SCATTERING OF HARMONIC WAVES BY A CAVITY IN CROSS-ANISOTROPIC MEDIA: COMPLEX FUNCTIONS APPROACH

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ABSTRACT:

An analytical method for calculation of dynamic response of a cylindrical cavity in infinite cross-anisotropic media using complex functions theory is presented. The basis of the method is grounded on solving the wave equations in the complex plane and frequency domain. Solution of the partial differential equations is found in series of the Hankel functions with unknown coefficients. Applying appropriate boundary conditions of the problem, a set of algebraic equations are achieved. Solving these equations, the unknown coefficients and consequently all desirable parameters such as stress, strain and displacement are calculated.
Numerical results including stress and displacement fields in vicinity of the cylinder subjected to the harmonic waves are presented. Also the effects of important cross-anisotropic parameters are mentioned.

KEYWORDS: wave scattering; analytical solution; dynamic response; complex functions; cylindrical cavity

1. INTRODUCTION

The problem of wave propagation and scattering in the infinite media is described by wave equations. In the most previous researches, the isotropic conditions were considered for the media. In this way, the exact solutions are well known such as Pao and Mow [1]; Eringen and Suhubi [2]. A few studies have been performed for the problem of wave propagation and scattering in cross-anisotropic media. Honarvar & Sinclair [3, 4] solved the problem of acoustic wave scattering from transversely isotropic cylinders. Also Fan et al. [5] solved this problem for the case that cylinder encased in a solid elastic medium. Both of them are on the basis of normal mode expansion method. Ahmad & Rahman [6, 7] solved the problem of acoustic wave scattering by transversely isotropic cylinders with the same method but with different potential functions.
This paper presents an analytical solution for the problem of scattering of harmonic waves in an infinite cross-anisotropic medium on the basis of complex functions theory. The use of complex functions in elastostatic problems was expanded by Muskhelishvili [8]. Nowinski [9] solved the problem of static stress concentration around holes subjected to uniaxial tension, using complex function theory. This method was used for solving the problem of wave scattering by a cavity in infinite elastic media by Liu et al. [10] and Han et al. [11]. The use of complex functions for solving the problem of wave scattering in cross-anisotropic medium is presented for the first time. In this way, first a group of potential functions (Φ, Ψ, X) is applied into displacement formulation of wave equations. Then, the governing equations are transformed into the complex plane. Solving an eigenvalue problem, two wave numbers (quasi-P, quasi-SV) are determined. The wave number SH is found directly. In the complex plane, solution of the resulting partial differential equations (calculation of potential functions) is found in series of the Hankel functions (complex sum of the Bessel functions) with unknown coefficients. These Hankel functions satisfy the radiation conditions. Applying appropriate boundary conditions of the problem, a set of algebraic equations are achieved. Solving these equations, the unknown coefficients and consequently potential functions are calculated. After evaluating the potential functions, all desirable parameters such as stress and displacement can be calculated in any point of the medium. Numerical results are presented including stress and displacement fields in the vicinity of cylinder in an infinite cross-anisotropic medium subjected to harmonic waves. Also the effects of important parameters such as E_h / E_v, G_v / E_v and ν_{hh} are mentioned.

2. GOVERNING EQUATIONS

The equilibrium equation of a medium is written as:

$$\sigma_{ij,j} + \rho b_i = \rho \ddot{u}_i \tag{2.1}$$

In this statement σ_{ij} is the stress tensor component, ρb_i is the body force, u_i is the displacement vector component and ρ is the mass density of medium. The dots $(\dot{\ })$ denote the differentiation respect to the time ($\frac{\partial}{\partial t}$). The constitutive equations in a cross-anisotropic medium are written as below:

$$\begin{cases} \sigma_x = C_{11}\frac{\partial u_x}{\partial x} + C_{12}\frac{\partial u_y}{\partial y} + C_{13}\frac{\partial u_z}{\partial z} \\ \sigma_y = C_{12}\frac{\partial u_x}{\partial x} + C_{11}\frac{\partial u_y}{\partial y} + C_{13}\frac{\partial u_z}{\partial z} \\ \sigma_z = C_{13}\frac{\partial u_x}{\partial x} + C_{13}\frac{\partial u_y}{\partial y} + C_{33}\frac{\partial u_z}{\partial z} \end{cases} \qquad \begin{cases} \sigma_{xy} = \left(\frac{C_{11} - C_{12}}{2}\right)\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right) \\ \sigma_{xz} = C_{44}\left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}\right) \\ \sigma_{yz} = C_{44}\left(\frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y}\right) \end{cases} \tag{2.2}$$

σ_x, σ_y, σ_z are the normal components of stress tensor and σ_{xy}, σ_{xz}, σ_{yz} are the shear components of it. Also u_x, u_y, u_z are the components of displacements in direction x, y and z receptively. The relation between coefficients C_{ij} and elastic parameters is written as following:

$$\begin{cases} C_{11} = E_h(1 - \nu_{hv}\nu_{vh})/\Delta \\ C_{12} = E_h(\nu_{hv}\nu_{vh} + \nu_{hh})/\Delta \\ C_{13} = E_h(1 + \nu_{hh})\nu_{vh}/\Delta \end{cases} \qquad \begin{cases} C_{33} = E_v(1 - \nu_{hh}^2)/\Delta \\ C_{44} = G_v \end{cases} \tag{2.3}$$

