FOUR LECTURES ON THE APPLICATIONS OF THE FOURIER TRANSFORM TO HAUSDORFF DIMENSION
U. B. TEWARI DISTINGUISHED LECTURE SERIES IN MATHEMATICS
NOVEMBER 7-11, 2024, IIT KANPUR

PERTTI MATTILA

These notes give an overview of the lectures. I have also listed several results which I probably don't have time to discuss. In the lectures I plan to explain some of the basic proof ideas and methods. Below I shall give the most essential preliminaries on which the lectures will be based. Basic general back-ground material can be found in Sections 1, 6 of [F3], Sections 1, 4, 8, 12 of [M4], Sections 2, 3 of [M5], Sections 1-3, 8 of [W2]. Those books also contain proofs of many results mentioned below which had appeared prior 2015.

1. Preliminaries on Hausdorff dimension, energy integrals and the Fourier transform

First I give a quick review of the Hausdorff dimension and its relations to energy-integrals and the Fourier transform. Most of the details can be found in [M4] and [M5].

For $A \subset \mathbb{R}^n$, let $\mathcal{M}(A)$ be the set of Borel measures μ such that $0 < \mu(A) < \infty$ and μ has compact support $\mathrm{spt}\mu \subset A$.

The Fourier transform of $\mu \in \mathcal{M}(\mathbb{R}^n)$ is defined by

$$\widehat{\mu}(x) = \int e^{-2\pi i x\cdot y}\, d\mu y,\ x \in \mathbb{R}^n.$$

Many of the basic formulas of classical Fourier analysis extend to measures, with appropriate assumptions, see [M5] and [W2].

The s-dimensional *Hausdorff measure* $\mathcal{H}^s$, $s \geq 0$, is defined by

$$\mathcal{H}^s(A) = \lim_{\delta\to 0} \mathcal{H}^s_\delta(A),\ A \subset \mathbb{R}^n,$$

where, for $0 < \delta \leq \infty$,

$$\mathcal{H}^s_\delta(A) = \inf\{\sum_j d(E_j)^s : A \subset \bigcup_j E_j, d(E_j) < \delta\}.$$

Here $d(E)$ denotes the diameter of the set E.

Then $\mathcal{H}^n$ is a constant multiple of the Lebesgue measure $\mathcal{L}^n$ on $\mathbb{R}^n$.

The *Hausdorff dimension* of $A \subset \mathbb{R}^n$ is

$$\dim A = \inf\{s : \mathcal{H}^s(A) = 0\} = \sup\{s : \mathcal{H}^s(A) = \infty\}.$$

The following is a useful tool for lower bounds for the Hausdorff dimension, $B(x,r)$ is the closed ball with centre x and radius r:

Theorem 1.1 (Frostman's lemma)**.** *Let $0 \le s \le n$. For a Borel set $A \subset \mathbb{R}^n, \mathcal{H}^s(A) > 0$ if and only there is $\mu \in \mathcal{M}(A)$ such that*

$$\mu(B(x,r)) \le r^s \quad \textit{for all } x \in \mathbb{R}^n, r > 0. \tag{1.1}$$

In particular,

$$\dim A = \sup\{s : \textit{there is } \mu \in \mathcal{M}(A) \textit{ such that (1.1) holds}\}.$$

Such measures μ are often called Frostman measures.

The *s-energy*, $s > 0$, of $\mu \in \mathcal{M}(\mathbb{R}^n)$ is

$$I_s(\mu) = \iint |x-y|^{-s}\,d\mu y\,d\mu x = \int k_s * \mu\,d\mu,$$

where k_s is the *Riesz kernel*:

$$k_s(x) = |x|^{-s}, \quad x \in \mathbb{R}^n.$$

Integration of Frostman's lemma, that is using the simple formula

$$\int |x-y|^{-s}\,d\mu y = s\int_0^\infty r^{-s-1}\mu(B(x,r))\,dr,$$

gives that if μ satisfies (1.1), then $I_t(\mu) < \infty$ for $0 < t < s$. On the other hand, if $I_s(\mu) < \infty$, then a restriction of μ to a set of large μ measure satisfies (1.1). So these conditions are very close to each other and we have

$$\dim A = \sup\{s : \text{there is } \mu \in \mathcal{M}(A) \text{ such that } I_s(\mu) < \infty\}. \tag{1.2}$$

The s-energy of $\mu \in \mathcal{M}(\mathbb{R}^n)$ can be written in terms of the Fourier transform:

$$I_s(\mu) = c(n,s)\int |\widehat{\mu}(x)|^2|x|^{s-n}\,dx.$$

This follows by Plancherel's formula and the fact that the distributional Fourier transform of k_s is $c(n,s)k_{n-s}$:

$$I_s(\mu) = \int k_s * \mu\,d\mu = c(n,s)\int k_{n-s}|\widehat{\mu}|^2.$$

Thus we have

$$\dim A = \sup\{s < n : \exists \mu \in \mathcal{M}(A) \text{ such that } \int |\widehat{\mu}(x)|^2|x|^{s-n}\,dx < \infty\}. \tag{1.3}$$

Notice that if $I_s(\mu) < \infty$, then $|\widehat{\mu}(x)|^2 < |x|^{-s}$ for most x with large norm. However, this need not hold for all x with large norm.

