A COMPLEX MODAL SUPERPOSITION METHOD FOR THE SEISMIC ANALYSIS OF STRUCTURES WITH SUPPLEMENTAL DAMPERS

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ABSTRACT:

A response spectrum method for the seismic analysis of structures with added dampers is herein introduced. The method is based on a modal decomposition of the equations of motion and the derivation of relationships between spectral accelerations, velocities, and displacements valid for high damping ratios. It involves the calculation of the complex natural frequencies, mode shapes, and participation factors of the system and the use of an acceleration response spectrum. It differs from similar methods proposed in the past in that it properly accounts for the high damping ratios observed in structures with added dampers and employs unique formulations to determine the peak relative velocities and absolute accelerations of the system. A numerical example is provided to illustrate the application of the procedure and compare the results attained with it and a time-history analysis. Through this example, it is shown that the proposed method is simple to use and leads to results that are close to the results from a time-history analysis.

KEYWORDS: Response spectrum method**,** earthquake response analysis**,** passive structural control

1. INTRODUCTION

The use of damping devices to improve the seismic performance of structures is nowadays widely accepted by the engineering profession and is becoming thus a common occurrence. Although beneficial, the addition of damping devices to a structure may nonetheless change significantly its original properties, particularly when the damping introduced by the added dampers is much higher than the damping in the structure and the added damping is not evenly distributed throughout the structure. Examples of these changes are: (1) a significant increase in the values of the modal damping ratios of the structure; (2) a change in the relative values of these modal damping ratios among the various vibration modes, such that some modes may contribute little to the total response before the dampers are added but may contribute substantially afterwards; and (3) mode shapes and natural frequencies that deviate appreciably from the mode shapes and natural frequencies of the undamped structure. Such changes, in turn, render the conventional response spectrum method and other approximate methods of analysis inadequate. The reason is that it can no longer be assumed that the system is classically damped. That is, it can no longer be assumed that pre-multiplication and post-multiplication by the undamped mode shapes may transform the damping matrix of the system into a diagonal matrix since in such a case the off-diagonal elements of the transformed damping matrix are of the same order of magnitude as the diagonal elements. Hence, these off-diagonal elements cannot be considered negligibly small. It is always possible to perform a reliable analysis through the step-by-step integration of the equations of motion, but this approach becomes cumbersome when the seismic input is specified in the form of a design spectrum or during the preliminary analyses carried out to select the size and location of the dampers.

With the purpose of facilitating the analysis of structures with added dampers, this paper introduces a response spectrum method that effectively accounts for the aforementioned changes and is thus adequate for the seismic analysis of structures with added dampers. The method is based on a complex modal decomposition of the equations of motion and the derivation of relationships between spectral accelerations, velocities, and displacements valid for high damping ratios. It involves the calculation of the complex natural frequencies, mode shapes, and participation factors of the system and the use of an acceleration response spectrum. It is intended for structures that are designed to remain in their linear range of behavior at all times and viscous dampers with a force-velocity relationship that is also approximately linear. It differs from other response spectrum methods proposed in the past for the analysis of nonclassically damped systems (i.e., Singh, 1980, Villaverde, 1980, Gupta and Jaw, 1986) in that it properly accounts for the high damping ratios encountered in structures with added dampers and employs unique formulations to compute the peak relative velocities and absolute accelerations in a structure.

2. DERIVATION

Consider a linearly damped linear structure subjected to a ground acceleration $\ddot{x}_g(t)$. The equation of motion for this system when the structural displacements are expressed relative to the ground may be written as

$$[M]_c\{\ddot{x}(t)\}_c + [C]_c\{\dot{x}(t)\}_c + [K]_c\{x(t)\}_c = -[M]_c\{r\}_c\ddot{x}_g(t) \tag{2.1}$$

where $[M]_c$, $[C]_c$, $[K]_c$, $\{r\}_c$, and $\{x(t)\}_c$ respectively denote the mass matrix, damping matrix, stiffness matrix, influence vector, and displacement vector of the structure-damper system. According to the method suggested by Foss (1952) and described in texts such as Hurty and Rubinstein (1964), a modal decomposition is possible if the equation is first reduced to a first-order equation of the form

$$[A]\{\dot{q}(t)\} + [B]\{q(t)\} = \{Q(t)\} \tag{2.2}$$

where

$$[A] = \begin{bmatrix} [0] & [M]_c \\ [M]_c & [C]_c \end{bmatrix}; \qquad [B] = \begin{bmatrix} -[M]_c & [0] \\ [0] & [K]_c \end{bmatrix} \tag{2.3}$$

and

$$\{q(t)\} = \begin{Bmatrix} \{\dot{x}(t)\}_c \\ \{x(t)\}_c \end{Bmatrix}; \qquad \{Q(t)\} = \begin{Bmatrix} \{0\} \\ -[M]_c\{r\}_c\ddot{x}_g(t) \end{Bmatrix} \tag{2.4}$$

Eqn. 2.2 is of size $2n \times 2n$, with n being the number of degrees of freedom of the structure, and is usually referred to as the reduced equation of motion.