E_v and E_h and are the Young's modulus in the vertical and any horizontal direction. ν_{hh} and ν_{vh} is Poisson's ratio as the corresponding operators of lateral expansion due to horizontal direct stress in a horizontal direction and due to horizontal direct stress in a vertical direction, respectively. G_v is modulus of shear deformation in a vertical plane. The other parameters are defined as below:

$$\frac{E_h}{E_v} = \frac{\nu_{hv}}{\nu_{vh}} \qquad G_h = \frac{E_h}{2(1 + \nu_{hh})} \qquad \Delta = (1 + \nu_{hh})(1 - \nu_{hh} - 2\nu_{vh}\nu_{hv}) \tag{2.4}$$

From Eqns. 2.1 and 2.2 the equilibrium equations in the absence of body forces in term of displacement vector components are derived as

$$\begin{cases} C_{11}\dfrac{\partial^2 u_x}{\partial x^2}+\dfrac{(C_{11}-C_{12})}{2}\dfrac{\partial^2 u_x}{\partial y^2}+C_{44}\dfrac{\partial^2 u_x}{\partial z^2}+\left(\dfrac{C_{11}+C_{12}}{2}\right)\dfrac{\partial^2 u_y}{\partial x\partial y}+(C_{13}+C_{44})\dfrac{\partial^2 u_z}{\partial x\partial z}=\rho\ddot{u}_x \\ \dfrac{(C_{11}-C_{12})}{2}\dfrac{\partial^2 u_y}{\partial x^2}+C_{11}\dfrac{\partial^2 u_y}{\partial y^2}+C_{44}\dfrac{\partial^2 u_y}{\partial z^2}+\left(\dfrac{C_{11}+C_{12}}{2}\right)\dfrac{\partial^2 u_x}{\partial x\partial y}+(C_{13}+C_{44})\dfrac{\partial^2 u_z}{\partial y\partial z}=\rho\ddot{u}_y \\ C_{44}\dfrac{\partial^2 u_z}{\partial x^2}+C_{44}\dfrac{\partial^2 u_z}{\partial y^2}+C_{33}\dfrac{\partial^2 u_z}{\partial z^2}+(C_{13}+C_{44})\dfrac{\partial^2 u_x}{\partial x\partial z}+(C_{13}+C_{44})\dfrac{\partial^2 u_y}{\partial y\partial z}=\rho\ddot{u}_z \end{cases} \tag{2.5}$$

3. POTENTIAL FUNCTIONS

The relation between displacements and potential functions (Φ, Ψ, X) is defined as below:

$$\vec{u}=\nabla\Phi+\nabla\times\left(X\vec{e}_z\right)+a\nabla\times\nabla\times\left(\Psi\vec{e}_z\right) \tag{3.1}$$

in which "a" is the radius of the cylindrical cavity. In the Cartesian coordinates:

$$u_x=\frac{\partial\Phi}{\partial x}+\frac{\partial X}{\partial y}+a\frac{\partial^2\Psi}{\partial x\partial z} \qquad u_y=\frac{\partial\Phi}{\partial y}-\frac{\partial X}{\partial x}+a\frac{\partial^2\Psi}{\partial y\partial z} \qquad u_z=\frac{\partial\Phi}{\partial z}-a\nabla^2\Psi \tag{3.2}$$

where u_x, u_y, u_z are the Cartesian components of the u_i. Therefore, Eqn. 2.5 is converted to the two groups of equations ($\Phi=\phi e^{-i\omega t}e^{iK_z z}$, $\Psi=\psi e^{-i\omega t}e^{iK_z z}$, and $X=\chi e^{-i\omega t}e^{iK_z z}$):

$$\begin{cases} C_{11}\nabla^2\phi+(C_{13}+2C_{44})\dfrac{\partial^2\phi}{\partial z^2}+\rho\omega^2\phi+a\dfrac{\partial}{\partial z}\left[(C_{11}-C_{13}-C_{14})\nabla^2\psi+C_{44}\dfrac{\partial^2\psi}{\partial z^2}+\rho\omega^2\psi\right]=0 \\ \dfrac{\partial}{\partial z}\left[(C_{13}+2C_{44})\nabla^2\phi+C_{33}\dfrac{\partial^2\phi}{\partial z^2}+\rho\omega^2\phi\right]-a\nabla^2\left[C_{44}\nabla^2\psi+(C_{33}-C_{13}-C_{44})\dfrac{\partial^2\psi}{\partial z^2}+\rho\omega^2\psi\right]=0 \end{cases} \tag{3.3}$$

$$\left(\frac{C_{11}-C_{12}}{2}\right)\nabla^2\chi+C_{44}\frac{\partial^2\chi}{\partial z^2}+\rho\omega^2\chi=0 \tag{3.4}$$

where ∇^2 is the two-dimensional Laplace operator:

$$\nabla^2=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2} \tag{3.5}$$

and ω is the frequency of the waves.