The following classical theorem of Davies, see [F3, Theorem 5.4] and [F, Corollary 2.10.48], often reduces the study of general Borel sets to sets with positive and finite Hausdorff measure:

Theorem 1.2. *If $A \subset \mathbb{R}^n$ is a Borel set with $\mathcal{H}^s(A) = \infty$, then A contains a compact set C with $0 < \mathcal{H}^s(C) < \infty$.*

If $A \subset \mathbb{R}^n$ is $\mathcal{H}^s$ measurable with $\mathcal{H}^s(A) < \infty$, then (see [M4, Theorem 6.2]) for $\mathcal{H}^s$ almost all $x\varepsilon \in A$,

$$2^{-s} \leq \limsup_{r \to 0} \mathcal{H}^s(A \cap B(x,r)) \leq 1.$$

In general, lim sup cannot be replaced lim inf. We shall say that A is *Ahlfors-David regular* if for some positive numbers s and C,

$$r^s/C \leq \mathcal{H}^s(A \cap B(x,r)) \leq Cr^s \text{ for } x \in A, 0 < r < d(A).$$

The following well-known facts are often useful, we shall always identify absolutely continuous measures with their Radon-Nikodym derivatives:

Proposition 1.3. *Let $\mu \in \mathcal{M}(\mathbb{R}^n)$.*

1. *If $\widehat{\mu} \in L^2(\mathbb{R}^n)$, then $\mu \in L^2(\mathbb{R}^n)$.*
2. *If $\widehat{\mu} \in L^1(\mathbb{R}^n)$, then μ is a continuous function.*

Proposition 1.4. *Let $\mu, \mu_j \in \mathcal{M}(\mathbb{R}^n), j = 1, 2, \dots$. If $p > 1$ and $\mu_j \in L^p(\mathbb{R}^n)$ with $\|\mu_j\|_p \leq C$ for all j and $\mu_j \to \mu$ weakly, then $\mu \in L^p(\mathbb{R}^n)$ with $\|\mu\|_p \leq C$.*

This fails for $p = 1$.

If $\mu \in \mathcal{M}(\mathbb{R}^n)$, the smooth functions $\phi_\epsilon * \mu, \epsilon > 0$, converge weakly to μ, where ϕ_ϵ is smooth with support in $B(0, \epsilon)$ and $\int \phi_\epsilon = 1$. This with the aid of Proposition 1.4 often allows to reduce the arguments to smooth functions instead of measures.

We let $O(n)$ be the orthogonal group of $\mathbb{R}^n$ and θ_n its Haar measure.

We denote by $f_\sharp\mu$ the push-forward of a measure μ by a mapping f; $f_\sharp\mu(A) = \mu(f^{-1}(A))$. By $\mu \ll \nu$ we mean that μ is absolutely continuous with respect to ν.

2. Hausdorff dimension and projections

We shall now discuss the question: how do orthogonal projections onto lines, or m-planes, affect the Hausdorff dimension? Set

$$P_e(x) = e \cdot x, \quad x \in \mathbb{R}^n, e \in S^{n-1}.$$

The first two items of the following theorem are the classical Marstrand's projection theorem from 1954, [M]. The third was proved by Falconer and O'Neil [FO] in 1999 and by Peres and Schlag [PS] in 2000:

Theorem 2.1. *Let $A \subset \mathbb{R}^n$ be a Borel set.*

1. *If $\dim A \leq 1$, then*
$$\dim P_e(A) = \dim A \quad \textit{for almost all } e \in S^{n-1}.$$
2. *If $\dim A > 1$, then*
$$\mathcal{L}^1(P_e(A)) > 0 \quad \textit{for almost all } e \in S^{n-1}.$$
3. *If $\dim A > 2$, then $P_e(A)$ has non-empty interior for almost all $e \in S^{n-1}$.*

Proof. For $\mu \in \mathcal{M}(A)$ let $\mu_e \in \mathcal{M}(P_e(A))$ be the push-forward of μ under P_e: $\mu_e(B) = \mu(P_e^{-1}(B))$.

To prove (1) let $0 < s < \dim A$ and choose, using (1.2), $\mu \in \mathcal{M}(A)$ such that $I_s(\mu) < \infty$. Then

$$\begin{aligned}\int_{S^{n-1}} I_s(\mu_e)\, de &= \int_{S^{n-1}} \iint |e\cdot(x-y)|^{-s}\, d\mu x\, d\mu y\, de \\ &= \iiint_{S^{n-1}} |e\cdot(\tfrac{x-y}{|x-y|})|^{-s}\, de |x-y|^{-s}\, d\mu x\, d\mu y = c(s)I_s(\mu) < \infty,\end{aligned}$$

where for $v \in S^{n-1}$, $c(s) = \int_{S^{n-1}} |e\cdot v|^{-s}\, de < \infty$ as $s < 1$. Referring again to (1.2) we see that $\dim P_e(A) \geq s$ for almost all $e \in S^{n-1}$. By the arbitrariness of $s, 0 < s < \dim A$, we obtain $\dim P_e(A) \geq \dim A$ for almost all $e \in S^{n-1}$. The opposite inequality follows from the fact that the projections are Lipschitz mappings.

To prove (2) choose by (1.3) a measure $\mu \in \mathcal{M}(A)$ such that $\int |x|^{1-n}|\widehat{\mu}(x)|^2\, dx < \infty$. Directly from the definition of the Fourier transform we see that $\widehat{\mu_e}(t) = \widehat{\mu}(te)$ for $t \in \mathbb{R}, e \in S^{n-1}$. Integrating in polar coordinates we obtain

$$\int_{S^{n-1}} \int_{-\infty}^{\infty} |\widehat{\mu_e}(t)|^2\, dt\, de = 2\int_{S^{n-1}} \int_0^{\infty} |\widehat{\mu}(te)|^2\, dt\, de = 2\int |x|^{1-n}|\widehat{\mu}(x)|^2\, dx < \infty.$$

Thus for almost all $e \in S^{n-1}$, $\widehat{\mu_e} \in L^2(\mathbb{R})$ which means that μ_e is absolutely continuous with L^2 density and hence $\mathcal{L}^1(p_e(A)) > 0$.

