# A Unified Form for Response of 3D Generally Damped Linear Systems under Multiple Seismic Loads through Modal Analysis

**Y. Chu[1], J. Song[2] , Z. Liang[3], and G. C. Lee[4]**

[1]*Ph.D. Student,* [2]*Senior Research Scientist,* [3]*Research Associate Professor,* [4]*Professor*
*Dept. of Civil, Structural and Environmental Engineering, University at Buffalo, Buffalo, NY, USA*
*Email: yilunchu@buffalo.edu, songj@buffalo.edu, zliang@buffalo.edu, gclee@buffalo.edu*

**ABSTRACT :**

Accurate estimation of the seismic response of structures is an important issue in aseismic design. Under earthquake excitations, the deformation and internal force distributions of structures are quite complex. This is particularly true for irregular structures under multiple directional seismic loads. In most current engineering practice, only those response quantities which are linear combinations of the nodal displacements are considered to be critical for design. However, this may not always be the case, especially for those systems enhanced with high damping devices. For example, the relative nodal velocities are important for the design of the dampers and the nodal absolute accelerations determine the strength demand of the nonstructural components. This paper presents a time-history-based formulation of the responses for generally damped linear systems excited by multiple directional seismic loads through modal analysis. Solutions to the structural nodal displacements, the nodal velocity and absolute acceleration responses are derived. It is shown that the modal response quantities required in the nodal velocity and absolute acceleration are identical to those that are required in the structural displacement responses. Moreover, most response quantities of interest for seismic evaluation and design can be obtained through the linear transformation of the structural displacement, velocity or absolution acceleration as long as the structure remains in the elastic range (e.g. inter-story drift, story shear, nodal relative velocity, floor acceleration, etc.). A unified form capable of representing these response quantities is therefore established.

**KEYWORDS:** Modal analysis, generally damped structures, multicomponent seismic analysis, unified form

## 1. INTRODUCTION

Generally damped linear systems are those dynamic linear systems with non-classical damping mechanism and/or with over-damped modes. This type of systems is often seen when structures are enhanced with added damping devices. At present, most modal design and analysis approaches for structures with added damping devices subjected to seismic excitation are performed by using two-dimensional (2D) model with uni-directional loading (Trall-Nash 1981, Veletsos and Ventura 1986, Zhou et al. 2004 and Song et al. 2008a,b). The 2D models with uni-directional excitation cannot deal with the spatially coupling effects between motions in perpendicular directions and the torsional responses. Also, the correlations between ground motion components are not considered, yet they may be of significance to structural responses. In current codes (e.g. Caltrans 2004 and IBC 2003), in order to allow the extra responses resulting from the multiple excitations, a 30% or 40% rule arising from the orthogonal effects is applied. The safety margin of the percentage rule has not been examined through careful studies. It is used simply because the required computational effort is rather prohibitive and, at the same time, there has been a lack of knowledge necessary to formulate a simple, rational approach. Also, the discussion of the response quantity is generally limited to the structural nodal displacements. As complex structural systems become more popular, it is necessary to design and analyze the structures using three-dimensional (3D) model subjected to multi-components earthquake excitations and consider more response quantities, in addition to displacement in order to achieve safer design. As a result, the applicability and feasibility of the currently used modal analysis approach need to be further examined. Stemmed from the previous study of the authors (Song et al. 2008a,b) on 2D generally damped structures subjected to uni-direction ground excitation, this paper is specially engaged in establishing the theoretical base of modal analysis approach for 3D generally damped linear structures subjected to multiple excitations, from which a unified form able to represent most structural response quantities of interest is derived.

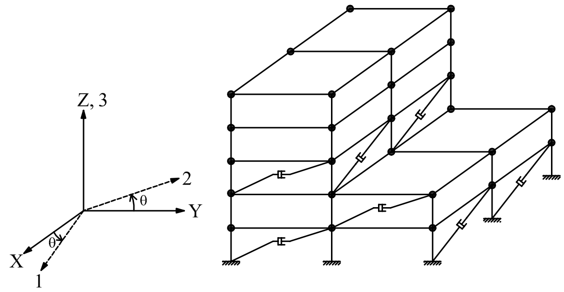


Figure 1 3D MDOF structure subjected to 3-component ground motion

## 2. FEATURES OF 3D GENERALLY DAMPED SYSTEMS

For a 3-Dimensional (3D) discrete generally damped linear structure with $N$ degrees-of-freedom (DOF) subjected to a three-component $\ddot{\mathbf{u}}_g(t)$ ($\in \mathbb{R}^3$ -- belongs to 3-dimension vector in real field) as shown in Fig. 1, the motion of the structure is governed by

