VISCOELASTIC LOVE-TYPE SURFACE WAVES

R.D. Borcherdt[1]

[1] *Dr., U.S. Geological Survey, Menlo Park, CA USA*
Email: Borcherdt@usgs.gov

ABSTRACT :

The general theoretical solution for Love-Type surface waves in viscoelastic media provides theoretical expressions for the physical characteristics of the waves in elastic as well as anelastic media with arbitrary amounts of intrinsic damping. The general solution yields dispersion and absorption-coefficient curves for the waves as a function of frequency and the amount of intrinsic damping for any chosen viscoelastic model. Numerical results valid for a variety of viscoelastic models provide quantitative estimates of the physical characteristics of the waves pertinent to models of Earth materials ranging from small amounts of damping in the Earth's crust to moderate and large amounts of damping in soft soils and water-saturated sediments. Numerical results, presented herein, are valid for a wide range of solids and applications.

KEYWORDS: Viscoelastic, Love-wave, surface-wave, damping, absorption, wave propagation

1 INTRODUCTION

Love in 1911 (1944) first established the solution for surface waves with horizontal particle motion in elastic media comprised of a layer over a half space. The solution was later extended to multiple layers of elastic media using a matrix formulation introduced by Thompson (1950) and implemented by Haskell (1953). Borcherdt (2008) extended the solution for Love-Type surface waves to arbitrary multiple layered viscoelastic media with the resulting solution being valid for an infinite number of viscoelastic models, such as Voight, Maxwell, and Standard Linear solids. This paper discusses the general solution for viscoelastic media. It describes the inferred physical characteristics of Viscoelastic Love-Type surface waves. It provides numerical estimates of dispersion and absorption curves that are valid for a wide range in Earth materials with various amounts of damping.

2. ANALYTIC SOLUTION AND DISPLACEMENTS

The problem of the steady-state response of a stack of $n-1$ layers overlying a half space is specified by Borcherdt (2008) using notation for the media and boundaries as shown in Figure (1.1), where thickness d_m and depth z_{m-1} of the top of the m^{th} layer and material parameters for m^{th} viscoelastic layer are density, ρ_m, wave speeds and reciprocal quality factors for homogenous S and P waves , namely v_{HS_m}, v_{HP_m}, $Q_{HS_m}^{-1}$, and $Q_{HP_m}^{-1}$.

Using this notation, application of the boundary conditions of vanishing stress at the free surface and continuity of stress and displacement at the boundaries to assumed displacement solutions allows the complex displacement, $\vec{u}_m(z)$ in the m^{th} viscoelastic layer to be expressed in terms of the complex amplitude, D_{11}, of the solutions at the free surface (Borcherdt, 2008) as

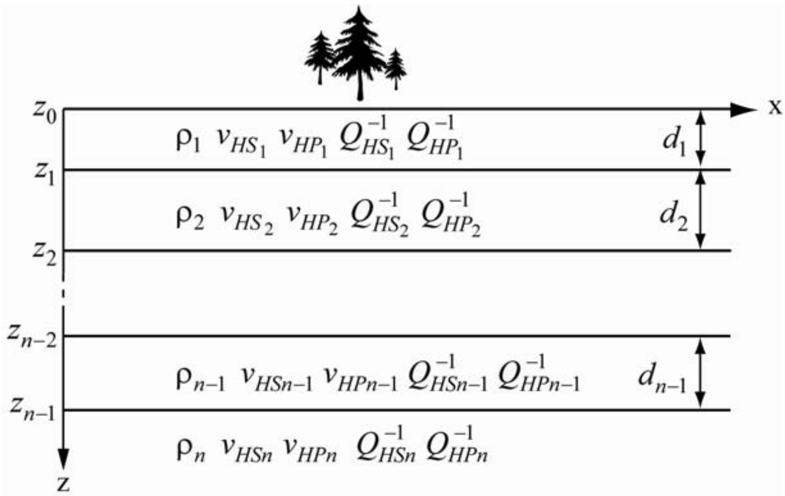

Figure (1.1). Diagram illustrating notation for the problem of a Love-Type Surface wave on multiple layered viscoelastic media.