The solution of the homogeneous equation of motion; i.e., the solution of

$$[A]\{\dot{q}(t)\} + [B]\{q(t)\} = \{0\} \tag{2.5}$$

is of the form

$$\{q(t)\} = \{s\}e^{\lambda t} \tag{2.6}$$

and, thus, substitution of this solution into Eqn. 2.5 leads to the eigenvalue problem

$$([B] + \lambda[A])\{s\} = \{0\} \tag{2.7}$$

The solution of this eigenvalue problem, in turn, leads to a set of $2n$ complex-valued eigenvalues λ_i and a set of $2n$ complex-valued eigenvectors $\{s\}_i$. It can be shown that when the system is underdamped, these eigenvalues and eigenvectors result in pairs of complex conjugates (Inman and Andry, 1980). It can also be shown that the resulting eigenvectors are orthogonal with respect to the matrices $[A]$ and $[B]$. That is,

$$\{s\}_i^T[A]\{s\}_j = 0, \quad i \neq j \qquad \{s\}_i^T[B]\{s\}_j = 0, \quad i \neq j \tag{2.8}$$

Furthermore, it can be shown that the eigenvalues λ_i and eigenvectors $\{s\}_i$ may be written alternatively as

$$\lambda_i = -\xi_i\omega_i + j\omega_i\sqrt{1-\xi_i^2} = -\xi_i\omega_i + j\omega_{di} \qquad \{s\}_i = \begin{Bmatrix} \lambda_i\{w\}_i \\ \{w\}_i \end{Bmatrix} \tag{2.9}$$

where ω_i, ω_{di}, and ξ_i respectively represent the natural frequency, damped natural frequency, and damping ratio in the ith mode of the system, j denotes the unit imaginary number, and $\{w\}_i$ is a complex-valued mode shape of size n that defines the relative amplitudes and phase angles of the various masses of the system when it vibrates freely in its ith mode.

Because of the orthogonality of the eigenvectors $\{s\}_i$ with respect to the matrices $[A]$ and $[B]$, the matrix that contains all such eigenvectors represents a transformation matrix that decouples the reduced equation of motion. Accordingly, if $[s]$ represents the matrix that contains the $2n$ eigenvectors of the system, and if $\{z(t)\}$ is a vector of unknown modal coordinates, under the transformation

$$\{q(t)\} = [s]\{z(t)\} = \sum_{i=1}^{2n}\{s\}_i z_i(t) \tag{2.10}$$

and after premultiplication by the transpose of $[s]$ and the use of the aforementioned orthogonality properties, Eqn. 2.2 may be transformed into the set of independent equations

$$A_i\dot{z}_i(t) + B_i z_i(t) = Q_i(t), \quad i = 1, 2, \cdots, 2n \tag{2.11}$$

where $z_i(t)$ is the ith element of $\{z(t)\}$ and A_i, B_i, and $Q_i(t)$ are complex-valued scalars defined by

$$A_i = \{s\}_i^T [A]\{s\}_i \qquad B_i = \{s\}_i^T [B]\{s\}_i \qquad Q_i(t) = \{s\}_i^T \{Q(t)\}_i \tag{2.12}$$

Eqn. 2.11 is a first-order differential equation with constant coefficients. As such, its solution may be obtained in terms of Duhamel's integral. Accordingly and under the assumption of zero initial conditions, $z_i(t)$ may be expressed as

$$z_i(t) = (1/A_i)\int_{\tau=0}^{t} Q_i(\tau)e^{\lambda_i(t-\tau)}d\tau \tag{2.13}$$

