0 < \alpha < 1$ exists and it is unique.

(d) As n tends to ∞, the Bayes estimate of λ with respect to the squared error loss function, tends to the maximum likelihood estimate of λ.

20. A $M \times M$ matrix $\mathbf{P} = ((p_{ij}))$, is called a stochastic matrix if $p_{ij} \geq 0$, and $\sum_{j=1}^{M} p_{ij} = 1$, for $i = 1, \ldots, M$. A stochastic matrix $\mathbf{P}$ is called a doubly stochastic if $\sum_{i=1}^{M} p_{ij} = 1$, for $j = 1, \ldots, M$. Which of the following statements is(are) correct:

(a) If $\mathbf{P}$ is a stochastic matrix, then $\mathbf{P}^m$ is also a stochastic matrix for $m = 1, 2, \ldots$.

(b) If $\mathbf{P}$ is a doubly stochastic matrix, then $\mathbf{P}^m$ is also a doubly stochastic matrix for $m = 1, 2, \ldots$.

(c) If $\mathbf{P}$ is a stochastic matrix, then there exists a stochastic matrix $\mathbf{Q}$, such that $\mathbf{P} = \mathbf{Q}^2$.

(d) If $\mathbf{P}$ is a stochastic matrix (not necessarily symmetric), then it has at least one real eigenvalue.

Space for rough work