$$\vec{u}_m(z) = D_{11}\sum_{j=1}^{2}\left(\left(\mathbf{F}_{\mathbf{m\text{-}1}_{11}} - \frac{(-1)^j\,\mathbf{F}_{\mathbf{m\text{-}1}_{21}}}{\mathrm{M}_m\, b_{\beta m}}\right)\exp\left[(-1)^{j+1} b_{\beta m}\left(z - z_{m-1}\right)\right]\right)\exp\left[i\left(\omega t - kx\right)\right]\hat{x}_2 \tag{1.2}$$

and in the half space as

$$\vec{u}_n(z) = 2D_{11}\mathbf{F}_{\mathbf{n\text{-}1}_{11}}\exp\left[k_I x - b_{\beta n}\left(z - z_{n-1}\right)\right]\exp\left[i\left(\omega t - k_R x\right)\right]\hat{x}_2, \tag{1.3}$$

where

$$\mathbf{F}_{\mathbf{n\text{-}1}_{11}} = -\mathbf{F}_{\mathbf{n\text{-}1}_{21}}\big/\mathrm{M}_n\, b_{\beta n}\ , \tag{1.4}$$

$$\mathbf{F}_{\mathbf{n\text{-}1}} \equiv \mathbf{f}_{\mathbf{n\text{-}1}}\,\mathbf{f}_{\mathbf{n\text{-}2}}\ldots\mathbf{f}_{\mathbf{1}}, \tag{1.5}$$

$$\mathbf{f}_{\mathbf{m}} \equiv \begin{pmatrix} \cos\left[-ib_{\beta m}\, d_m\right] & \dfrac{\sin\left[-ib_{\beta m}\, d_m\right]}{-iM_m\, b_{\beta m}} \\ iM_m\, b_{\beta m}\sin\left[-ib_{\beta m}\, d_m\right] & \cos\left[-ib_{\beta m}\, d_m\right] \end{pmatrix}, \tag{1.6}$$

and

$$b_\beta \equiv \textit{principal value}\sqrt{k^2 - k_S^2}\ . \tag{1.7}$$

The existence of values for the complex wave number k that satisfy (1.4) establishes solutions for a Love-Type surface wave in multilayered viscoelastic media overlying a viscoelastic half space exists (Borcherdt, 2008). Hence, with the magnitude of the phase velocity v_L and the absorption coefficient a_L for a Love-Type surface wave of circular frequency ω expressed in terms of the real and imaginary parts of the complex wave number k by $v_L = \omega/k_R$ and $a_L = -k_I$, the problem of finding the solution for a Love-Type surface wave reduces to the problem of finding pairs of values for the wave speed and absorption coefficient (v_L, a_L) that satisfy (1.4) for a given set of viscoelastic material parameters characterizing each layer and the half space.

The corresponding physical particle displacements specified by the real part of the complex displacements (1.2) and (1.3) in the m^{th} layer from Borcherdt (2008) are given by

$$\vec{u}_{m_R}(z) = \exp[k_I x]\, H_m \sin\left[\omega t - k_R x + (\pi/2) + \delta_m\right]\hat{x}_2 \tag{1.8}$$

$$H_m \equiv \sqrt{E_{1m}{}^2 + E_{2m}{}^2 + 2E_{1m}E_{2m}\cos[g_{1m} - g_{2m}]}, \tag{1.9}$$

$$\delta_m \equiv \arctan\left[\frac{E_{1m}\sin[g_{1m}] + E_{2m}\sin[g_{2m}]}{E_{1m}\cos[g_{1m}] + E_{2m}\cos[g_{2m}]}\right], \tag{1.10}$$

$$E_{jm} \equiv |D_{jm}|\exp\left[(-1)^{j+1} b_{\beta m_R}\, z\right], \tag{1.11}$$

$$g_{jm} \equiv (-1)^{j+1} b_{\beta m_I}\, z + \arg D_{jm}, \tag{1.12}$$

in the half space by

$$\begin{aligned}\frac{\vec{u}_{n_R}(z)}{|D_{11}|} = 2\left|\mathbf{F}_{n\text{-}\mathbf{1}_{11}}\right|\exp\left[k_I x - b_{\beta n_R}(z - z_{n-1})\right] \\ \sin\left[\omega t - k_R x + (\pi/2) - b_{\beta n_I}(z - z_{n-1}) + \arg D_{11} + \arg \mathbf{F}_{n\text{-}\mathbf{1}_{11}}\right]\hat{x}_2,\end{aligned} \tag{1.13}$$

and at the free surface by

$$\frac{\vec{u}_{1_R}(z = z_0)}{|D_{11}|} = 2\exp[k_I x]\sin\left[\omega t - k_R x + (\pi/2) + \arg D_{11}\right]\hat{x}_2 . \tag{1.14}$$