Eqns. 2.10 and 2.13 constitute the modal solution of Eqn. 2.2. Eqn. 2.10, however, may be written explicitly in terms of complex conjugates as

$$\{q(t)\} = \sum_{i=1}^{n}\{s\}_i z_i(t) + \sum_{i=1}^{n}\{\bar{s}\}_i \bar{z}_i(t) = 2\sum_{i=1}^{n}\mathrm{Re}[\{s\}_i z_i(t)] \tag{2.14}$$

where a bar above a variable indicates its complex conjugate, and "Re" reads as "the real part of." Furthermore, in view of the first formula in Eqn. 2.4 and the second in Eqn. 2.9, it is possible to express Eqn. 2.14 alternatively as

$$\begin{Bmatrix} \{\dot{x}(t)\}_c \\ \{x(t)\}_c \end{Bmatrix} = 2\sum_{i=1}^{n}\mathrm{Re}\left[\begin{Bmatrix} \lambda_i\{w\}_i \\ \{w\}_i \end{Bmatrix} z_i(t)\right] \tag{2.15}$$

whose lower part leads to the following explicit solution of Eqn. 2.1:

$$\{x(t)\}_c = 2\sum_{i=1}^{n}\mathrm{Re}[\{w\}_i z_i(t)] \tag{2.16}$$

Similarly, the substitution of the second formulas in Eqn. 2.9 and 2.4 into the third formula in Eqn. 2.12 yields

$$Q_i(t) = -\{w\}_i^T [M]_c \{r\}_c \ddot{x}_g(t) \tag{2.17}$$

while the first formula in Eqn. 2.12 in combination with the first formula in Eqn. 2.3 and the second in Eqn. 2.9 leads to

$$A_i = \{w\}_i^T (2\lambda_i [M]_c + [C]_c)\{w\}_i = 2(-\xi_i\omega_i + j\omega_{di})M_{ci} + C_{ci} \tag{2.18}$$

where M_{ci} and C_{ci} are a generalized mass and a generalized damping constant defined by

$$M_{ci} = \{w\}_i^T [M]_c \{w\}_i \qquad C_{ci} = \{w\}_i^T [C]_c \{w\}_i \tag{2.19}$$

Furthermore, if C_{ci} is expressed in terms of the damping ratio and natural frequency in the ith mode of the system as $C_{ci} = 2\xi_i\omega_i M_{ci}$, then A_i may be put into the form

$$A_i = 2(-\xi_i\omega_i + j\omega_{di})M_{ci} + 2\xi_i\omega_i M_{ci} = 2j\omega_{di}M_{ci} \tag{2.20}$$

In the light of Eqns. 2.17 and 2.20, Eqn. 2.13 may be therefore expressed as

$$z_i(t) = \frac{1}{A_i}\int_{\tau=0}^{t} Q_i(\tau)e^{\lambda_i(t-\tau)}d\tau = \frac{\Gamma_{ci}}{2j\omega_{di}}\int_{\tau=0}^{t}\ddot{x}_g(\tau)e^{\lambda_i(t-\tau)}d\tau \tag{2.21}$$

where Γ_{ci} is a complex participation factor defined as

$$\Gamma_{ci} = \{w\}_i^T [M]_c \{r\}_c / \{w\}_i^T [M]_c \{w\}_i \tag{2.22}$$

Correspondingly, Eqn. 2.16 may be written as

$$\{x(t)\}_c = -\sum_{i=1}^{n}\mathrm{Re}[\frac{1}{\omega_{di}}\{w'\}_i \int_{\tau=0}^{t}\ddot{x}_g(\tau)e^{\lambda_i(t-\tau)}d\tau] \tag{2.23}$$

where

$$\{w'\}_i = j\Gamma_{ci}\{w\}_i \tag{2.24}$$

However, if $\{w'\}_i$ and λ_i are expressed explicitly in terms of their real and imaginary parts, if it is considered that $e^{\lambda_i(t-\tau)} = e^{-\xi_i\omega_i}e^{j\omega_{di}(t-\tau)}$, and if the second exponential function is expanded in terms of its sine and cosine components, it may also be written as

$$\{x(t)\}_c = -\sum_{i=1}^{n} \mathrm{Re}\{\frac{1}{\omega_{di}}(\{u'\}_i + j\{v'\}_i)\int_{\tau=0}^{t} \ddot{x}_g(\tau)e^{-\xi_i\omega_i(t-\tau)}[\cos\omega_{di}(t-\tau) + j\sin\omega_{di}(t-\tau)]d\tau\}$$

$$= -\sum_{i=1}^{n}[\frac{1}{\omega_{di}}\{u'\}_i\int_{\tau=0}^{t}\ddot{x}_g(\tau)e^{-\xi_i\omega_i(t-\tau)}\cos\omega_{di}(t-\tau)d\tau - \frac{1}{\omega_{di}}\{v'\}_i\int_{\tau=0}^{t}\ddot{x}_g(\tau)e^{-\xi_i\omega_i(t-\tau)}\sin\omega_{di}(t-\tau)d\tau] \quad (2.25)$$

where $\{u'\}_i$ and $\{v'\}_i$ respectively denote the real and imaginary parts of $\{w'\}_i$. Furthermore, from elementary structural dynamics it is known that the relative displacement $y_i(t)$ at time t of a single-degree-of-freedom system with natural frequency ω_i, damping ratio ξ_i, and damped natural frequency ω_{di}, when the system is subjected to zero initial conditions and a ground acceleration $\ddot{x}_g(t)$, is given by

$$y_i(t) = -(1/\omega_{di})\int_0^t \ddot{x}_g(\tau)e^{-\xi_i\omega_i(t-\tau)}\sin\omega_{di}(t-\tau)d\tau \quad (2.26)$$