4. COMPLEX FUNCTIONS

The complex variables are introduced:

$$\zeta = x + iy \qquad \bar{\zeta} = x - iy \qquad \zeta = re^{i\theta} \tag{4.1}$$

Using relations below,

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial \zeta} + \frac{\partial}{\partial \bar{\zeta}} \qquad \frac{\partial}{\partial y} = i\left(\frac{\partial}{\partial \zeta} - \frac{\partial}{\partial \bar{\zeta}}\right) \tag{4.2}$$

$$\frac{\partial^2}{\partial x^2} = \frac{\partial^2}{\partial \zeta^2} + 2\frac{\partial^2}{\partial \zeta \partial \bar{\zeta}} + \frac{\partial^2}{\partial \bar{\zeta}^2} \qquad \frac{\partial^2}{\partial y^2} = -\left(\frac{\partial^2}{\partial \zeta^2} - 2\frac{\partial^2}{\partial \zeta \partial \bar{\zeta}} + \frac{\partial^2}{\partial \bar{\zeta}^2}\right) \tag{4.3}$$

the Laplace operator in the complex plane is obtained as

$$\nabla^2 = 4\frac{\partial^2}{\partial \zeta \partial \bar{\zeta}} \tag{4.4}$$

Also, the potential functions are considered as below:

$$\phi = \sum_{n=-\infty}^{+\infty} a_n H_n^{(1)}(\delta r)e^{in\theta} \qquad \psi = \sum_{n=-\infty}^{+\infty} b_n H_n^{(1)}(\delta r)e^{in\theta} \qquad \chi = \sum_{n=-\infty}^{+\infty} c_n H_n^{(1)}(\beta r)e^{in\theta} \tag{4.5}$$

$H_n^{(1)}(\ell r)$ is the Hankel function of first kind and order n:

$$H_n^{(1)}(\ell r) = J_n(\ell r) + iY_n(\ell r) \tag{4.6}$$

where $J_n(\ell r)$ and $Y_n(\ell r)$ are the Bessel functions of first and second kind respectively and of order n. This type of the Hankel function satisfies the radiation boundary condition (Sommerfield condition). Also a_n, b_n are the unknown coefficients. Based on Eqns. 4.4, 4.5 and relations below,

$$\frac{\partial}{\partial \zeta}\left[H_n^{(1)}(\ell r)e^{in\theta}\right] = \frac{\ell}{2}H_{n-1}^{(1)}(\ell r)e^{i(n-1)\theta} \tag{4.7}$$

$$\frac{\partial}{\partial \bar{\zeta}}\left[H_n^{(1)}(\ell r)e^{in\theta}\right] = -\frac{\ell}{2}H_{n+1}^{(1)}(\ell r)e^{i(n+1)\theta} \tag{4.8}$$

Eqn. 3.3 is converted to an eigenvalue problem as

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{Bmatrix} \phi \\ \psi \end{Bmatrix} = 0 \tag{4.9}$$

The coefficients a_{ij} are

$$a_{11} = C_{11}\delta^2 + (C_{13} + 2C_{44})K_z^2 - \rho\omega^2$$
$$a_{12} = aiK_z\left[(C_{11} - C_{13} - C_{44})\delta^2 + C_{44}K_z^2 - \rho\omega^2\right] \tag{4.10-1}$$
$$a_{21} = iK_z\left[(C_{13} + 2C_{44})\delta^2 + C_{33}K_z^2 - \rho\omega^2\right]$$
$$a_{22} = a\delta^2\left[C_{44}\delta^2 + (C_{33} - C_{13} - C_{44})K_z^2 - \rho\omega^2\right] \tag{4.10-2}$$

Solving this eigenvalue problem, two values for δ will be found. (δ_1 for quasi-P wave (qP), δ_2 for quasi-SV wave (qSV)). Also from Eqn. 3.4, the wave number β for SH wave is achieved directly as

$$\beta = \left[2\frac{\rho\omega^2 - C_{44}K_z^2}{(C_{11} - C_{12})}\right]^{1/2} \tag{4.11}$$

Finally, the potential functions are written as below:

$$\begin{cases} \phi = \sum_{n=-\infty}^{+\infty} X_n H_n^{(1)}(\delta_1 r)e^{in\theta} + q_2 \sum_{n=-\infty}^{+\infty} Y_n H_n^{(1)}(\delta_2 r)e^{in\theta} \\ \psi = q_1 \sum_{n=-\infty}^{+\infty} X_n H_n^{(1)}(\delta_1 r)e^{in\theta} + \sum_{n=-\infty}^{+\infty} Y_n H_n^{(1)}(\delta_2 r)e^{in\theta} \end{cases} \tag{4.12}$$

$$\chi = \sum_{n=-\infty}^{+\infty} Z_n H_n^{(1)}(\beta r)e^{in\theta} \tag{4.13}$$

in which:

$$q_1 = -a_{11}/a_{12} = -a_{21}/a_{22} \qquad \text{for} \quad \delta = \delta_1 \tag{4.14}$$
$$q_2 = -a_{12}/a_{11} = -a_{22}/a_{21} \qquad \text{for} \quad \delta = \delta_2 \tag{4.15}$$

The unknown coefficients, X_n, Y_n and Z_n are calculated from boundary conditions of the problem. Now, knowing the potential functions, displacements and stresses and can be calculated from the known potential functions and then the boundary value problem can be solved.