$$\mathbf{M}\ddot{\mathbf{u}}(t) + \mathbf{C}\dot{\mathbf{u}}(t) + \mathbf{K}\mathbf{u}(t) = \mathbf{f}(t) \quad \in \mathbb{R}^N \tag{1}$$

in which $\mathbf{M} \in \mathbb{R}^{N\times N}$, $\mathbf{C} \in \mathbb{R}^{N\times N}$ and $\mathbf{K} \in \mathbb{R}^{N\times N}$ are the mass, viscous damping and stiffness matrices, respectively. $\mathbf{u}(t) = [\mathbf{u}_{\mathrm{x}}^{\mathrm{T}}(t) \quad \mathbf{u}_{\mathrm{y}}^{\mathrm{T}}(t) \quad \mathbf{u}_{\mathrm{z}}^{\mathrm{T}}(t) \quad \mathbf{u}_{\theta_{\mathrm{x}}}^{\mathrm{T}}(t) \quad \mathbf{u}_{\theta_{\mathrm{y}}}^{\mathrm{T}}(t) \quad \mathbf{u}_{\theta_{\mathrm{z}}}^{\mathrm{T}}(t)]^{\mathrm{T}} \in \mathbb{R}^N$ is a generalized displacement vector representing the translational and rotational DOFs for each node. $\mathbf{f}(t) = -\mathbf{MJT}\ddot{\mathbf{u}}_g(t) \in \mathbb{R}^N$ and $\mathbf{J} = [\mathbf{J}^{(\mathbf{x})} \quad \mathbf{J}^{(\mathbf{y})} \quad \mathbf{J}^{(\mathbf{z})}] \in \mathbb{R}^{N\times 3}$ is the influence matrix, which contains three resultant displacement vectors of the mass to a static application of a unit ground displacement along three structure reference axes X, Y and Z, respectively. $\ddot{\mathbf{u}}_{\mathrm{g}}(t) = [\ddot{u}_{\mathrm{g}1}(t) \quad \ddot{u}_{\mathrm{g}2}(t) \quad \ddot{u}_{\mathrm{g}3}(t)]^{\mathrm{T}}$ is the acceleration vector consisting of three orthogonal components along reference axes 1, 2 and 3, respectively. $\mathbf{T} \in \mathbb{R}^3$ is a transformation matrix and has the from of

$$\mathbf{T} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{2}$$

When the structure is generally damped, it cannot be decoupled in the $N$ dimensional physical space. As a result, it is necessary to use the $2N$ dimensional state space to perform the eigen analysis. Namely, Eq. (1) can be cast into a set of first-order linear equation as given by

$$\mathbf{A}\dot{\mathbf{v}}(t) + \mathbf{B}\mathbf{v}(t) = \mathbf{f}_{\mathrm{S}}(t) \qquad \in \mathbb{R}^{2N} \tag{3}$$

where $\mathbf{A} = \begin{pmatrix} \mathbf{0} & \mathbf{M} \\ \mathbf{M} & \mathbf{C} \end{pmatrix} \in \mathbb{R}^{2N\times 2N}$, $\mathbf{B} = \begin{pmatrix} -\mathbf{M} & \mathbf{0} \\ \mathbf{0} & \mathbf{K} \end{pmatrix} \in \mathbb{R}^{2N\times 2N}$, $\mathbf{v}(t) = \begin{Bmatrix} \dot{\mathbf{u}}(t) \\ \mathbf{u}(t) \end{Bmatrix} \in \mathbb{R}^{2N}$, $\mathbf{f}_{\mathrm{S}}(t) = \begin{Bmatrix} \mathbf{0} \\ \mathbf{f}(t) \end{Bmatrix} \in \mathbb{R}^{2N}$ (4)

It can be shown that $\mathbf{A}$ and $\mathbf{B}$ are non-singular matrices for a completely constrained structure with non-zero nodal masses; that is, both $\mathbf{A}^{-1}$ and $\mathbf{B}^{-1}$ exist (Song et al. 2008a). Let $\lambda \in \mathbb{C}$ be an admissible eigenvalue.