For the proof of (3) one takes $2 < s < \dim A$ and $\mu \in \mathcal{M}(A)$ such that $I_s(\mu) < \infty$, whence $\int |x|^{s-n}|\widehat{\mu}(x)|^2\, dx < \infty$. Then as above and by the Schwartz inequality

$$\begin{aligned}&\int_{S^{n-1}} \int_{|t|\geq 1} |\widehat{\mu_e}(t)|\, dt\, de = 2\int_{|x|\geq 1} |x|^{1-n}|\widehat{\mu}(x)|\, dx \\ &\leq 2\int_{|x|\geq 1} |x|^{2-s-n}|\, dx \int_{|x|\geq 1} |x|^{s-n}\widehat{\mu}(x)|^2\, dx < \infty\end{aligned}$$

since $2 - s - n < -n$. Thus for almost all $e \in S^{n-1}$, $\widehat{\mu_e} \in L^1(\mathbb{R})$ which implies that μ_e is absolutely continuous with continuous density. Hence $P_e(A)$ has non-empty interior. $\square$

Also part (2) can rather easily be proven without the Fourier transform, but I don't know any proof for (3) without it.

The conditions on $\dim A$ are necessary.

The higher dimensional analogue is also true. There $G(n,m)$ is the Grassmannian manifold of linear m-dimensional subspaces of $\mathbb{R}^n$, almost all $V \in G(n,m)$ refers to its its orthogonally invariant Borel probability measure, and $P_V : \mathbb{R}^n \to V$ is the orthogonal projcetion:

Theorem 2.2. *Let $A \subset \mathbb{R}^n$ be a Borel set.*

(1) *If $\dim A \leq m$, then*

$$\dim P_V(A) = \dim A \quad \text{for almost all } V \in G(n,m).$$

(2) *If $\dim A > m$, then*

$$\mathcal{H}^m(P_V(A)) > 0 \quad \text{for almost all } V \in G(n,m).$$

(3) *If $\dim A > 2m$, then $P_V(A)$ has non-empty interior for almost all $V \in G(n,m)$.*

I don't know if (3) is sharp when $m \geq 2$.

How much more one can say about the size of the sets of exceptional lines. Kaufman [Ka] proved in 1968 the first item of the following theorem, Falconer [F1] in 1982 the second and Peres and Schlag [PS] in 2000 the third.

Theorem 2.3. *Let $A \subset \mathbb{R}^n$ be a Borel set.*

(1) *If* $\dim A \leq 1$, *then*

$$\dim\{e \in S^{n-1} : \dim P_e(A) < \dim A\} \leq \dim A. \tag{2.1}$$

(2) *If* $\dim A > 1$, *then*

$$\dim\{e \in S^{n-1} : \mathcal{L}^1(P_e(A)) = 0\} \leq n - \dim A. \tag{2.2}$$

(3) *If* $\dim A > 2$, *then*

$$\dim\{e \in S^{n-1} : \mathrm{Int}(P_e(A)) \neq \emptyset\} \leq n + 1 - \dim A. \tag{2.3}$$

The proofs are not too difficult modifications of those for Theorem 2.1. The higher dimensional analogues can be found in [M5, Corollay 5.12]

The following result of Ren and Wang [RW] solved a conjecture of Oberlin. The proof uses an argument of Orponen and Shmerkin [OS] to deduce this from the solution of Furstenberg's problem, see Section 5.

Theorem 2.4. *Let $A \subset \mathbb{R}^2$ be a Borel set. Then for $0 \leq u \leq \min\{\dim A, 1\}$,*

$$\dim\{e \in S^1 : \dim P_e(A) < u\} \leq \max\{2u - \dim A, 0\}. \tag{2.4}$$

The upper bound is sharp, as the constructions in [KM] show.

Analogues of Marstrand's projection theorem have recently been proved for various restricted families of projections. Let $\gamma : [0,1] \to S^2$ be a curve with sufficient curvature properties, that is, $\gamma(\theta), \gamma'(\theta), \gamma''(\theta))$ span $\mathbb{R}^3$ for all $\theta \in [0,1]$, and let p_θ be the projection onto line spanned by $\gamma(\theta)$. A typical example is $\gamma(\theta) = \frac{1}{\sqrt{2}}(\cos\theta, \sin\theta, 1)$, but not $\gamma(\theta) = \frac{1}{\sqrt{2}}(\cos\theta, \sin\theta, 0)$ for which Theorem 2.5 clearly fails. The first part of the following theorem is due to Käenmäki, Orponen and Venieri [KOV], Pramanik, Yang, and Zahl [PYZ], Gan, Guth, and Maldague [GGM], and the second to Harris [H1]:

Theorem 2.5. *Let $A \subset \mathbb{R}^3$ be a Borel set.*

(1) *If* $\dim A \leq 1$, *then*

$$\dim p_\theta(A) = \dim A \quad \textit{for almost all } \theta.$$

(2) *If* $\dim A > 1$, *then*

$$\mathcal{H}^1(p_\theta(A)) > 0 \quad \textit{for almost all } \theta.$$

The corresponding problem of the projections π_θ onto the planes orthogonal to $\gamma(\theta)$ was solved by Gan, Guo, Guth, Harris, Maldague and Wang in [GGGHMW]:

Theorem 2.6. *Let $A \subset \mathbb{R}^3$ be a Borel set.*

(1) *If* $\dim A \leq 2$, *then*

$$\dim \pi_\theta(A) = \dim A \quad \textit{for almost all } \theta.$$

(2) *If* $\dim A > 2$, *then*

$$\mathcal{H}^2(\pi_\theta(A)) > 0 \quad \textit{for almost all } \theta.$$

See [GGGHMW], [H2] and [H3] for many recent related results and references. The proofs involve several powerful methods; good-bad decomposition of measures (good part assings small measure to sets in a wave-packet decomposition of the space), decoupling inequalities (for a treatise on the decoupling theory, see Demeter's book [D]) and high-low methods (decomposing a function as a sum of a function with high frequencies and a function with low frequencies). The papers [KOV] and [PYZ] use variants of Wolff's circular maximal function estimates.

In [H2] Harris proved quantitative L^p analogs of Theorem 2.5, with $p = 6/5$, and Theorem 2.6, with $p = 4/3$, and showed that the latter yields an $L^{4/3}$ inequality for Kakeya maximal functions of certain restricted type.