Similarly, by taking the first derivative with respect to time of Eqn. 2.26, it is known that the corresponding relative velocity is given by

$$\dot{y}_i(t) = -\int_0^t \ddot{x}_g(\tau)e^{-\xi_i\omega_i(t-\tau)}[\cos\omega_{di}(t-\tau) - (\xi_i/\sqrt{1-\xi_i^2})\sin\omega_{di}(t-\tau)]d\tau \quad (2.27)$$

Therefore, the integrals in Eqn. 2.25 may be expressed as

$$\int_{\tau=0}^{t}\ddot{x}_g(\tau)e^{-\xi_i\omega_i(t-\tau)}\cos\omega_{di}(t-\tau)d\tau = -\dot{y}_i(t) - \xi_i\omega_i y_i(t) \quad (2.28)$$

$$\int_{\tau=0}^{t}\ddot{x}_g(\tau)e^{-\xi_i\omega_i(t-\tau)}\sin\omega_{di}(t-\tau)d\tau = -\omega_{di}y_i(t) \quad (2.29)$$

and Eqn. 2.25 may be written in terms of such displacement and velocity responses as

$$\{x(t)\}_c = \sum_{i=1}^{n}\{x(t)\}_{ci} = \sum_{i=1}^{n}[\{a'\}_i\dot{y}_i(t) - \{b'\}_i y_i(t)] \quad (2.30)$$

where $\{x(t)\}_{ci}$ denotes a vector that contains the displacements of the system in its ith mode, and

$$\{a'\}_i = (1/\omega_{di})\{u'\}_i \qquad \{b'\}_i = \{v'\}_i - (\xi_i/\sqrt{1-\xi_i^2})\{u'\}_i \quad (2.31)$$

The maximum values of the displacement $y_i(t)$ and the velocity $\dot{y}_i(t)$ may be determined from the displacement and velocity response spectra of the ground acceleration $\ddot{x}_g(t)$. However, because these maximum values do not occur at the same time, the maximum values of the modal displacements $\{x(t)\}_{ci}$ cannot be determined directly from such spectra. It is possible, nonetheless, to obtain an approximation using the response spectra in question if the $y_i(t)$ and $\dot{y}_i(t)$ terms in Eqn. 2.30 are combined using the square-root-of-the-sum-of-the-squares (SRSS) rule. That is, the maximum values of the displacements in each of the modes of the system may be approximated as

$$\max\{x(t)\}_{ci} = [\{a'^2\}_i SV_i^2 + \{b'^2\}_i SD_i^2]^{1/2} \quad (2.32)$$

where SV_i and SD_i respectively denote the ordinates corresponding to a natural frequency ω_i and damping ratio ξ_i in the velocity and displacement response spectra of the ground acceleration $\ddot{x}_g(t)$. In turn, such maximum modal responses may be combined using, again, the SRSS rule to obtain an estimate of the maximum displacements.

The relative velocities of the system may be similarly obtained by considering the upper part of Eqn. 2.15. Accordingly, these velocities may be expressed as

$$\{\dot{x}(t)\}_c = 2\sum_{i=1}^{n}\mathrm{Re}[\lambda_i\{w\}_i z_i(t)] \quad (2.33)$$

which in the light of Eqns. 2.24 and 2.21 may also be written as

$$\{\dot{x}(t)\}_c = -\sum_{i=1}^{n}\mathrm{Re}[\lambda_i\{w'\}_i\int_{\tau=0}^{t}\ddot{x}_g(\tau)e^{\lambda_i(t-\tau)}d\tau] \quad (2.34)$$

Thus, if λ_i, $\{w'\}_i$, and $e^{j\omega_{di}(t-\tau)}$ are expanded into their real and imaginary components, one has that

$$\{\dot{x}(t)\}_c = \sum_{i=1}^{n}[\frac{1}{\omega_{di}}(\xi_i\omega_i\{u'\}_i + \omega_{di}\{v'\})\int_{\tau=0}^{t}\ddot{x}_g(\tau)e^{-\xi_i\omega_i(t-\tau)}\cos\omega_{di}(t-\tau)d\tau -$$

$$\frac{1}{\omega_{di}}(\xi_i\omega_i\{v'\}_i - \omega_{di}\{u'\})\int_{\tau=0}^{t}\ddot{x}_g(\tau)e^{-\xi_i\omega_i(t-\tau)}\sin\omega_{di}(t-\tau)d\tau] \tag{2.35}$$