Associated with each eigenvalue $\lambda$ is an admissible eigenvector $\boldsymbol{\psi} \in \mathbb{C}^{2N}$. The associated eigenvalue problem of Eq. (3) is given by

$$(\lambda \mathbf{A} + \mathbf{B})\boldsymbol{\psi} = \mathbf{0} \tag{5}$$

From the theory of linear algebra, the solution to the above eigenvalue problem leads to a set of $2N$ eigenvalues $\lambda_i$ and $2N$ associated complex eigenvectors $\boldsymbol{\psi}_i$. Normally, $\lambda_i$ and $\boldsymbol{\psi}_i$ appear in complex conjugated pairs, which will correspond to under-damped vibration modes. However, $\lambda_i$ and $\boldsymbol{\psi}_i$ can also appear in real value, which correspond to over-damped modes. Suppose that there are $N_C$ complex modes (simply termed as modes) and $N_\mathrm{P}$ over-damped modes in the system ($2N_C + N_\mathrm{P} = 2N$). As a result, the following eigenvalue (or spectral) matrix and eigenvector matrix can be formed

$$\boldsymbol{\Lambda} = \mathrm{diag}\left(\lambda_1, \lambda_2 \cdots \lambda_{N_\mathrm{C}}, \lambda_1^*, \lambda_2^* \cdots \lambda_{N_\mathrm{C}}^*, \lambda_1^\mathrm{P}, \lambda_2^\mathrm{P} \cdots \lambda_{N_\mathrm{P}}^\mathrm{P}\right) \quad \in \mathbb{C}^{2N\times 2N} \tag{6}$$

$$\boldsymbol{\Psi} = \left(\boldsymbol{\psi}_1, \boldsymbol{\psi}_2 \cdots \boldsymbol{\psi}_{N_C}, \boldsymbol{\psi}_1^*, \boldsymbol{\psi}_2^* \cdots \boldsymbol{\psi}_{N_C}^*, \boldsymbol{\psi}_1^\mathrm{P}, \boldsymbol{\psi}_2^\mathrm{P} \cdots \boldsymbol{\psi}_{N_\mathrm{P}}^\mathrm{P}\right) = \begin{pmatrix} \boldsymbol{\Phi}\boldsymbol{\Lambda} \\ \boldsymbol{\Phi} \end{pmatrix} \in \mathbb{C}^{2N\times 2N} \tag{7}$$

in which $\boldsymbol{\Phi} = \left(\boldsymbol{\varphi}_1, \boldsymbol{\varphi}_2 \cdots \boldsymbol{\varphi}_{N_\mathrm{C}}, \boldsymbol{\varphi}_1^*, \boldsymbol{\varphi}_2^* \cdots \boldsymbol{\varphi}_{N_C}^*, \boldsymbol{\varphi}_1^\mathrm{P}, \boldsymbol{\varphi}_2^\mathrm{P} \cdots \boldsymbol{\varphi}_{N_\mathrm{P}}^\mathrm{P}\right) \in \mathbb{C}^{N\times 2N}$, $\lambda_i = -\xi_i \omega_i + \mathrm{j}\omega_{\mathrm{d}i}$, $\lambda_i^\mathrm{P} = -\omega_i^\mathrm{P}$ and $\mathrm{j} = \sqrt{-1}$ is the imaginary unit. $\omega_i \in \mathbb{R}$ and $\xi_i \in \mathbb{R}$ are the $i$th modal frequency and modal damping ratio, respectively, $\omega_{\mathrm{d}i} = \sqrt{1-\xi_i^2}\,\omega_i \in \mathbb{R}$ is the $i$th modal damped frequency and $\omega_i^\mathrm{P}$ is defined as $i$th over-damped natural circular frequency. The $i$th mode shape $\boldsymbol{\varphi}_i$ can be represented as $\boldsymbol{\varphi}_i = [\varphi_{ix} \quad \varphi_{iy} \quad \varphi_{iz} \quad \varphi_{i\theta_x} \quad \varphi_{i\theta_y} \quad \varphi_{i\theta_z}]^\mathrm{T}$, from which it can be observed that the modal motion is not limited to only one global reference direction. This explains the spatial coupling phenomenon in the physical space. It can also be shown that the eigenvector matrix $\boldsymbol{\Psi}$ can diagonalize the two system matrices $\mathbf{A}$ and $\mathbf{B}$ (Chu 2008), leading to following two diagonal square matrices