Next, let

$$\pi_t : \mathbb{R}^4 \to \mathbb{R}^2, \pi_t(x, y) = x + ty, x, y \in \mathbb{R}^2, t \in \mathbb{R}.$$

This family is connected with Besicovitch sets and the Kakeya conjecture in $\mathbb{R}^3$. The following theorem is due to D. M. Oberlin [Ob1]:

Theorem 2.7. *Let* $A \subset \mathbb{R}^4$ *be a Borel set.*

(1) *If* $\dim A \leq 3$, *then* $\dim \pi_t(A) \geq \dim A - 1$ *for almost all* $t \in \mathbb{R}$.
(2) *If* $\dim A > 3$, *then* $\mathcal{L}^2(\pi_t(A)) > 0$ *for almost all* $t \in \mathbb{R}$.

For $g \in O(n)$, define

$$S_g : \mathbb{R}^{2n} \to \mathbb{R}^n, \; S_g(x, y) = x - g(y).$$

Then S_g is essentially the orthogonal projection onto the n-plane $\{(x, -g^{-1}(x)) : x \in \mathbb{R}^n\}$.

The following theorem was proved in [M6]. A product set version will be applied to general intersection is Section 4.

Theorem 2.8. *Let* $A \subset \mathbb{R}^{2n}$ *be a Borel set.*

(1) *If* $\dim A > n + 1$, *then* $\mathcal{L}^n(S_g(A)) > 0$ *for almost all* $g \in O(n)$.
(2) *If* $n \leq \dim A \leq n + 1$, *then* $\dim S_g(A) \geq \dim A - 1$ *for almost all* $g \in O(n)$.
(3) *If* $n - 1 \leq \dim A \leq n$, *then* $\dim S_g(A) \geq n - 1$ *for almost all* $g \in O(n)$.
(4) *If* $\dim A \leq n - 1$, *then* $\dim S_g(A) \geq \dim A$ *for almost all* $g \in O(n)$.

The bounds in the last two theorems are sharp. The proofs use mainly Fourier transform estimates and are much easier than those of Theorems 2.5 and 2.6.

For $x \in \mathbb{R}^n$ define the *radial projection*

$$\pi_x : \mathbb{R}^n \setminus \{x\} \to S^{n-1}, \quad \pi_x(y) = \frac{y - x}{|y - x|}.$$

Then by standard proofs the statements of Marstrand's projection theorem are valid for almost all $x \in \mathbb{R}^n$. Orponen proved in [O] the following sharp estimate for the exceptional set of $x \in \mathbb{R}^n$.

Theorem 2.9. *If μ and ν are Frostman measures of exponent $s > n-1$ in $\mathbb{R}^n$, then for ν almost all y, the radial projection $\pi_{y\sharp}\mu$ has an L^p density for some $p = p(s) > 1$.*

Moreover, if $A \subset \mathbb{R}^n$ is a Borel set with $\dim A > n-1$, then

$$\dim\{x \in \mathbb{R}^n : \mathcal{H}^{n-1}(\{\pi_x(A)) = 0\}) \leq 2(n-1) - \dim A.$$

The first statement is typical to all of the above theorems where positive measure is concluded: they are obtained by L^p estimates, often with $p = 2$ but sometimes (as in [H2]) only with some $1 < p < 2$, for the push-forwards of Frostman measures or measures with finite energy. This is essential for many applications

3. Hausdorff dimension and distance sets

The distance set of a Borel set $A \subset \mathbb{R}^n, n \geq 2$, is

$$D(A) = \{|x-y| : x, y \in A\}.$$

Falconer conjectured in [F2] that $\mathcal{L}^1(A) > 0$ if $\dim A > n/2$, and he proved that this is true if $\dim A > (n+1)/2$. This conjecture is still open.

First, Falconer's result follows immediately from the rather easy estimate

$$\mu \times \mu(\{(x,y) : r \leq |x-y| \leq r+\delta\}) \lesssim I_s(\mu) r^{s-1}\delta \tag{3.1}$$

if $\mu \in \mathcal{M}(B^n(0,1)), s \geq (n+1)/2, 0 < \delta < r$, because this implies that the distance measure

$$\delta(\mu)(B) := \mu \times \mu(\{(x,y) : |x-y| \in B\}),\ B \subset \mathbb{R},$$

is absolutely continuous with L^∞ density provided $I_s(\mu) < \infty, s \geq (n+1)/2$.

By showing that $\delta(\mu)$ is even a continuous function if $I_s(\mu) < \infty, s > (n+1)/2$, we proved with Sjölin in [MS]

Theorem 3.1. *If $A \subset \mathbb{R}^n$ is a Borel set with $\dim A > (n+1)/2$, then $D(A)$ has non-empty interior.*

It is not known if for this the bound $(n+1)/2$ could be improved.

The bound $(n+1)/2$ is sharp (at least for $n = 2, 3$) for the L^∞ estimate (3.1), but further progress follows by L^2 estimates.

For $\mu \in \mathcal{M}(\mathbb{R}^n)$, set

$$\sigma(\mu)(r) = \int_{S^{n-1}} |\widehat{\mu}(rv)|^2\, dv.$$

I proved in [M2] that if

$$\int_1^\infty \sigma(\mu)(r)^2 r^{n-1}\, dr < \infty,$$

then $\delta(\mu) \in L^2(\mathbb{R})$. Then if $I_s(\mu) < \infty$ and $s \geq (n-2)/2, \varepsilon > 0$, Wolff proved in [W1] for $n = 2$ and Erdoğan for $n \geq 3$ that

$$\sigma(\mu)(r) \lesssim r^{-s/2-(n-2)/4+\varepsilon}, r > 1. \tag{3.2}$$

The combination of these two results gives that $\delta(\mu) \in L^2(\mathbb{R}^2)$ if $I_s(\mu) < \infty, s > n/2+1/3$, which implies that $\mathcal{L}^1(DA)) > 0$ if $\dim A > n/2 + 1/3$.