5. DISPLACEMENT ANS STRESS FIELDS

Using Eqn. 3.2, the displacement field is:

$$\begin{cases} u_r + iu_\theta = -\delta_1(1 + aiK_z q_1)\sum_{n=-\infty}^{+\infty} X_n H_{n+1}^{(1)}(\delta_1 r)e^{in\theta} - \delta_2(q_2 + aiK_z)\sum_{n=-\infty}^{+\infty} Y_n H_{n+1}^{(1)}(\delta_2 r)e^{in\theta} + i\beta\sum_{n=-\infty}^{+\infty} Z_n H_{n+1}^{(1)}(\beta r)e^{in\theta} \\ u_r - iu_\theta = \delta_1(1 + aiK_z q_1)\sum_{n=-\infty}^{+\infty} X_n H_{n-1}^{(1)}(\delta_1 r)e^{in\theta} + \delta_2(q_2 + aiK_z)\sum_{n=-\infty}^{+\infty} Y_n H_{n-1}^{(1)}(\delta_2 r)e^{in\theta} + i\beta\sum_{n=-\infty}^{+\infty} W_n H_{n-1}^{(1)}(\beta r)e^{in\theta} \\ u_z = (iK_z + aq_1\delta_1^2)\sum_{n=-\infty}^{+\infty} X_n H_n^{(1)}(\delta_1 r)e^{in\theta} + (iK_z q_2 + a\delta_2^2)\sum_{n=-\infty}^{+\infty} Y_n H_{n+1}^{(1)}(\delta_2 r)e^{in\theta} \end{cases} \tag{5.1}$$

u_r, u_θ, u_z are the radial, tangential and normal components of displacement vector respectively.
For calculating the stresses, Eqn. 2.2 is written as below:

$$\begin{cases} \sigma_r+\sigma_\theta=(C_{11}+C_{12})\left(\dfrac{\partial u_x}{\partial x}+\dfrac{\partial u_y}{\partial y}\right)+2C_{13}\dfrac{\partial u_z}{\partial z} \\ \sigma_\theta-\sigma_r+2i\sigma_{r\theta}=-(C_{11}-C_{12})\left(\dfrac{\partial}{\partial x}-i\dfrac{\partial}{\partial y}\right)(u_x-iu_y)e^{2i\theta} \\ \sigma_{rz}=C_{44}\left[\left(\dfrac{\partial u_x}{\partial z}+\dfrac{\partial u_z}{\partial x}\right)\cos\theta+\left(\dfrac{\partial u_y}{\partial z}+\dfrac{\partial u_z}{\partial y}\right)\sin\theta\right] \end{cases} \tag{5.2}$$

where σ_r, σ_θ are the radial and hoop components of stress tensor and $\sigma_{r\theta}$, σ_{rz} are the shear components of it. These relations in complex plane and in term of potential function are

$$\begin{cases} \sigma_r-i\sigma_{r\theta}=\dfrac{(C_{11}+C_{12})}{2}\nabla^2\phi+C_{13}\dfrac{\partial^2\phi}{\partial z^2}+\dfrac{a}{2}(C_{11}+C_{12}-2C_{13})\dfrac{\partial}{\partial z}\nabla^2\psi+2(C_{11}-C_{12})\dfrac{\partial^2}{\partial\zeta^2}\left(\phi+i\chi+a\dfrac{\partial\psi}{\partial z}\right)e^{2i\theta} \\ \sigma_r+i\sigma_{r\theta}=\dfrac{(C_{11}+C_{12})}{2}\nabla^2\phi+C_{13}\dfrac{\partial^2\phi}{\partial z^2}+\dfrac{a}{2}(C_{11}+C_{12}-2C_{13})\dfrac{\partial}{\partial z}\nabla^2\psi+2(C_{11}-C_{12})\dfrac{\partial^2}{\partial\bar{\zeta}^2}\left(\phi-i\chi+a\dfrac{\partial\psi}{\partial z}\right)e^{-2i\theta} \\ \sigma_{rz}=C_{44}\left[\left(e^{i\theta}\dfrac{\partial}{\partial\zeta}+e^{-i\theta}\dfrac{\partial}{\partial\bar{\zeta}}\right)\left(2\dfrac{\partial\phi^s}{\partial z}+a\dfrac{\partial^2\psi}{\partial z^2}-a\nabla^2\psi\right)+i\left(e^{i\theta}\dfrac{\partial}{\partial\zeta}-e^{-i\theta}\dfrac{\partial}{\partial\bar{\zeta}}\right)\dfrac{\partial\chi}{\partial z}\right] \end{cases} \tag{5.3}$$