$$\hat{\mathbf{a}} = \boldsymbol{\Psi}^\mathrm{T}\mathbf{A}\boldsymbol{\Psi} = \mathrm{diag}\left(a_1, a_2 \cdots a_{N_\mathrm{C}}, a_1^*, a_2^* \cdots a_{N_\mathrm{C}}^*, a_1^\mathrm{P}, a_2^\mathrm{P} \cdots a_{N_\mathrm{P}}^\mathrm{P}\right) \quad \in \mathbb{C}^{2N\times 2N} \tag{8}$$

$$\hat{\mathbf{b}} = \boldsymbol{\Psi}^\mathrm{T}\mathbf{B}\boldsymbol{\Psi} = \mathrm{diag}\left(b_1, b_2 \cdots b_{N_\mathrm{C}}, b_1^*, b_2^* \cdots b_{N_\mathrm{C}}^*, b_1^\mathrm{P}, b_2^\mathrm{P} \cdots b_{N_\mathrm{P}}^\mathrm{P}\right) \quad \in \mathbb{C}^{2N\times 2N} \tag{9}$$

where $a_i = \boldsymbol{\psi}_i^\mathrm{T}\mathbf{A}\boldsymbol{\psi}_i$, $b_i = \boldsymbol{\psi}_i^\mathrm{T}\mathbf{B}\boldsymbol{\psi}_i = -a_i\lambda_i$, $a_i^\mathrm{P} = (\boldsymbol{\psi}_i^\mathrm{P})^\mathrm{T}\mathbf{A}\boldsymbol{\psi}_i^\mathrm{P}$ and $b_i^\mathrm{P} = (\boldsymbol{\psi}_i^\mathrm{P})^\mathrm{T}\mathbf{B}\boldsymbol{\psi}_i^\mathrm{P} = -a_i^\mathrm{P}\lambda_i^\mathrm{P}$ (10)

### *2.1 Modal coordinate transformation*

Let the response vector $\mathbf{v}(t) \in \mathbb{R}^{2N}$ be represented in the following transformation

$$\mathbf{v}(t) = \begin{Bmatrix} \mathbf{u}(t) \\ \dot{\mathbf{u}}(t) \end{Bmatrix} = \boldsymbol{\Psi}\,\mathbf{w}(t) \tag{11}$$

where $\mathbf{w}(t) = \left[w_1(t),\ w_2(t), \cdots w_{N_\mathrm{C}}(t),\ w_1^*(t),\ w_2^*(t) \cdots w_{N_\mathrm{C}}^*(t),\ w_1^\mathrm{P}(t),\ w_2^\mathrm{P}(t) \cdots w_{N_\mathrm{P}}^\mathrm{P}(t)\right]^\mathrm{T} \in \mathbb{C}^{2N}$ (12)

is the complex modal coordinate vector in the time domain. Using Eq. (11), the state space Eq. (3) can be decoupled into a set of uncoupled 2$N$ independent modal equations as

$$\dot{w}_i(t)-\lambda_i w_i(t)=\frac{\boldsymbol{\varphi}_i^{\mathrm{T}}\mathbf{f}(t)}{a_i}\quad\in\mathbb{C}\qquad(i=1,2\ldots N_{\mathrm{C}})\tag{13}$$

$$\dot{w}_i^*(t)-\lambda_i^* w_i^*(t)=\frac{\boldsymbol{\varphi}_i^{\mathrm{H}}\mathbf{f}(t)}{a_i^*}\quad\in\mathbb{C}\qquad(i=1,2\ldots N_{\mathrm{C}})\tag{14}$$

and

$$\dot{w}_i^{\mathrm{P}}(t)-\lambda_i^{\mathrm{P}} w_i^{\mathrm{P}}(t)=\frac{(\boldsymbol{\varphi}_i^{\mathrm{P}})^{\mathrm{T}}\mathbf{f}(t)}{a_i^{\mathrm{P}}}\quad\in\mathbb{R}\qquad(i=1,2\cdots N_{\mathrm{P}})\tag{15}$$