For $n = 2$ the power $s/2$ in (3.2) is the best possible. In higher dimensions the best exponent is not known but Du and Zhang [DZ] (see [Du] for some examples) proved the estimate

$$\sigma(\mu)(r) \lesssim r^{-(n-1)s/n+\varepsilon} \tag{3.3}$$

when $I_s(\mu) < \infty$.

As above, this leads to

Theorem 3.2. *If $A \subset \mathbb{R}^n$ is a Borel set with $\dim A > n/2 + 1/4 + 1/(8n-4)$, then $\mathcal{L}^1(DA)) > 0$.*

When $n = 2$ the borderline $4/3$ is the best possible to get $\delta(\mu) \in L^2(\mathbb{R}^2)$ by an example of Guth, Iosevich, Ou and Wang, [GIOW]. These authors proved the best result so far in the plane:

Theorem 3.3. *If $A \subset \mathbb{R}^2$ is a Borel set with $\dim A > 5/4$, then $\mathcal{L}^1(DA)) > 0$.*

The proof uses Orponen's radial projection estimates, recall Theorem 2.9, Liu's formula [L]:

$$\int_0^\infty |f * \sigma_t(x)|^2 t\, dt = \int_0^\infty |f * \widehat{\sigma_r}(x)|^2 r\, dr, \tag{3.4}$$

σ_t is the normalized length measure on the circle $\{x : |x| = t\}$, and decoupling.

In higher dimensions Du, Iosevich, Ou, Wang and Zhang [DIOWZ] proved, by a rather similar method as that of [GIOW]

Theorem 3.4. *If n is even and $A \subset \mathbb{R}^n$ is a Borel set with $\dim A > n/2 + 1/4$, then $\mathcal{L}^1(DA)) > 0$.*

For odd integers n Theorem 3.2 is still the best known result.

Also estimates on the dimension of the distance sets have been studied extensively. The main question in the plane is: if $\dim A = 1$, how big must $\dim D(A)$ be? The best known result is due to Shmerkin and Wang, see [SW].

Theorem 3.5. *If $A \subset \mathbb{R}^2$ is a Borel set with $\dim A = 1$, then $\dim D(A) \geq (\sqrt{5} - 1)/2$ (the golden mean).*

They also showed that if $A \subset \mathbb{R}^n$ is an Ahlfors-David regular Borel set with $\dim A = n/2$, then $\dim D(A) = 1$. The paper [SW] gives an up-to-date survey of other related results.

4. Plane sections and intersections

What can we say about the dimensions if we intersect a subset of $\mathbb{R}^n$ with $(n-m)$-dimensional planes? We have the following result proved by Marstrand in the plane in [M], in general dimensions it was proved in [M1]:

Theorem 4.1. *Let $m < s < n$ and let $A \subset \mathbb{R}^n$ be $\mathcal{H}^s$ measurable with $0 < \mathcal{H}^s(A) < \infty$. Then*

(1) *for $\mathcal{H}^s$ almost all $x \in A$, $\dim(A \cap (V + x)) = s - m$ for almost all $V \in G(n, n-m)$,*

(2) *for almost all* $V \in G(n, n-m)$,

$$\mathcal{H}^m(\{x \in V^\perp : \dim(A \cap (V+x)) = s-m\}) > 0.$$

These statements are essentially equivalent.

Obviously this is stronger than the corresponding projection theorem 2.2(2). On the other hand, we shall now see that quantitative projection theorems imply such section theorems.

Let $P_\lambda : \mathbb{R}^n \to \mathbb{R}^m, \lambda \in \Lambda$, be orthogonal projections, where Λ is a compact metric space. Suppose that $\lambda \mapsto P_\lambda x$ is continuous for every $x \in \mathbb{R}^n$. Let also ω be a finite non-zero Borel measure on Λ. These assumptions are just to guarantee that the measurability of the various functions appearing in the proofs can easily be checked.

Theorem 4.2. *Let* $s > m$ *and* $p > 1$. *Suppose that* $P_{\lambda\sharp}\mu \ll \mathcal{L}^m$ *for* ω *almost all* $\lambda \in \Lambda$ *and that there exists a positive number* C *such that*

$$\iint P_{\lambda\sharp}\mu(u)^p \, d\mathcal{L}^m u \, d\omega\lambda < C\mu(B^n(0,1)) \tag{4.1}$$

whenever $\mu \in \mathcal{M}(B^n(0,1))$ *is such that* $\mu(B(x,r)) \le r^s$ *for* $x \in \mathbb{R}^n, r > 0$.

If $A \subset \mathbb{R}^n$ *is* $\mathcal{H}^s$ *measurable and* $0 < \mathcal{H}^s(A) < \infty$, *then for* $\mathcal{H}^s \times \omega$ *almost all* $(x, \lambda) \in A \times \Lambda$,

$$\dim P_\lambda^{-1}\{P_\lambda x\} \cap A = s - m, \tag{4.2}$$

and for ω *almost all* $\lambda \in \Lambda$,

$$\mathcal{L}^m(\{u \in \mathbb{R}^m : \dim P_\lambda^{-1}\{u\} \cap A = s-m\}) > 0. \tag{4.3}$$

This was proved in [M7]. It gives a general version of Theorem 4.1. It applies to many restricted families of projections, discussed in Section 2, and the related plane sections.

Next we give a version of Theorem 4.2 for product sets and measures. It is used to prove Theorem 4.4 on general intersections. Now $P_\lambda : \mathbb{R}^n \times \mathbb{R}^l \to \mathbb{R}^m, \lambda \in \Lambda$, are orthogonal projections.