But as in the case of the displacements, the integrals in Eqn. 2.35 may be expressed as indicated by Eqns. 2.28 and 2.29. Therefore, Eqn. 2.35 may also be written as

$$\{\dot{x}(t)\}_c = \sum_{i=1}^{n}\{x(t)\}_{ci} = -\sum_{i=1}^{n}\{p'\}_i\,\dot{y}_i(t) + \{q'\}_i\,y_i(t)\} \tag{2.36}$$

where $\{\dot{x}(t)\}_{ci}$ denotes a vector that contains the relative velocities of the system in its ith mode, and

$$\{p'\}_i = \{v'\}_i + (\xi_i/\sqrt{1-\xi_i^2})\{u'\}_i \qquad \{q'\}_i = (\omega_i/\sqrt{1-\xi_i^2})\{u'\}_i \tag{2.37}$$

As in the case of the displacements, the maximum values of $y_i(t)$ and $\dot{y}_i(t)$ may be obtained from the response spectra of the excitation $\ddot{x}_g(t)$. But also as in the case of the displacements, the maximum values of the modal velocities $\{\dot{x}(t)\}_{ci}$ cannot be obtained using the spectral values. Nonetheless, an approximation may be obtained by combining the two terms in the right-hand side of Eqn. 2.36 using the SRSS rule. That is, it may be considered that the modal velocities are approximately equal to

$$\max\{\dot{x}(t)\}_{ci} = [\{p'^2\}_i SV_i^2 + \{q'^2\}_i SD_i^2]^{1/2} \tag{2.38}$$

In like fashion, an estimate of the maximum velocities may be obtained by combining the maximum modal velocities using the same rule.

In determining the floor accelerations, it should be noted that in the case of high damping ratios the damping forces are not negligibly small in comparison with the elastic forces. Hence, the floor accelerations cannot be considered approximately equal to the elastic forces divided by the corresponding floor masses, as is the practice in the case of conventional structures. To determine the floor accelerations, then, it is necessary to obtain them directly from the equation of motion, i.e., Eqn. 2.1. That is, one needs to consider that

$$\{\ddot{u}(t)\}_c = \{\ddot{x}(t)\}_c + \{r\}_c\ddot{x}_g(t) = -[M]_c^{-1}([C]_c\{\dot{x}(t)\}_c + [K]_c\{x(t)\}_c) \tag{2.39}$$

which in terms of modal responses may also be expressed as

$$\{\ddot{u}(t)\}_c = [M]_c^{-1}\sum_{i=1}^{n}\{F_I(t)\}_{ci} = -[M]_c^{-1}\sum_{i=1}^{n}[\{F_D(t)\}_{ci} + \{F_S(t)\}_{ci}] \tag{2.40}$$

where $\{F_I\}_{ci}$, $\{F_D\}_{ci}$ and $\{F_S\}_{ci}$ are vectors of modal inertia forces, modal damping forces, and modal elastic forces respectively given by

$$\{F_I(t)\}_{ci} = -\{F_D(t)\}_{ci} - \{F_S(t)\}_{ci} \qquad \{F_D(t)\}_{ci} = [C]_c\{\dot{x}(t)\}_{ci} \qquad \{F_S(t)\}_{ci} = [K]_c\{x(t)\}_{ci} \tag{2.41}$$

Note, however, that in view of Eqns. 2.36 and 2.32, the vectors of modal damping forces and modal elastic forces may be expressed as

$$\{F_D(t)\}_{ci} = -[C]_c\{p'\}_i\,\dot{y}_i(t) - [C]_c\{q'\}_i\,y_i(t) \qquad \{F_S(t)\}_{ci} = [K]_c\{a'\}_i\,\dot{y}_i(t) - [K]_c\{b'\}_i\,y_i(t) \tag{2.42}$$

and, in consequence, the vector of modal inertia forces may be written as

$$\{F_I(t)\}_{ci} = ([C]_c\{p'\}_i - [K]_c\{a'\})_i\,\dot{y}_i(t) + ([C]_c\{q'\}_i + [K]_c\{b'\}_i)\,y_i(t) \tag{2.43}$$