In Eq. (14), the superscript "H" denotes Hermitian transpose. After substituting the solutions of Eqs. (13) to (15) into Eq. (11), the structural displacement and velocity response vector can be expressed as

$$\mathbf{u}(t)=\sum_{i=1}^{N_{\mathrm{C}}}\left[\boldsymbol{\varphi}_i w_i(t)+\boldsymbol{\varphi}_i^* w_i^*(t)\right]+\sum_{i=1}^{N_{\mathrm{P}}}\boldsymbol{\varphi}_i^{\mathrm{P}} w_i^{\mathrm{P}}(t)\tag{16}$$

$$\dot{\mathbf{u}}(t)=\sum_{i=1}^{N_{\mathrm{C}}}\left[\lambda_i\boldsymbol{\varphi}_i w_i(t)+\lambda_i^*\boldsymbol{\varphi}_i^* w_i^*(t)\right]+\sum_{i=1}^{N_{\mathrm{P}}}\lambda_i^{\mathrm{P}}\boldsymbol{\varphi}_i^{\mathrm{P}} w_i^{\mathrm{P}}(t)\tag{17}$$

### *2.2 Structural residual matrices*

By taking the derivative with respect to time of both sides of Eq. (16), together with the use of Eqs. (13) to (15), and equating the resulting form to Eq. (17), the following expression results

$$\sum_{i=1}^{N_{\mathrm{C}}}\left[\frac{\boldsymbol{\varphi}_i\boldsymbol{\varphi}_i^{\mathrm{T}}}{a_i}+\frac{\boldsymbol{\varphi}_i^*\boldsymbol{\varphi}_i^{\mathrm{H}}}{a_i^*}\right]\mathbf{f}(t)+\sum_{i=1}^{N_{\mathrm{P}}}\left[\frac{\boldsymbol{\varphi}_i^{\mathrm{P}}(\boldsymbol{\varphi}_i^{\mathrm{P}})^{\mathrm{T}}}{a_i^{\mathrm{P}}}\right]\mathbf{f}(t)=\mathbf{0}\quad\in\mathbb{C}^{N\times N}\tag{18}$$

Since $\mathbf{f}(t)$ is arbitrary, it implies that

$$\sum_{i=1}^{N_{\mathrm{C}}}\left[\frac{\boldsymbol{\varphi}_i\boldsymbol{\varphi}_i^{\mathrm{T}}}{a_i}+\frac{\boldsymbol{\varphi}_i^*\boldsymbol{\varphi}_i^{\mathrm{H}}}{a_i^*}\right]+\sum_{i=1}^{N_{\mathrm{P}}}\left[\frac{\boldsymbol{\varphi}_i^{\mathrm{P}}(\boldsymbol{\varphi}_i^{\mathrm{P}})^{\mathrm{T}}}{a_i^{\mathrm{P}}}\right]=\boldsymbol{\Phi}\,\hat{\mathbf{a}}^{-1}\boldsymbol{\Phi}^{\mathrm{T}}=\mathbf{0}\quad\in\mathbb{C}^{N\times N}\tag{19}$$

Further, denote

$$\mathbf{R}_i=\mathbf{R}_i^{\mathrm{R}}+\mathrm{j}\mathbf{R}_i^{\mathrm{I}}=\frac{\boldsymbol{\varphi}_i\boldsymbol{\varphi}_i^{\mathrm{T}}}{a_i}\in\mathbb{C}^{N\times N}\quad,\quad\mathbf{R}_i^*=\mathbf{R}_i^{\mathrm{R}}-\mathrm{j}\mathbf{R}_i^{\mathrm{I}}=\frac{\boldsymbol{\varphi}_i^*\boldsymbol{\varphi}_i^{\mathrm{H}}}{a_i^*}\in\mathbb{C}^{N\times N}\qquad(i=1,2\cdots N_{\mathrm{C}})\tag{20}$$

and

$$\mathbf{R}_i^{\mathrm{P}}=\frac{\boldsymbol{\varphi}_i^{\mathrm{P}}(\boldsymbol{\varphi}_i^{\mathrm{P}})^{\mathrm{T}}}{a_i^{\mathrm{P}}}\quad\in\mathbb{R}^{N\times N}\quad(i=1,2\cdots N_{\mathrm{P}})\tag{21}$$

$\mathbf{R}_i$, $\mathbf{R}_i^*$ and $\mathbf{R}_i^{\mathrm{P}}$ are referred to as the residual matrices corresponding to the eigenvalues $\lambda_i$, $\lambda_i^*$ and $\lambda_i^{\mathrm{P}}$, respectively. Eq. (19) shows an important property for a linear system, that is, the summation of all residual matrices equal to the null matrix. Note that all residual matrices only depend on the structural system parameters and are independent of the normalization manner of the modes.