Expression (1.13) indicates that the physical displacement in an anelastic viscoelastic half space shows a superimposed sinusoidal dependence on depth that decays exponentially with distance from the interface. It indicates that the displacement in the half space (1.13) is out of phase with that at the surface (1.14) by an amount that depends on the thickness of intervening layers and their viscoelastic material parameters. It indicates that the physical displacement attenuates with absorption coefficient $a_L = -k_I$ along both the interface at depth and the free surface. The expression also shows that for elastic media with $Q_{HS_n}^{-1} = Q_{HP_n}^{-1} = 0$ that no superimposed sinusoidal dependence exists, displacement at the surface is in phase with that in the half space, and no attenuation occurs in the direction of phase propagation.

The solution for a Love-Type surface wave in viscoelastic media is most readily considered for the case of a single layer overlying a half space. For the case that $n = 2$, (1.4) reduces to the complex period equation (Borcherdt 2008)

$$\omega\frac{d_1}{v_L} = F\left[\frac{\rho_1}{\rho_2}, Q_{HS_1}^{-1}, Q_{HS_2}^{-1}, \frac{a_L}{a_{HS_1}}, \frac{v_L}{v_{HS_1}}\right] \tag{1.15}$$

where the function $F[\ ]$ is defined by

$$F\left[\frac{\rho_1}{\rho_2}, Q_{HS_1}^{-1}, Q_{HS_2}^{-1}, \frac{a_L}{a_{HS_1}}, \frac{v_L}{v_{HS_1}}\right] \equiv -i\left(\begin{array}{l}\left(1-i\dfrac{a_L}{a_{HS_1}}\dfrac{v_L}{v_{HS_1}}\dfrac{Q_{HS_1}^{-1}}{1+\chi_{HS_1}}\right)^2 \\ -\left(\dfrac{v_L}{v_{HS_1}}\left(1-i\dfrac{Q_{HS_1}^{-1}}{1+\chi_{HS_1}}\right)\right)^2\end{array}\right)^{-1/2}$$

$$\left(\arctan\left[\left(-\frac{i\,\rho_2}{\rho_1}\frac{{v_{HS_2}}^2}{{v_{HS_1}}^2}\frac{1+\chi_{HS_2}}{1+\chi_{HS_1}}\frac{1-i\,Q_{HS_1}^{-1}}{1-i\,Q_{HS_2}^{-1}}\right)\left(\frac{\sqrt{\left(1-i\dfrac{a_L}{a_{HS_1}}\dfrac{v_L}{v_{HS_1}}\dfrac{Q_{HS_1}^{-1}}{1+\chi_{HS_1}}\right)^2-\left(\dfrac{v_L}{v_{HS_1}}\dfrac{v_{HS_1}}{v_{HS_2}}\left(1-i\dfrac{Q_{HS_2}^{-1}}{1+\chi_{HS_2}}\right)\right)^2}}{\sqrt{\left(1-i\dfrac{a_L}{a_{HS_1}}\dfrac{v_L}{v_{HS_1}}\dfrac{Q_{HS_1}^{-1}}{1+\chi_{HS_1}}\right)^2-\left(\dfrac{v_L}{v_{HS_1}}\left(1-i\dfrac{Q_{HS_1}^{-1}}{1+\chi_{HS_1}}\right)\right)^2}}\right)\right]\mp n\pi\right). \tag{1.16}$$

Hence, for a given set of material parameters ρ_2/ρ_1, v_{HS_2}/v_{HS_1}, ${Q_{HS_1}}^{-1}$, and ${Q_{HS_2}}^{-1}$ (1.15) and (1.16) show explicitly that pairs of values of v_L/v_{HS_1} and a_L/a_{HS_1} that yield a non-negative real number for the right-hand side of (1.15) represent the solution for a Love-Type surface wave of circular frequency ω for viscoelastic media with a layer of thickness d_1. For elastic media with ${Q_{HS_1}}^{-1} = {Q_{HS_2}}^{-1} = 0$, the period equation simplifies to the familiar period equation for elastic media, namely

$$\frac{\omega}{v_L}d_1 = \left(\sqrt{\frac{v_L^2}{v_{HS_1}^2}-1}\right)^{-1}\left(\arctan\left[\frac{\rho_2}{\rho_1}\frac{v_{HS_2}^2}{v_{HS_1}^2}\left(\sqrt{1-\frac{v_L^2}{v_{HS_2}^2}}\Big/\sqrt{\frac{v_L^2}{v_{HS_1}^2}-1}\right)\right]\mp n\pi\right) \tag{1.17}$$

showing that a solution exists for values of v_L that satisfy $v_{HS_1} < v_L < v_{HS_2}$.