As before, the maximum values of $y_i(t)$ and $\dot{y}_i(t)$ may be obtained from the response spectra of the excitation $\ddot{x}_g(t)$ and the maximum values of the modal inertia forces may be estimated by combining the terms associated with $y_i(t)$ and $\dot{y}_i(t)$ using the SRSS rule; i.e., according to

$$\max\{F_I(t)\}_{ci} = [([C]_c\{p'\}_i - [K]_c\{a'\}_i)^2 SV_i^2 + ([C]_c\{q'\}_i + [K]_c\{b'\}_i)^2 SD_i^2]^{1/2} \tag{2.44}$$

As before, too, the maximum values of the inertia forces may be approximately determined by combining the maximum modal values using the SRSS rule. The maximum values of the floor accelerations may be considered approximately equal to

$$\max\{\ddot{u}(t)\}_c = [M]_c^{-1}\max\{F_I(t)\}_c \tag{2.45}$$

It may be noted that Eqns. 2.32, 2.38, and 2.44 are expressed in terms of a spectral displacement and a spectral velocity. Thus, its application requires knowing the velocity and displacement response spectra of the excitation. In many cases, however, the velocity response spectrum is not directly available. To overcome this problem in such cases, it is possible to use the empirical formula proposed by Sadek *et al.* (2000) to estimate spectral velocities is terms of the corresponding pseudovelocities. That is, if PSV_i denotes the pseudovelocity corresponding to the spectral velocity SV_i, it may be considered that

$$SV_i = r_i PSV_i = r_i \omega_i SD_i \tag{2.46}$$

in which $r_i = a_{vi} T_i^{b_{vi}}$ where $a_{vi} = 1.095 + 0.647\xi_i - 0.382\xi_i^2$ $\quad b_{vi} = 0.193 + 0.838\xi_i - 0.621\xi_i^2$ (2.47)

Furthermore, given an acceleration response spectrum, the spectral displacements may be determined from the equation that relates the spectral acceleration SA_i to the spectral displacement SD_i and the spectral velocity SV_i. That is, from the equation

$$SA_i = \omega_i^2 [SD_i^2 + (4\xi_i^2 / \omega_i^2) SV_i^2]^{1/2} \tag{2.48}$$

which after incorporating Eqn. 2.46 becomes

$$SA_i = \omega_i^2 SD_i \sqrt{1 + 4r_i^2 \xi_i^2} \tag{2.49}$$

Eqn. 2.48 is derived by considering first that the absolute acceleration at time t of a single-degree-of-freedom system with natural frequency ω_i, damping ratio ξ_i, and damped natural frequency ω_{di}, which may be determined by taking the first derivative with respect to time of Eqn. 2.27 and adding the ground acceleration $\ddot{x}_g(t)$, is given by

$$\ddot{a}_i(t) = \ddot{y}_i(t) + \ddot{x}_g(t) = \omega_{di} \int_0^t \ddot{x}_g(\tau) e^{-\xi_i \omega_i (t-\tau)} [(1 - \frac{\xi_i^2}{1-\xi_i^2}) \sin \omega_{di}(t-\tau) + \frac{2\xi_i}{\sqrt{1-\xi_i^2}} \cos \omega_{di}(t-\tau)] d\tau \tag{2.50}$$

which after using Eqn.s 2.28 and 2.29 may also be written as

$$\ddot{a}_i(t) = -\omega_i^2 \left[y_i(t) + (2\xi_i / \omega_i) \dot{y}_i(t) \right] \tag{2.51}$$

Then, it is considered that the maximum values of the absolute acceleration, relative velocity, and relative displacement are respectively equal to the spectral acceleration, spectral velocity and spectral displacement, and that the maximum values of the displacement and velocity terms in Eqn. 2.51 may be combined using the SRSS rule. Observe that for small damping ratios, Eqn. 2.48 is reduced to the classical relationship $SA_i = \omega_i^2 SD_i$.

It is also worthwhile to recall that the applicability of the SRSS rule is limited to systems with well-separated natural frequencies. Although most structures with added dampers possess well-separated natural frequencies, it is important to keep in mind that it is necessary to use a combination rule of the double-sum type whenever a structure has, instead, closely spaced natural frequencies. The reader is referred to Villaverde (1988), Maldonado and Singh (1991), Sinha and Igusa (1995), and Zhou et al. (2004) for modal combination rules applicable to systems with closely spaced natural frequencies and nonclassical damping.