### *2.3 Expansion of the inverse of the mass matrix*

Considering Eq. (4) and (7) in Eq. (9),

$$\mathbf{\Psi}^{\mathrm{T}}\mathbf{B}\mathbf{\Psi}=\begin{pmatrix}\mathbf{\Phi\Lambda}\\ \mathbf{\Phi}\end{pmatrix}^{\mathrm{T}}\begin{pmatrix}-\mathbf{M} & \mathbf{0}\\ \mathbf{0} & \mathbf{K}\end{pmatrix}\begin{pmatrix}\mathbf{\Phi\Lambda}\\ \mathbf{\Phi}\end{pmatrix}=-\mathbf{\Lambda\Phi}^{\mathrm{T}}\mathbf{M\Phi\Lambda}+\mathbf{\Phi}^{\mathrm{T}}\mathbf{K\Phi}=\hat{\mathbf{b}}=-\hat{\mathbf{a}}\,\mathbf{\Lambda} \tag{22}$$

If the masses at all $N$ structural DOFs are non-zero, $\mathbf{M}$ is a positive-definite matrix and $\mathbf{M}^{-1}$ exists. From Eqs, (19) and (22), $\mathbf{M}^{-1}$ can be expanded in terms of the system parameters as (Song et al. 2008a)

$$\mathbf{M}^{-1}=\mathbf{\Phi}\hat{\mathbf{a}}^{-1}\mathbf{\Lambda\Phi}^{\mathrm{T}}=-2\sum_{i=1}^{N_{\mathrm{C}}}\left(\xi_i\omega_i\mathbf{R}_i^{\mathrm{R}}+\omega_{\mathrm{d}i}\mathbf{R}_i^{\mathrm{I}}\right)-\sum_{i=1}^{N_{\mathrm{P}}}\omega_i^{\mathrm{P}}\mathbf{R}_i^{\mathrm{P}} \tag{23}$$

## 3. DEVELOPMENT OF THE UNIFIED FORM

The seismic displacement and velocity responses have been given by Eqs. (16) and (17), respectively. However, it is noted that the modal responses $w_i(t)$, $w_i^*(t)$ and $w_i^{\mathrm{P}}(t)$ are complex-valued and have no physical interpretations. In addition, the computation efforts for solving Eqs. (16) and (17) are demanding since they are complex-valued. Thus, responses expressed by the form shown in Eqs. (16) and (17) are not preferred. In the following, a unified formulation in terms of real numbers for most response quantities is developed.

### *3.1 Displacement response*

With the help of the Laplace transform applied to Eqs. (11) to (15) and with certain simple manipulation of the resulting expressions, Eq. (16) in Laplacian form can be represented as

$$\mathbf{U}(s)=-\sum_{i=1}^{N_{\mathrm{C}}}\left(\frac{\mathbf{A}_{\mathrm{D}i}s}{s^2+2\xi_i\omega_i s+\omega_i^2}+\frac{\mathbf{B}_{\mathrm{D}i}}{s^2+2\xi_i\omega_i s+\omega_i^2}\right)\mathbf{T}\ddot{\mathbf{U}}_{\mathrm{g}}(s)-\sum_{i=1}^{N_{\mathrm{P}}}\frac{\mathbf{A}_{\mathrm{D}i}^{\mathrm{P}}}{s+\omega_i^{\mathrm{P}}}\mathbf{T}\ddot{\mathbf{U}}_{\mathrm{g}}(s)\quad\in\mathbb{C}^N \tag{24}$$

in which $\mathbf{A}_{\mathrm{D}i}=2\mathbf{R}_i^{\mathrm{R}}\mathbf{MJ}\in\mathbb{R}^N$, $\mathbf{B}_{\mathrm{D}i}=2\omega_i\left(\xi_i\mathbf{R}_i^{\mathrm{R}}-\sqrt{1-\xi_i^2}\mathbf{R}_i^{\mathrm{I}}\right)\mathbf{MJ}\in\mathbb{R}^N$, $\mathbf{A}_{\mathrm{D}i}^{\mathrm{P}}=\mathbf{R}_i^{\mathrm{P}}\mathbf{MJ}\in\mathbb{R}^N$. Further denote

$$\mathbf{Q}_i(s)=H_i(s)\ddot{\mathbf{U}}_{\mathrm{g}}(s)\quad\in\mathbb{C}^3\quad\text{and}\quad\mathbf{Q}_{\mathrm{V}i}(s)=H_{\mathrm{V}i}(s)\ddot{\mathbf{U}}_{\mathrm{g}}(s)\quad\in\mathbb{C}^3 \tag{25a,b}$$