Theorem 4.3. *Let* $s, t > 0$ *with* $s + t > m$ *and* $p > 1$. *Suppose that* $P_{\lambda\sharp}(\mu \times \nu) \ll \mathcal{L}^m$ *for* ω *almost all* $\lambda \in \Lambda$ *and there exists a positive number* C *such that*

$$\iint P_{\lambda\sharp}(\mu \times \nu)(u)^p \, d\mathcal{L}^m u \, d\omega\lambda < C\mu(B^n(0,1))\nu(B^l(0,1)) \tag{4.4}$$

whenever $\mu \in \mathcal{M}(B^n(0,1)), \nu \in \mathcal{M}(B^l(0,1))$ *are such that* $\mu(B(x,r)) \le r^s$ *for* $x \in \mathbb{R}^n, r > 0$, *and* $\nu(B(y,r)) \le r^t$ *for* $y \in \mathbb{R}^l, r > 0$.

If $A \subset \mathbb{R}^n$ *is* $\mathcal{H}^s$ *measurable with* $0 < \mathcal{H}^s(A) < \infty$ *and* $B \subset \mathbb{R}^l$ *is* $\mathcal{H}^t$ *measurable with* $0 < \mathcal{H}^t(B) < \infty$, *then for* $\mathcal{H}^s \times \mathcal{H}^t \times \omega$ *almost all* $(x, y, \lambda) \in A \times B \times \Lambda$,

$$\dim P_\lambda^{-1}\{P_\lambda(x,y)\} \cap (A \times B) \ge s + t - m, \tag{4.5}$$

and for ω *almost all* $\lambda \in \Lambda$,

$$\mathcal{L}^m(\{u \in \mathbb{R}^m : \dim P_\lambda^{-1}\{u\} \cap (A \times B) \ge s + t - m\}) > 0. \tag{4.6}$$

Equalities need not hold in (4.5) and (4.6), but they hold if, for example, A or B is Ahlfors-David regular.

The intersections $A \cap (g(B) + z), g \in O(n), z \in \mathbb{R}^n$, can be written essentially as level sets of the restricted projections

$$S_g : \mathbb{R}^{2n} \to \mathbb{R}^n, S_g(x, y) = x - g(y) :$$

letting $\Pi(x, y) = x$ we have

$$A \cap (g(B) + z) = \Pi(S_g^{-1}(A \times B)).$$

Then a quantitative analogue of the projection theorem 2.8 for product measures combined with Theorem 4.3 leads to the following:

Theorem 4.4. *Let $s, t > 0$ with $s + (n-1)t/n > n$ or $s > (n+1)/2$. If $A \subset \mathbb{R}^n$ is $\mathcal{H}^s$ measurable with $0 < \mathcal{H}^s(A) < \infty$ and $B \subset \mathbb{R}^n$ is $\mathcal{H}^t$ measurable with $0 < \mathcal{H}^t(B) < \infty$, then for $\mathcal{H}^s \times \mathcal{H}^t \times \theta_n$ almost all $(x, y, g) \in A \times B \times O(n)$,*

$$\dim A \cap (g(B - y) + x) \geq s + t - n, \tag{4.7}$$

and for θ_n almost all $g \in O(n)$,

$$\mathcal{L}^n(\{z \in \mathbb{R}^n : \dim A \cap (g(B) + z) \geq s + t - n\}) > 0. \tag{4.8}$$

Again, equalities need not hold, see [F4], but they hold if, for example, A or B is Ahlfors-David regular.

I believe that this theorem should be true with the optimal hypothesis $s + t > n$.

See [M4, Chapter 7] and [M5] for discussions and references for the intersection problems.

5. Besicovitch and Furstenberg sets

We say that a Borel set in $\mathbb{R}^n, n \geq 2$, is a *Besicovitch set*, or a Kakeya set, if it has zero Lebesgue measure and it contains a line segment of unit length in every direction. This means that for every $e \in S^{n-1}$ there is $b \in \mathbb{R}^n$ such that $\{te + b : 0 < t < 1\} \subset B$. It is not obvious that Besicovitch sets exist but they do in every $\mathbb{R}^n, n \geq 2$, as was proved by Besicovitch in 1919:

Theorem 5.1. *For any $n \geq 2$ there exists a Borel set $B \subset \mathbb{R}^n$ such that $\mathcal{L}^n(B) = 0$ and B contains a whole line in every direction. Moreover, there exist compact Besicovitch sets in $\mathbb{R}^n$.*

Proof. It is enough to prove this in the plane, then $B \times \mathbb{R}^{n-2}$ is fine in $\mathbb{R}^n$. We shall use projections and duality between points and lines. More precisely, parametrize the lines, except those parallel to the y-axis, by $(a, b) \in \mathbb{R}^2$:

$$l(a, b) = \{(x, ax + b) : x \in \mathbb{R}\}.$$

Then if $C \subset \mathbb{R}^2$ is some parameter set and $B = \cup_{(a,b)\in C} l(a, b)$, one checks that

$$B \cap \{(t, y) : y \in \mathbb{R}\} = \{t\} \times \pi_t(C)$$

where

$$\pi_t : \mathbb{R}^2 \to \mathbb{R}^2, \quad \pi_t(a, b) = ta + b,$$

is essentially an orthogonal projection. Suppose that we can find C such that $\pi(C) = [0,1]$, where $\pi(a,b) = a$, and $\mathcal{L}^1(\pi_t(C)) = 0$ for almost all t. Then $\mathcal{L}^2(B) = 0$ by Fubini's theorem and taking the union of four rotated copies of B gives the desired set. It is not trivial that such sets C exist but they do. For example, a suitably rotated copy of the product of a standard Cantor set with dissection ratio $1/4$ with itself is such. □

The idea to construct Besicovitch sets using duality between lines and points is due to Besicovitch from 1964. It was further developed by Falconer in [F3].

Conjecture 5.2 (Kakeya conjecture)**.** All Besicovitch sets in $\mathbb{R}^n$ have Hausdorff dimension n.