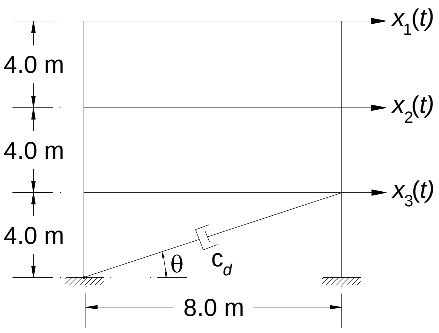

Figure 3.1. Three-story frame building considered in comparative analysis

Table 3.1. Dynamic properties of undamped frame

Mode	1	2	3
Frequency^2 (rad^2/s^2)	511.204	5025.43	14008.
Natural period (s)	0.278	0.089	0.053
Participation factor	1.0	1.0	1.0
	1.284	-0.389	0.105
Mode shape	0.973	0.214	-0.187
	0.436	0.377	0.186

3. COMPARATIVE ANALYSIS

To illustrate the application of the proposed procedure and assess its accuracy, the 3-story reinforced concrete frame building shown in Figure 3.1 is analyzed under the ground motion recorded in the North-South direction at the Takatori station during the 1995 Kobe earthquake. The plan dimensions of the building are 8.0 m × 8.0 m. The dimensions of all the columns are 0.6 m × 0.6 m and the dimensions of all the beams are 0.5 m × 0.8 m. The compressive strength of the concrete is 27.579 GN/m^2 and its modulus of elasticity is 24,821,128 kN/m^2. Considering a combined dead and live load of 6.3 kN for the floors and 4.7 kN for the roof, the frame has masses of 10.275 Mg lumped at the floor joints and 7.666 Mg at the roof joints. The first floor of the building is implemented with a supplemental linear damper with a damping constant C_d of 3500 kN-s/m. Without the damper, the building's damping matrix is considered proportional to its stiffness matrix and with a damping ratio of 2 percent in its first mode of vibration. The undamped frame exhibits the dynamic properties shown in Table 3.1. Without the damper, the damping ratios in the second and third modes of the frame are 0.063 and 0.105, respectively

Table 3.2. Dynamic properties of damped frame with added damper

Mode	1	2	3
Eigenvalue, λ_i	-5.86873+24.7311j	-62.0416+41.6766j	-17.5131+98.3091j
Natural period (s)	0.247	0.084	0.063
Damping ratio	0.231	0.830	0.175
Participation factor	-8.059+42.754j	-96.196+37.728j	34.763+50.682j
Mode shape, $\{w\}_i$	-0.00487-0.03030j -0.00669-0.02138j -0.00690-0.00734j	0.00166-0.00091j -0.00194-0.00473j -0.01071-0.00588j	0.00117+0.00536j -0.00043-0.00762j -0.00338+0.00061j

Table 3.3. Spectral values corresponding to frame with added damper

Mode	T_i (s)	ξ_i	SA_i (m/s^2)	a_{vi}	b_{vi}	r_i	SD_i (m)	SV_i (m/s)
1	0.247	0.231	7.738	1.224	0.353	0.747	0.0113	0.215
2	0.084	0.830	6.063	1.369	0.461	0.437	0.0009	0.029
3	0.063	0.175	6.015	1.197	0.321	0.493	0.0006	0.029

The damping matrix of the structure with the damper is obtained by adding to the damping matrix without the damper the horizontal force exerted by the damper on the first floor when this floor is subjected to a unit horizontal relative velocity. When the damper is oriented as shown in Figure 3.1, this force is equal to $F_D = C_d \cos^2\theta = 3500 \cos^2 26.57° = 2800$ kN. With the damping matrix constructed this way, with the solution of the eigenvalue problem $[B]\{s\} = -\lambda[A]\{s\}$, and with the consideration of Eqns. 2.9 and 2.22, the dynamic properties of the frame result as shown in Table 3.2. These dynamic properties correspond to the three modes of the system that are not a complex conjugate of another mode.

Now, from the acceleration response spectrum of the ground motion being considered, the spectral accelerations corresponding to the natural frequencies and damping ratios given in Table 3.2 are as listed in Table 3.3. Similarly, the constants a_{vi}, b_{vi} and r_i defined by Eqn. 2.47 take the values indicated in Table 3.3. In terms of these constants, the aforementioned spectral accelerations, and Eqns. 2.49 and 2.46, the corresponding spectral velocities and spectral displacements are as shown in this same table.