where
$$H_i(s)=-\frac{1}{s^2+2\xi_i\omega_i s+\omega_i^2}\quad\in\mathbb{C}\text{ and }\quad H_{\mathrm{V}i}(s)=-\frac{s}{s^2+2\xi_i\omega_i s+\omega_i^2}\quad\in\mathbb{C} \tag{26a,b}$$

are displacement and velocity transfer function of a under-damped SDOF system with the *i*th modal damping ratio $\xi_i$ and the *i*th modal natural frequency $\omega_i$, respectively. In fact, Eqs. (25a) and (25b) can be considered as the Laplace transformation of the displacement vector and the velocity vector governed by the following differential equation set expressed in the time domain with zero initial conditions, respectively

$$\ddot{\mathbf{q}}_i(t)+2\xi_i\omega_i\dot{\mathbf{q}}_i(t)+\omega_i^2\mathbf{q}_i(t)=-\ddot{\mathbf{u}}_{\mathrm{g}}(t) \tag{27}$$

where $\mathbf{q}_i(t) = \mathrm{L}^{-1}\left[\mathbf{Q}_i(s)\right] = [q_{1i}(t)\; q_{2i}(t)\; q_{3i}(t)]^{\mathrm{T}} \in \mathbb{R}^3$ . Also, denote

$$\mathbf{Q}_i^{\mathrm{P}}(s) = H_i^{\mathrm{P}}(s)\ddot{\mathbf{U}}_{\mathrm{g}}(s) \quad \in \mathbb{C}^3 \tag{28}$$

in which

$$H_i^{\mathrm{P}}(s) = -\frac{1}{s+\omega_i^{\mathrm{P}}} \quad \in \mathbb{C} \tag{29}$$

Similarly, Eq. (29) is the Laplace transformation of the following first-order differential equation with zero initial condition.

$$\dot{\mathbf{q}}_i^{\mathrm{P}}(t) + \omega_i^{\mathrm{P}}\mathbf{q}_i^{\mathrm{P}}(t) = -\ddot{\mathbf{u}}_{\mathrm{g}}(t) \tag{30}$$

where $\mathbf{q}_i^{\mathrm{P}}(t) = \mathrm{L}^{-1}\left[\mathbf{Q}_i^{\mathrm{P}}(s)\right] = [q_{1i}^{\mathrm{P}}(t) \quad q_{2i}^{\mathrm{P}}(t) \quad q_{3i}^{\mathrm{P}}(t)]^{\mathrm{T}} \in \mathbb{R}^3$ is the $i$th over-damped modal response vector.

As a result, the inverse Laplace transform of Eq. (24) which is the displacement vector $\mathbf{u}(t,\theta)$ can be expressed as (Chu 2008).

$$\mathbf{u}(t,\theta) = \sum_{i=1}^{N_{\mathrm{C}}}\left[\mathbf{A}_{\mathrm{D}i}\mathbf{T}\dot{\mathbf{q}}_i(t) + \mathbf{B}_{\mathrm{D}i}\mathbf{T}\mathbf{q}_i(t)\right] + \sum_{i=1}^{N_{\mathrm{P}}}\mathbf{A}_{\mathrm{D}i}^{\mathrm{P}}\mathbf{T}\mathbf{q}_i^{\mathrm{P}}(t) \tag{31}$$

The variable $\theta$ is included in the expression to show that the response vector is also dependent on the seismic incident angle $\theta$, considered through the transformation matrix $\mathbf{T}$. Unlike Eq. (11), it may be observed that the displacement responses can be expressed in terms of real-valued quantities. It is a linear combination of the modal displacement and velocity responses of a set of under-damped SDOF systems as well as those of a set of over-damped first order system subjected to three orthogonal ground motions. Also, from Eq. (31), it is clear that the modal structural response vector (including modal responses of all structural DOFs) for each mode consists of two terms which are related to the modal velocity response, $\mathbf{A}_{\mathrm{D}i}\mathbf{T}\dot{\mathbf{q}}_i(t)$ and modal displacement response, $\mathbf{B}_{\mathrm{D}i}\mathbf{T}\mathbf{q}_i(t)$ , respectively. Comparing Eq. (31) with the corresponding formula used for classically-damped system (Clough and Penzien 1993), the former term is introduced due to the non-classical damping effect. Since the corresponding elements in the coefficient vectors $\mathbf{A}_{\mathrm{D}i}$ and $\mathbf{B}_{\mathrm{D}i}$ for each mode may not be equal, the combined result of these two terms may shift the modal response phase along structural DOFs. Thus, it can be concluded that when the structure is non-classically damped, the modal structural responses for all DOFs may not vibrate in phase or out of phase, which is different from the behavior appearing in a classically-damped system. In Eq. (31), the last term related to the over-damped modal response $\mathbf{A}_{\mathrm{D}i}^{\mathrm{P}}\mathbf{T}\mathbf{q}_i^{\mathrm{P}}(t)$ are used to compute the modal responses contributed from the over-damped modes when they are present.