3. NUMERICAL CHARACTERISTICS OF LOVE-TYPE SURFACE WAVES

Quantitative descriptions of the physical characteristics of a Love-Type surface wave are specified by solutions for pairs of values for wave speed v_L and absorption coefficient a_L that satisfy (1.15). Corresponding estimates of the normalized absorption coefficient ratio a_L/a_{HS_1} as a function of the wave-speed ratio v_L/v_{HS_1} for values of the ratio that satisfy $1 < v_L/v_{HS_1} < v_{HS_2}/v_{HS_1}$ are shown for a given set of viscoelastic material parameters for the fundamental mode of a Love-Type surface wave in Figure (1.18). Values chosen for the material parameters to solve (1.15) are $\rho_2/\rho_1 = 1.283$, $v_{HS_2}/v_{HS_1} = 1.297$, $Q_{HS_2}^{-1} = 0.01$, and a set of values for intrinsic absorption in the layer as indicated in the figure ranging from low loss, $Q_{HS_1}^{-1} = 0.01$, for crustal material to significant loss, $Q_{HS_1}^{-1} = 0.5$, for soft soils. The curves indicate that the dependence of the absorption-coefficient ratio on the wave-speed ratio varies significantly as the amount of intrinsic absorption in the layer increases.

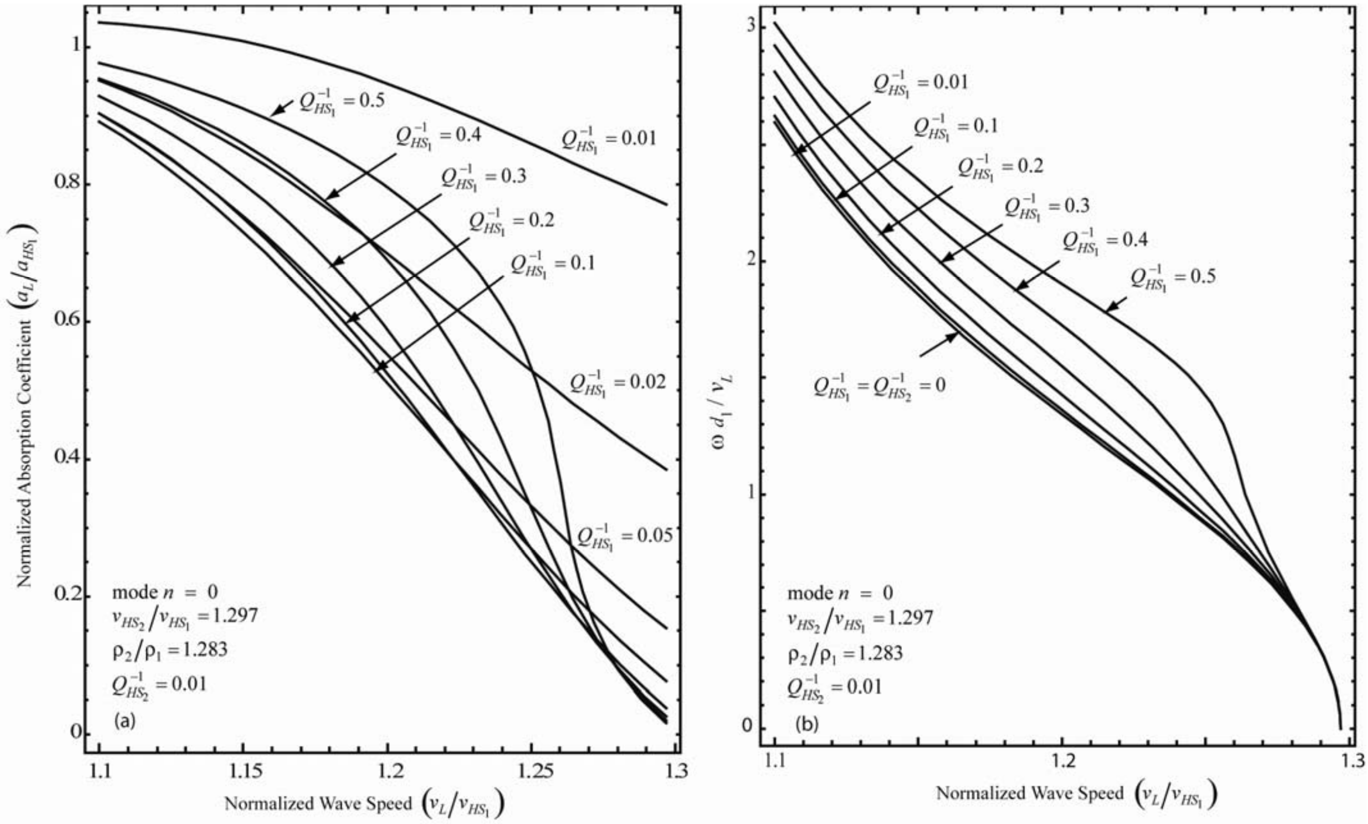

Figure (1.18). (a) Normalized absorption coefficient and (b) $\omega d_1/v_L$ versus normalized wave speed for the fundamental mode of a Love-Type surface wave in viscoelastic media with material parameters as indicated.