(The second and third relations have been constructed from subtracting the second and third relations of Eqn. 5.2.). Finally, Eqn. 5.3 is written in term of the Hankel functions as below:

$$\begin{aligned} \sigma_r-i\sigma_{r\theta}=&\left[-\frac{(C_{11}+C_{12})}{2}\delta_1^2-C_{13}K_z^2-\frac{a}{2}iK_zq_1\delta_1^2(C_{11}+C_{12}-2C_{13})\right]\sum_{n=-\infty}^{+\infty}X_nH_n^{(1)}(\delta_1 r)e^{in\theta} \\ &+\frac{(C_{11}-C_{12})}{2}\delta_1^2(1+aiK_zq_1)\sum_{n=-\infty}^{+\infty}X_nH_{n-2}^{(1)}(\delta_1 r)e^{in\theta} \\ &+\left[-\frac{(C_{11}+C_{12})}{2}q_2\delta_2^2-C_{13}q_2K_z^2-\frac{a}{2}iK_z\delta_2^2(C_{11}+C_{12}-2C_{13})\right]\sum_{n=-\infty}^{+\infty}Y_nH_n^{(1)}(\delta_2 r)e^{in\theta} \\ &+\frac{(C_{11}-C_{12})}{2}\delta_2^2(q_2+aiK_z)\sum_{n=-\infty}^{+\infty}Y_nH_{n-2}^{(1)}(\delta_2 r)e^{in\theta}+i\frac{(C_{11}-C_{12})}{2}\beta^2\sum_{n=-\infty}^{+\infty}Z_nH_{n-2}^{(1)}(\beta r)e^{in\theta} \end{aligned} \tag{5.4-1}$$

$$\begin{aligned} \sigma_r+i\sigma_{r\theta}=&\left[-\frac{(C_{11}+C_{12})}{2}\delta_1^2-C_{13}K_z^2-\frac{a}{2}iK_zq_1\delta_1^2(C_{11}+C_{12}-2C_{13})\right]\sum_{n=-\infty}^{+\infty}X_nH_n^{(1)}(\delta_1 r)e^{in\theta} \\ &+\frac{(C_{11}-C_{12})}{2}\delta_1^2(1+aiK_zq_1)\sum_{n=-\infty}^{+\infty}X_nH_{n+2}^{(1)}(\delta_1 r)e^{in\theta} \\ &+\left[-\frac{(C_{11}+C_{12})}{2}q_2\delta_2^2-C_{13}q_2K_z^2-\frac{a}{2}iK_z\delta_2^2(C_{11}+C_{12}-2C_{13})\right]\sum_{n=-\infty}^{+\infty}Y_nH_n^{(1)}(\delta_2 r)e^{in\theta} \\ &+\frac{(C_{11}-C_{12})}{2}\delta_2^2(q_2+aiK_z)\sum_{n=-\infty}^{+\infty}Y_nH_{n+2}^{(1)}(\delta_2 r)e^{in\theta}-i\frac{(C_{11}-C_{12})}{2}\beta^2\sum_{n=-\infty}^{+\infty}Z_nH_{n+2}^{(1)}(\beta r)e^{in\theta} \end{aligned} \tag{5.4-2}$$

$$\begin{aligned}\sigma_{rz} &= C_{44}\delta_1\left[iK_z - \frac{a}{2}K_z^2 q_1 + \frac{a}{2}q_1\delta_1^2\right]\sum_{n=-\infty}^{+\infty} X_n\left[H_{n-1}^{(1)}(\delta_1 r) - H_{n+1}^{(1)}(\delta_1 r)\right]e^{in\theta}\\ &+C_{44}\delta_2\left[iK_z q_2 - \frac{a}{2}K_z^2 + \frac{a}{2}\delta_2^2\right]\sum_{n=-\infty}^{+\infty} Y_n\left[H_{n-1}^{(1)}(\delta_2 r) - H_{n+1}^{(1)}(\delta_2 r)\right]e^{in\theta}\\ &-C_{44}\frac{K_z\beta}{2}\sum_{n=-\infty}^{+\infty} Z_n\left[H_{n-1}^{(1)}(\beta r) + H_{n+1}^{(1)}(\beta r)\right]e^{in\theta}\end{aligned} \tag{5.4-3}$$

6. BOUNDARY VALUE PROBLEM

Now consider a cylindrical cavity in an infinite medium is subjected to a harmonic incident wave. The radius of cavity is “a” and on its surface the stress is vanished. Theses boundary conditions should be satisfied at the cavity surface:

$$\begin{cases}\sigma_r - i\sigma_{r\theta} = 0\\ \sigma_r + i\sigma_{r\theta} = 0 \qquad \text{at} \quad r = a\\ \sigma_{rz} = 0\end{cases} \tag{6.1}$$