Perhaps the main interest of this conjecture among Fourier analysts is that it would follow from Stein's restriction conjecture:

$$\|\widehat{f}\|_{L^q(\mathbb{R}^n)} \leq C(n,q)\|f\|_{L^\infty(S^{n-1})} \quad \text{for } q > 2n/(n-1).$$

The Kakeya conjecture is open for $n \geq 3$. In the plane Davies proved the following in 1971 in [D].

Theorem 5.3. *For every Besicovitch set $B \subset \mathbb{R}^2$, $\dim B = 2$. In particular, the Kakeya conjecture is true in the plane.*

The proof of this is, up to some technicalities, reversing the above argument for the proof of Theorem 5.1 and using Marstrand's projection Theorem 2.1(1).

Although the Kakeya conjecture in the plane follows rather easily by projection theorems, it seems that one does not get far with them in higher dimensions.

Many people have made progress on the Kakeya conjecture since Fefferman in 1971 solved the multiplier problem for the ball by Kakeya methods and Bourgain [B1] in 1991 introduced powerful new methods to prove partial results for the Kakeya conjecture. The most recent work is due to Wang and Zahl [WZ]. They proved that the Besicovitch sets in $\mathbb{R}^3$ have Assouad dimension 3 and that the Ahlfors-David regular Besicovitch sets in $\mathbb{R}^3$ have Hausdorff dimension 3.

Furstenberg sets are kind of fractal versions of Besicovitch sets. They originate in Furstenberg's paper [Fu]. We say that $F \subset \mathbb{R}^2$ is a Furstenberg s-set, $0 < s \leq 1$, if for every $e \in S^1$ there is a line L_e in direction e such that $\dim F \cap L_e \geq s$. What can be said about the dimension of F? Wolff [W2], Section 11.1, proved some partial estimates, showed that there is such an F with $\dim F = 3s/2 + 1/2$, and conjectured that $\dim F \geq 3s/2 + 1/2$ would hold for all Furstenberg s-sets.

Ren and Wang solved the Furstenberg problem in [RW] by proving

Theorem 5.4. *For any Furstenberg s-set $F \subset \mathbb{R}^2$ we have*

$$\dim F \geq (3s+2)/2.$$

More generally, they proved the case $(3s+2)/2$ of the following theorem, the other two cases were already known and easier. The Hausdorff dimension of the line set $\mathcal{L}$ is defined using a natural metric on the 2-dimensional space of lines in the plane.

Theorem 5.5. *Let $0 \leq s \leq 1, 0 \leq t \leq 2$. If $F \subset \mathbb{R}^2$ has the property that there exists a family $\mathcal{L}$ of lines in the plane such that $\dim \mathcal{L} \geq t$ and $\dim F \cap L \geq s$ for every $L \in \mathcal{L}$, then*

$$\dim F \geq \min\{s + t, (3s + 2)/2, t + 1\}.$$

The proof is based on the paper [GSW] of Guth, Solomon and Wang, where the problem was solved for well-spaced sets, and the paper [OS] by Orponen and Shmerkin, where the problem was solved for Ahlfors-David regular sets. These two classes are sort of opposite to each other.

The proof in [OS] begins with addditive combinatorics and related projections dealing with finite unions discs of radius δ. This topic was pioneered by Bourgain in [B2] and related papers. These are used to prove dimension estimates for Furstenberg sets.

An essential feature of [GSW], and further of [RW], is the application of high-low methods of Fourier analysis (recall Section 2), to prove incidence estimates for collections $\mathcal{B}$ of δ-discs and $\mathcal{T}$ of δ-tubes (δ-neighborhoods line segments), that is, estimates for the cardinalities of sets like $\{T \in \mathcal{T} : T \text{ meets at least } N \text{ discs } B \in \mathcal{B}\}$.

Orponen and Shmerkin [OS] showed how estimates on the dimension of Furstenberg sets lead to estimates on the exceptional sets of projections. Using this and Theorem 5.5 Ren and Wang proved the sharp estimate of Theorem 2.4.

References

[B1] J. Bourgain. Besicovitch type maximal operators and applications to Fourier analysis, *Geom. Funct. Anal.* **1**, 147-187.

[B2] J. Bourgain. The discretized sum-product and projection theorems, *J. Anal. Math.* **112** (2010), 193-236.

[BD] J. Bourgain and C. Demeter. The proof of the l^2 decoupling conjecture, *Ann. of Math* **182** (2015), 351 - 389.

[D] R.O. Davies. Some remarks on the Kakeya problem, *Proc. Cambridge Philos. Soc.* **69** (1971), 417-421.

[Du] X. Du, Upper bounds of Fourier decay rate of fractal measures. J. London Math. Soc. (2) 102 (2020), 1318–1336.

[DGO] X. Du, L. Guth, Y. Ou, H. Wang, B. Wilson, and R. Zhang. Weighted restriction estimates and application to Falconer distance set problem, *Amer. J. Math* 143 (2021), 175–211.

[DIOWZ] X. Du, A. Iosevich, Y. Ou, H. Wang, and R. Zhang, An improved result for Falconer's distance set problem in even dimensions.*Math. Ann.* **380** (2021), 189 1215-1231. (

[DZ] X. Du and R. Zhang, Sharp L^2 estimates of the Schrödinger maximal function in higher dimensions. Ann. of Math. (2) 189 (2019), 837–861.

[E] M. B. Erdoğan. A bilinear Fourier extension problem and applications to the distance set problem, *Int. Math. Res. Not.* **23** (2005), 1411-1425.

[F1] K.J. Falconer. Hausdorff dimension and the exceptional set of projections, *Mathematika* **29** (1982), 109-115.

[F2] K.J. Falconer. On the Hausdorff dimension of distance sets, *Mathematika* **32** (1985), 206-212.

[F3] K.J. Falconer. *The Geometry of Fractal Sets*, Cambridge University Press, Cambridge, 1985.

[F4] K.J. Falconer. Classes of sets with large intersection, *Mathematika* **32** (1985), 191-205.