In terms of the dynamic properties and spectral values given above, Eqns. 2.32, 2.38, and 2.45 and the combination of the maximum modal responses using the SRSS rule lead to the maximum relative displacements, maximum relative velocities and maximum absolute accelerations listed in Table 3.4. For comparison, the corresponding responses obtained through a time-history analysis with a time step of 0.005 seconds using the computer program SAP200 and with the so-called forced classical damping method are also shown in this table. As is well known, the analysis with this latter method is performed using the natural frequencies and mode shapes of the undamped system and damping ratios calculated under the assumption of classical damping. That is, damping ratios calculated by assuming that the transformed damping matrix is a diagonal matrix and that the diagonal elements in this matrix represent the damping constants of the system in each of its modes. For the case under consideration, the damping ratios obtained using this approach are shown in Table 3.5. Also shown in this table are the spectral accelerations, spectral velocities and spectral displacements corresponding to these damping ratios and the natural frequencies of the undamped frame. The spectral displacements and velocities in Table 3.5 are also calculated using Eqns. 2.49 and 2.46.

It may be seen, thus, that the results obtained with the proposed method are close to the results determined using a time-history analysis. This in spite the fact that only one ground motion is considered in the analysis and that the SRSS combination rule is supposed to be accurate for statistical averages only. It may also be seen that except for the first-floor acceleration, the results obtained with the forced classical damping method are also close to the time-history analysis results. This seems to coincide with the findings of Zhou et al. (2004).

Table 3.4. Peak responses obtained with proposed method, forced classical damping method, and SAP2000

Floor	Proposed method			Classical damping			SAP2000		
	Disp. (m)	Vel. (m/s)	Acc. (m/s^2)	Disp. (m)	Vel. (m/s)	Acc. (m/s^2)	Disp. (m/s)	Vel. (m/s)	Acc. (m/s^2)
3	0.0152	0.286	10.613	0.0182	0.308	10.037	0.0185	0.336	8.788
2	0.0107	0.222	8.024	0.0136	0.233	7.573	0.0143	0.235	7.977
1	0.0043	0.120	7.285	0.0073	0.105	4.177	0.0067	0.096	6.738

Table 3.5. Spectral values for frame with added damper but considered classically damped

Mode	T_i (s)	ξ_i	SA_i (m/s^2)	a_{vi}	b_{vi}	r_i	SD_i (m)	SV_i (m/s)
1	0.278	0.262	7.58	1.238	0.370	0.771	0.0138	0.240
2	0.089	0.517	6.08	1.327	0.460	0.436	0.0011	0.034
3	0.053	0.362	6.01	1.279	0.415	0.378	0.0004	0.018

4. CONCLUDING REMARKS

The derived expressions and the results from the comparative analysis seem to indicate that the suggested procedure is simple to apply and offers an accuracy that is comparable to the accuracy provided by the conventional response spectrum method. It is believed, thus, that it represents a convenient alternative for the seismic analysis of structures with added dampers.

REFERENCES

Foss, K. A. (1958). Coordinates which uncouple the equations of motion of damped linear dynamic systems. *Journal of Applied Mechanics*, ASME **25**, 361-364.

Gupta, A. K. and Jaw, J. W. (1986). Response spectrum method for nonclassically damped systems. *Nuclear Engineering and Design* **91**,161-169.

Hurty, W. C. and Rubinstein, M. F. (1964). *Dynamics of Structures*, Prentice-Hall, Inc., Englewood, N. J.

Inman, D. J. and Andry, Jr., A. N. (1980). Some results on the nature of eigenvalues of discrete damped linear systems. *Journal of Applied Mechanics, ASME* **47**, 927-930.

Maldonado, G. O. and Singh, M. P. (1991). An improved response spectrum method for calculating seismic design response. Part 2: Non-classically damped structures. *Earthquake Engineering and Structural Dynamics* **20:7**, 637-649.

Sadek, F., Mohraz, B., and Riley, M. A. (2000). Linear procedures for structures with velocity-dependant dampers. *Journal of Structural Engineering, ASCE* **126:8,** 887-895.

Singh, M. P. (1980). Seismic response by SRSS for nonproportional damping. *Journal of the Engineering Mechanics Division*, *ASCE* **106:6**, 1405-1419.

Sinha, R. and Igusa, T. (1995). CQC and SRSS methods for non-classically damped structures. *Earthquake Engineering and Structural Dynamics* **24**, 615-619.

Villaverde, R. (1980). Earthquake response of systems with nonproportional damping by the conventional response spectrum method. *Proc. 7th World Conference on Earthquake Engineering*, Istanbul, Turkey, **5**, 467-474.

Villaverde, R. (1988). Rosenblueth's modal combination rule for systems with non-classical damping. *Earthquake Engineering and Structural Dynamics* **16**, 315-328.

Zhou, X. Y., Yu, R. and Dong, D. (2004). Complex mode superposition algorithm for seismic responses of non-classically damped linear MDOF system. *Journal of Earthquake Engineering* **8:4**, 597-641.