All of the wave variables are the sum of incident and scattering components, therefore the potential functions are written as

$$\phi = \phi^{in} + \phi^{sc} \qquad \psi = \psi^{in} + \psi^{sc} \qquad \chi = \chi^{in} + \chi^{sc} \tag{6.2}$$

where ϕ^{in}, ψ^{in}, χ^{in} are the incident and ϕ^{sc}, ψ^{sc}, χ^{sc} are the scattered components of the potential functions. Therefore the boundary conditions at the cavity surface are

$$\begin{cases}(\sigma_r - i\sigma_{r\theta})^{in} + (\sigma_r - i\sigma_{r\theta})^{sc} = 0\\ (\sigma_r + i\sigma_{r\theta})^{in} + (\sigma_r + i\sigma_{r\theta})^{sc} = 0 \qquad \text{at} \quad r = a\\ \sigma_{rz}^{in} + \sigma_{rz}^{sc} = 0\end{cases} \tag{6.3}$$

Incident components of the stresses are derived from incident potential functions using Eqn. 5.3. Also scattering component of these variables is found in Eqns. 5.4-1 to 5.4-3. In the case of dilatational incident wave (P wave), the incident potential function is written as

$$\phi^{in} = \phi_0 e^{iK_x x} e^{iK_z z} \tag{6.4}$$

ϕ_0 is a coefficient and

$$K_x = K\cos\gamma \qquad K_z = K\sin\gamma \tag{6.5}$$

K is the wave number and is related to the wave frequency as following:

$$\omega = K\left(\frac{C_{11}}{\rho}\right)^{1/2} \tag{6.6}$$

Also γ is the angle of incident wave with axis of cylinder. The incident potential function can be written as

$$\phi^{in} = \phi_0 \sum_{n=-\infty}^{+\infty} i^n J_n(K_x r) e^{in\theta} e^{iK_z z} \tag{6.7}$$

Consequently, the incident stresses are found:

$$\begin{cases} (\sigma_r - i\sigma_{r\theta})^{in} = -\phi_0 \left[\frac{(C_{11}+C_{12})}{2} K_x^2 + C_{13} K_z^2 \right] \sum_{n=-\infty}^{+\infty} i^n J_n(K_x r) e^{in\theta} \\ \qquad + \phi_0 \frac{(C_{11}-C_{12})}{2} K_x^2 \sum_{n=-\infty}^{+\infty} i^n J_{n-2}(K_x r) e^{in\theta} \\ (\sigma_r + i\sigma_{r\theta})^{in} = -\phi_0 \left[\frac{(C_{11}+C_{12})}{2} K_x^2 + C_{13} K_z^2 \right] \sum_{n=-\infty}^{+\infty} i^n J_n(K_x r) e^{in\theta} \\ \qquad + \phi_0 \frac{(C_{11}-C_{12})}{2} K_x^2 \sum_{n=-\infty}^{+\infty} i^n J_{n+2}(K_x r) e^{in\theta} \\ \sigma_{rz}^{in} = \phi_0 i C_{44} K_x K_z \sum_{n=-\infty}^{+\infty} i^n \left[J_{n-1}(K_x r) - J_{n+1}(K_x r) \right] e^{in\theta} \end{cases} \tag{6.8}$$

Finally, the boundary value problem (Eqn. 6.3) is written as an algebraic equation:

$$\begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{bmatrix} \begin{Bmatrix} X_n \\ Y_n \\ Z_n \end{Bmatrix} = \begin{Bmatrix} n_1 \\ n_2 \\ n_3 \end{Bmatrix} \tag{6.9}$$

The coefficients m_{ij} are defined as

$$\begin{aligned} m_{11} &= \left[-\frac{(C_{11}+C_{12})}{2} \delta_1^2 - C_{13} K_z^2 - \frac{a}{2} i K_z q_1 \delta_1^2 (C_{11} + C_{12} - 2C_{13}) \right] H_n^{(1)}(\delta_1 a) \\ &+ \frac{(C_{11}-C_{12})}{2} \delta_1^2 (1 + aiK_z q_1) H_{n-2}^{(1)}(\delta_1 a) \\ m_{12} &= \left[-\frac{(C_{11}+C_{12})}{2} q_2 \delta_2^2 - C_{13} q_2 K_z^2 - \frac{a}{2} i K_z \delta_2^2 (C_{11} + C_{12} - 2C_{13}) \right] H_n^{(1)}(\delta_2 a) \\ &+ \frac{(C_{11}-C_{12})}{2} \delta_2^2 (q_2 + aiK_z) H_{n-2}^{(1)}(\delta_2 a) \\ m_{13} &= i \frac{(C_{11}-C_{12})}{2} \beta^2 H_{n-2}^{(1)}(\beta a) \end{aligned} \tag{6.10-1}$$

$$m_{21} = \left[-\frac{(C_{11}+C_{12})}{2}\delta_1^2 - C_{13}K_z^2 - \frac{a}{2} iK_z q_1 \delta_1^2 (C_{11}+C_{12}-2C_{13}) \right] H_n^{(1)}(\delta_1 a) + \frac{(C_{11}-C_{12})}{2}\delta_1^2 (1+aiK_z q_1) H_{n+2}^{(1)}(\delta_1 a)$$