Dispersion curves corresponding to the absorption curves in Figure (1.18)a are shown in Figure (1.18)b for the fundamental mode of Love-Type surface waves. The plots indicate that for the chosen material parameters, variations in the curves due to differences in the amount of intrinsic absorption in the layer are most evident for amounts of intrinsic absorption $Q_{HS_1}^{-1} \geq 0.1$. For low-loss amounts of intrinsic absorption in the layer with $Q_{HS_1}^{-1} < 0.1$ the deviations in the curves from those for an elastic solid are small and less discernible at the scale plotted.

The distribution of physical displacement with depth as inferred from (1.8) through (1.14) computed as a function of depth in units of fractions of a wavelength $\lambda \equiv 2\pi/k_R$ along the surface is shown from Borcherdt (2008) in Figure (1.19). The amplitude distribution corresponding to each value of $Q_{HS_1}^{-1}$ is that for which the wave-speed ratio $v_L/v_{HS_1} = 1.25$ and the viscoelastic material parameters are as indicated.

The curves in Figure (1.19) indicate that for the chosen material parameters, the amplitudes decrease by about 25 percent within a depth of about 17 percent of a wavelength below the surface. The curves indicate that deviations in the amplitude distributions due to increases in intrinsic absorption in the layer begin to become apparent for depths greater than about 20 percent of a wavelength and values of $Q_{HS_1}^{-1} > 0.1$. For depths greater than about 20 percent, the decrease in amplitude with depth increases with an increase in intrinsic absorption. For the material parameters chosen, the dependence of the normalized amplitude distribution on depth for low-loss media ($Q_{HS_1}^{-1} < 0.1$) is nearly indistinguishable at the scale plotted from that for corresponding elastic media.