[FO] K.J. Falconer and T. O'Neil. Convolutions and the geometry of multifractal measures *Math. Nachr.* **204** (1999), 61-82.

[F] H. Federer. *Geometric Measure Theory*, Springer-Verlag, 1969.

[Fu] H. Furstenberg. . Intersections of cantor sets and transversality of semigroups. Problems in analysis (Sympos. Salomon Bochner, Princeton Univ., Princeton, NJ, 1969), pages 41-59, 1970.

[GGGHMW] S. Gan, S. Guo, L. Guth, T. L. J. Harris, D. Maldague and H. Wang. On restricted projections to planes in $\mathbb{R}^3$, arXiv:2207.13844.

[GGM] S. Gan, L. Guth and D. Maldague. An exceptional set estimate for restricted projections to lines in $\mathbb{R}^3$, arXiv:2209.15152.

[GIOW] L. Guth, A. Iosevich, Y. Ou, and H. Wang. On Falconer's distance set problem in the plane, *Invent. Math.* **219** (2020), 779-830.

[GSW] L. Guth, N. Solomon, and H. Wang. Incidence estimates for well spaced tubes, *Geom. Funct. Anal.* **29** (2019), 1844-1863.

[H1] T. Harris. Length of sets under restricted families of projections onto lines, Contemp. Math., 792, arXiv:2208.06896.

[H2] T. Harris. An $L^{4/3}SL_2$ Kakeya maximal inequality, arXiv:2311.14667.

[H3] T. Harris. Exceptional sets for length under restricted families of projections onto lines in $\mathbb{R}^3$, arXiv:2408.04885.

[KOV] A. Käenmäki, T. Orponen and L. Venieri. A Marstrand-type restricted projection theorem in $\mathbb{R}^3$, arXiv:1708.04859.

[KZ] N. H. Katz and J. Zahl. An improved bound on the Hausdorff dimension of Besicovitch sets in $\mathbb{R}^3$, arXiv:1704.07210.

[Ka] R. Kaufman. On Hausdorff dimension of projections, *Mathematika* **15** (1968), 153-155.

[KM] R. Kaufman and P. Mattila. Hausdorff dimension and exceptional sets of linear transformations, *Ann. Acad. Sci. Fenn. A Math.* **1** (1975), 387-392.

[L] B. Liu. An L^2-identity and pinned distance problem, *Geom. Funct. Anal* **29** (2019), 283 - 294.

[M] J. M. Marstrand. Some fundamental geometrical properties of plane sets of fractional dimensions, *Proc. London Math. Soc.(3)* **4** (1954), 257-302.

[M1] P. Mattila. Hausdorff dimension, orthogonal projections and intersections with planes, *Ann. Acad. Sci. Fenn. A Math.* **1** (1975), 227-244.

[M2] P. Mattila. On the Hausdorff dimension and capacities of intersections, *Mathematika* **32**, (1985), 213- 217.

[M3] P. Mattila. Spherical averages of Fourier transforms of measures with finite energy; dimension of intersections and distance sets, *Mathematika* **34**, (1987), 207-228.

[M4] P. Mattila. *Geometry of Sets and Measures in Euclidean Spaces*, Cambridge University Press, Cambridge, 1995.

[M5] P. Mattila. *Fourier Analysis and Hausdorff Dimension*, Cambridge University Press, Cambridge, 2015.

[M6] P. Mattila. Hausdorff dimension and projections related to intersections, *Publ. Mat.* **66** (2022), 305-323.

[M7] P. Mattila. Hausdorff dimension of plane sections and general intersections, *Bull. London Math. Soc* **56** (2024), 1988-1998.

[MO] P. Mattila and T. Orponen. Hausdorff dimension, intersection of projections and exceptional plane sections, *Proc. Amer. Math. Soc.* **144** (2016), 3419-3430.

[MS] P. Mattila and P. Sjölin. Regularity of distance measures and sets, *Math. Nachr.* **204** (1997), 157-162.

[Ob1] D. M. Oberlin. Exceptional sets of projections, unions of k-planes, and associated transforms, *Israel J. Math.* **202** (2014), 331-342.

[Ob2] D. M. Oberlin. Some toy Furstenberg sets and projections of the four-corner Cantor set, *Proc. Amer. Math. Soc.* **142**, no. 4, 1209-1215.

[O] T. Orponen. , On the dimension and smoothness of radial projections, Anal. PDE 12 (2019), no. 5, 1273-1294.

[OS] T. Orponen and P. Shmerkin. Projections, Furstenberg sets, and the abc sum-product problem, arXiv:2301.10199.

[OSW] Tuomas Orponen, Pablo Shmerkin, and Hong Wang. Kaufman and Falconer estimates for radial projections and a continuum version of Beck's Theorem. arXiv:2209.00348.

[PS] Y. Peres and W. Schlag. Smoothness of projections, Bernoulli convolutions, and the dimension of exceptions, *Duke Math. J.* **102** (2000), 193-251.

[PYZ] M. Pramanik, T. Yang, J. Zahl. A Furstenberg-type problem for circles, and a Kaufmantype restricted projection theorem in $\mathbb{R}^3$. arXiv:2207.02259.

[RW] K. Ren and H. Wang. Furstenberg sets estimate in the plane, arXiv:2308.08819.

[SW] P. Shmerkin and H. Wang. On the distance sets spanned by sets of dimension $d/2$ in $\mathbb{R}^d$, arXiv:2112.09044.

[WZ] H. Wang and J. Zahl. The Assouad dimension of Kakeya sets in $\mathbb{R}^3$, arXiv:2401.12337.

[W1] T. W. Wolff. Decay of circular means of Fourier transforms of measures, *Int. Math. Res. Not.* **10** (1999), 547-567.

[W2] T. W. Wolff. Lectures on Harmonic Analysis, Amer. Math. Soc., University Lecture Series 29, 2003.