$$m_{22} = \left[-\frac{(C_{11}+C_{12})}{2} q_2\delta_2^2 - C_{13}q_2K_z^2 - \frac{a}{2} iK_z \delta_2^2 (C_{11}+C_{12}-2C_{13}) \right] H_n^{(1)}(\delta_2 a) + \frac{(C_{11}-C_{12})}{2}\delta_2^2 (q_2+aiK_z) H_{n+2}^{(1)}(\delta_2 a) \tag{6.10-2}$$

$$m_{23} = -i\frac{(C_{11}-C_{12})}{2}\beta^2 H_{n+2}^{(1)}(\beta a)$$

$$m_{31} = C_{44}\delta_1 \left[iK_z - \frac{a}{2}K_z^2 q_1 + \frac{a}{2} q_1 \delta_1^2 \right] \left[H_{n-1}^{(1)}(\delta_1 a) - H_{n+1}^{(1)}(\delta_1 a) \right]$$
$$m_{32} = C_{44}\delta_2 \left[iK_z q_2 - \frac{a}{2}K_z^2 + \frac{a}{2}\delta_2^2 \right] \left[H_{n-1}^{(1)}(\delta_2 a) - H_{n+1}^{(1)}(\delta_2 a) \right] \tag{6.10-3}$$
$$m_{33} = -C_{44}\frac{K_z\beta}{2}\left[H_{n-1}^{(1)}(\beta a) + H_{n+1}^{(1)}(\beta a) \right]$$

Also, the coefficients n_i are written as:

$$\begin{cases} n_1 = \phi_0 \left[\frac{(C_{11}+C_{12})}{2}K_x^2 + C_{13}K_z^2 \right] i^n J_n(K_x a) - \phi_0 \frac{(C_{11}-C_{12})}{2}K_x^2 i^n J_{n-2}(K_x a) \\ n_2 = \phi_0 \left[\frac{(C_{11}+C_{12})}{2}K_x^2 + C_{13}K_z^2 \right] i^n J_n(K_x a) - \phi_0 \frac{(C_{11}-C_{12})}{2}K_x^2 i^n J_{n+2}(K_x a) \\ n_3 = -\phi_0 i C_{44} i^n K_x K_z \left[J_{n-1}(K_x a) - J_{n+1}(K_x a) \right] \end{cases} \tag{6.11}$$

7. NUMERICAL RESULTS

Figure 1 represents the cylinder in an infinite cross-anisotropic medium. The cylinder is subjected to the P incident wave and its surface is free of stress. In the following examples, the parameters given in the Table 1 are assumed.

Table 1 Parameters used in applications

ρ (kg/m^3)	a (m)	ϕ_0	E_v (GPa)
2400	1	1	20

To investigate the effects of different parameters, a parametric study is performed for the same geometry and with the above-mentioned parameters. In this study, the cavity is subjected to the P incident wave. Figure 2 gives the ratio of hoop stress to the parameter K^2 with respect to the ratio of E_h/E_v and dimensionless wave number, Ka ($\gamma = 0^\circ$, r=a, $\theta = \pi/2$). It is seen that with increasing the ratio of E_h/E_v, the hoop stress increases. Figure 3 represents the value of hoop stress with respect to ν_{hh} and dimensionless wave number ($\gamma = 0^\circ$, r=a, $\theta = \pi/2$). It is seen the influence of this parameter (ν_{hh}) on the hoop stress is dominant near the static case and with increasing the wave number this effect vanished. Figure 4 shows the ratio of radial stress to the parameter K^2 with respect to the ratio of E_h/E_v and radial distance from the cylinder ($\gamma = 0^\circ$, $Ka = 1$, $\theta = \pi$). Also Figure 5 represents the value of radial stress with respect to ν_{hh} and radial distance from the cylinder ($\gamma = 0^\circ$, $Ka = 1$, $\theta = \pi$). As it can be seen, with increasing the ratio of E_h/E_v and values of ν_{hh} the radial stress increases, but the effect of ratio of the parameter ν_{hh} is very small in compare with the ratio of E_h/E_v.

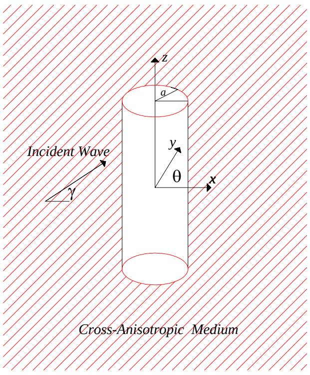

Figure 1 Cylinder in an infinite cross-anisotropic medium subjected to the harmonic wave

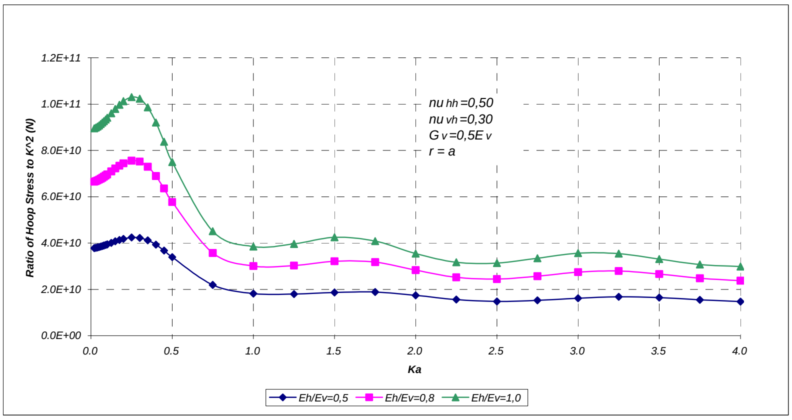

Figure 2 Ratio of hoop stress to the parameter K^2 with respect to the ratio of E_h/E_v and dimensionless wave number ($\theta = \pi/2$)