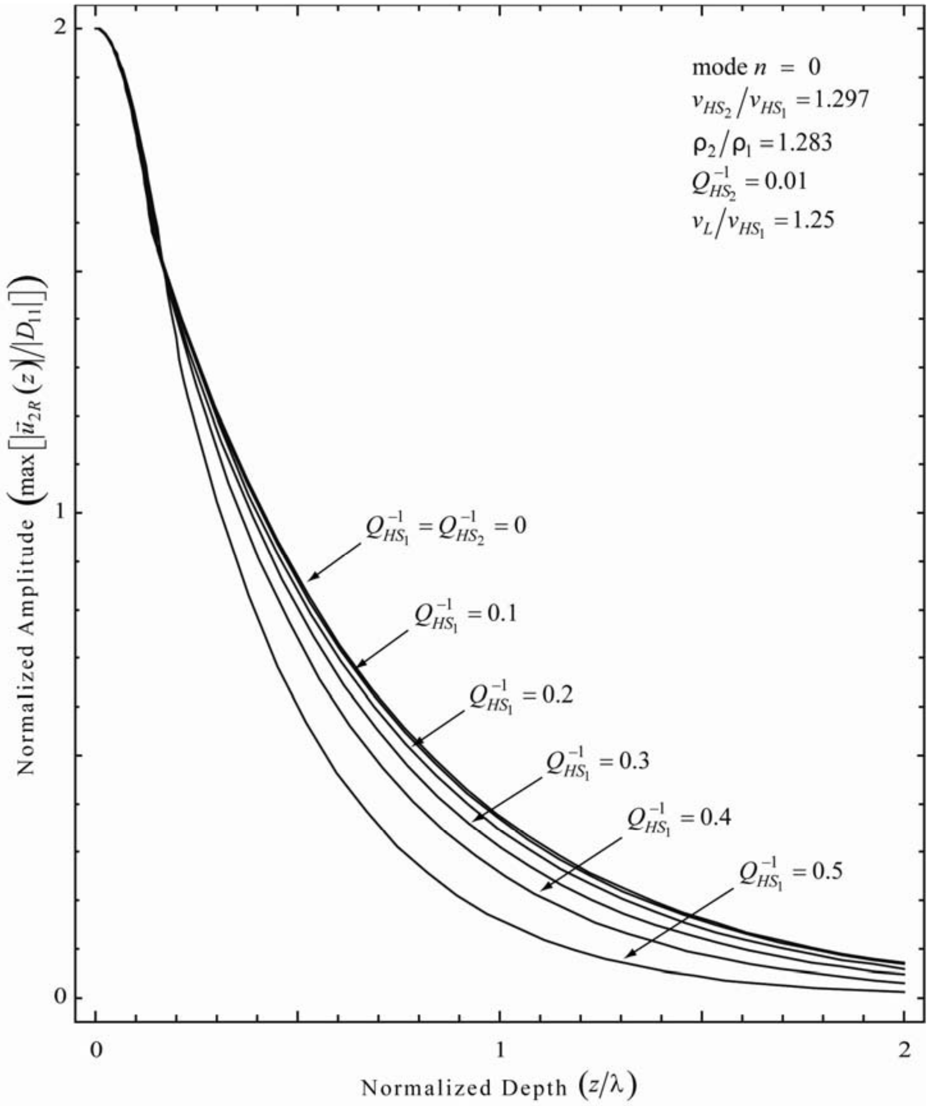

Figure (1.19). Normalized amplitude distribution for the fundamental mode of a Love-Type surface wave in viscoelastic media with material parameters as indicated.

The procedure used here to derive quantitative estimates of the physical characteristics of Love-Type surface waves on an arbitrary viscoelastic media with general material parameters $v_{HS_j}, Q_{HS_j}^{-1}, \rho_j$ for $j = 1, 2$ can be readily extended to any particular viscoelastic solid as might be characterized by combinations of springs and dashpots. As an example, if the viscoelastic model for the layer is chosen as a Standard Linear solid with material parameters τ_p, τ_e, M_r, and ρ_1 and the model for the half space is chosen as a Maxwell solid with material parameters μ, η, and ρ_2, then specification of circular frequency ω and these material parameters implies $Q_{HS_j}^{-1}$ and v_{HS_j} for $j = 1, 2$ from Borcherdt (2008) are easily specified as

$$Q_{HS_1}^{-1} = \frac{\omega\left(\tau_e - \tau_p\right)}{1 + \omega^2 \tau_p \tau_e} \tag{1.20}$$

and

$$v_{HS_1} = \sqrt{\frac{M_r}{\rho_1} \frac{1+\omega^2\tau_e\tau_p}{1+\omega^2\tau_p^2} 2\left(1+\left(\frac{\omega(\tau_e-\tau_p)}{(1+\omega^2\tau_e\tau_p)}\right)^2\right) \Bigg/ \left(1+\sqrt{1+\left(\frac{\omega(\tau_e-\tau_p)}{(1+\omega^2\tau_e\tau_p)}\right)^2}\right)} \qquad (1.21)$$

and in the half space by

$$Q_{HS_2}^{-1} = \frac{\mu}{\omega\eta} \qquad (1.22)$$

and

$$v_{HS_2} = \sqrt{\frac{\mu}{\rho_2} \frac{\omega^2\eta^2}{\mu^2+\omega^2\eta^2} 2\left(1+\left(\frac{\mu}{\omega\eta}\right)^2\right) \Bigg/ \left(1+\sqrt{1+\left(\frac{\mu}{\omega\eta}\right)^2}\right)} \qquad (1.23)$$

Hence, specification of the material parameters for these particular viscoelastic models together with frequency ω allows the general material parameters v_{HS_2}/v_{HS_1}, ρ_2/ρ_1, and $Q_{HS_j}^{-1}$ for $j=1, 2$ to be determined, from which curves similar to those in Figures (1.18) and (1.19) for a Love-Type surface wave can be readily derived.

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