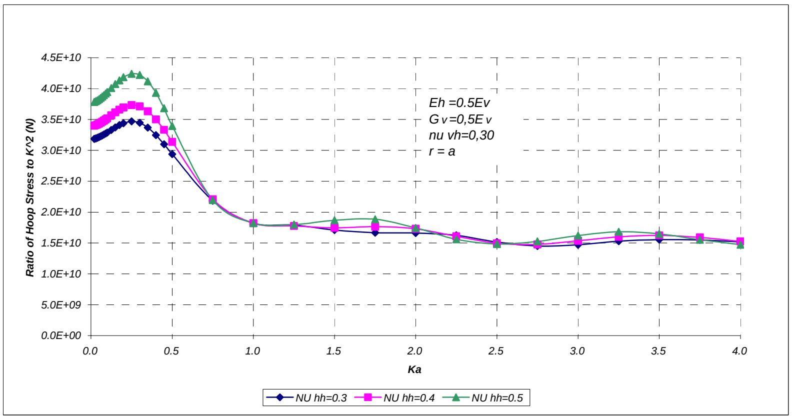

Figure 3 Ratio of hoop stress to the parameter K^2 with respect to ν_{hh} and dimensionless wave number ($\theta = \pi/2$)

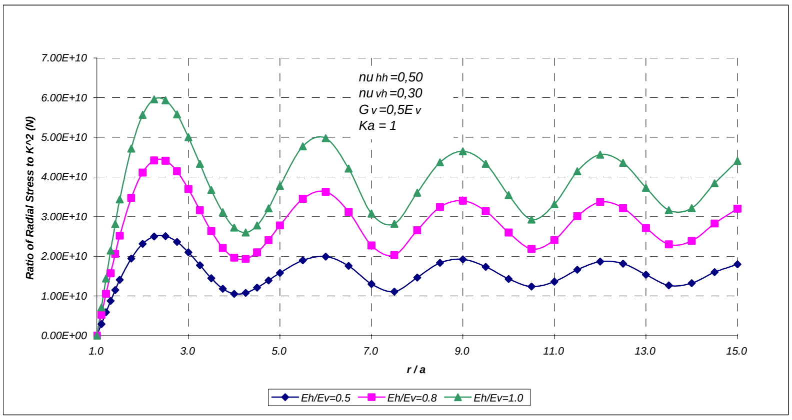

Figure 4 Ratio of radial stress to the parameter K^2 with respect to the ratio of E_h / E_v and radial distance from the cylinder ($\theta = \pi$)

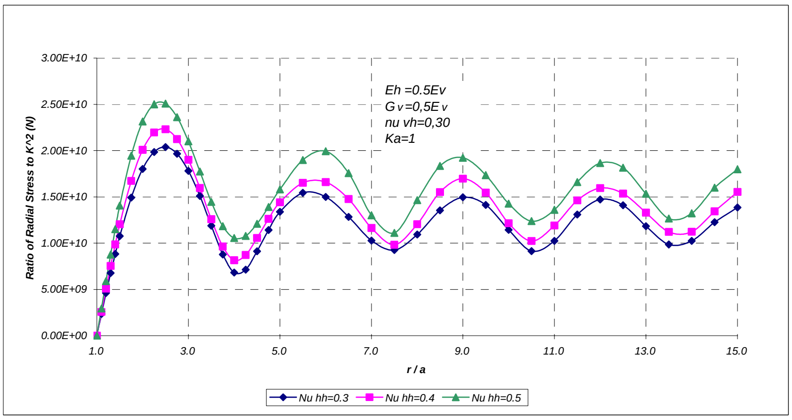

Figure 5 Ratio of radial stress to the parameter K^2 with respect to ν_{hh} and radial distance from the cylinder ($\theta = \pi$)

8. CONCLUSIONS

An analytical method for solving the problem of wave scattering in a cross-anisotropic medium was presented. This method includes the solution of wave equations by means of complex functions theory. Using potential functions, the solution is obtained in series of the Hankel functions by means of complex functions, calculated from boundary conditions of the problem. Many numerical results including stresses and displacement were obtained for a cylinder in a cross- anisotropic medium subjected to the harmonic waves. Also the effect of many parameters such as E_h/E_v, G_v/E_v and ν_{hh} were studied. It was found that the stress field is much more sensitive to the anisotropy in compare with displacement field. Also the ratios of E_h/E_v and G_v/E_v is more effective on the results in compare with the parameter ν_{hh}.

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