### *3.2 Velocity response*

Intuitively, the structural velocity response vector can be obtained directly by taking derivative of Eq. (31) with respect to time variable *t* as

$$\dot{\mathbf{u}}(t,\theta) = \sum_{i=1}^{N_{\mathrm{C}}}\left[\mathbf{A}_{\mathrm{D}i}\mathbf{T}\ddot{\mathbf{q}}_i(t) + \mathbf{B}_{\mathrm{D}i}\mathbf{T}\dot{\mathbf{q}}_i(t)\right] + \sum_{i=1}^{N_{\mathrm{P}}}\mathbf{A}_{\mathrm{D}i}^{\mathrm{P}}\mathbf{T}\dot{\mathbf{q}}_i^{\mathrm{P}}(t) \tag{32}$$

However, this formulation requires the incorporation of two additional modal responses $\ddot{\mathbf{q}}_i(t)$ and $\dot{\mathbf{q}}_i^{\mathrm{P}}(t)$ in the expression. A different approach to derive the expression of the relative velocity vector is given as follows. First, let us rearrange Eqs. (27) and (30), respectively, as

$$\ddot{\mathbf{q}}_i(t) = -2\xi_i\omega_i\dot{\mathbf{q}}_i(t) - \omega_i^2\mathbf{q}_i(t) - \ddot{\mathbf{u}}_{\mathrm{g}}(t) \tag{33}$$

$$\dot{\mathbf{q}}_i^{\mathrm{P}}(t) = -\omega_i^{\mathrm{P}}\mathbf{q}_i^{\mathrm{P}}(t) - \ddot{\mathbf{u}}_{\mathrm{g}}(t) \tag{34}$$

After substituting Eqs. (33) and (34) into Eq. (31), one will have

$$\begin{aligned}\dot{\mathbf{u}}(t,\theta) = &\sum_{i=1}^{N_{\mathrm{C}}}\left[\left(\mathbf{B}_{\mathrm{D}i} - 2\xi_i\omega_i\mathbf{A}_{\mathrm{D}i}\right)\mathbf{T}\dot{\mathbf{q}}_i(t) - \omega_i^2\mathbf{A}_{\mathrm{D}i}\mathbf{T}\mathbf{q}_i(t)\right] + \sum_{i=1}^{N_{\mathrm{P}}}\left(-\omega_i^P\mathbf{A}_{\mathrm{D}i}^{\mathrm{P}}\right)\mathbf{T}\mathbf{q}_i^{\mathrm{P}}(t) \\ &-\left[\sum_{i=1}^{N_{\mathrm{C}}}\mathbf{A}_{\mathrm{D}i} + \sum_{i=1}^{N_{\mathrm{P}}}\mathbf{A}_{\mathrm{D}i}^{\mathrm{P}}\right]\mathbf{T}\ddot{\mathbf{u}}_{\mathrm{g}}\end{aligned} \tag{35}$$

in which the last term vanishes after considering the summation of the residual matrices shown in Eq. (19), i.e.

$$\left[\sum_{i=1}^{N_{\mathrm{C}}}\mathbf{A}_{\mathrm{D}i} + \sum_{i=1}^{N_{\mathrm{P}}}\mathbf{A}_{\mathrm{D}i}^{\mathrm{P}}\right]\mathbf{T}\ddot{\mathbf{u}}_{\mathrm{g}} = \left[\sum_{i=1}^{N_{\mathrm{C}}}2\mathbf{R}_i^{\mathrm{R}} + \sum_{i=1}^{N_{\mathrm{P}}}\mathbf{R}_i^{\mathrm{P}}\right]\mathbf{MJT}\ddot{\mathbf{u}}_{\mathrm{g}} = \mathbf{0} \tag{36}$$

Therefore, the velocity vector $\dot{\mathbf{u}}(t,\theta)$ can be expressed as

$$\dot{\mathbf{u}}(t,\theta) = \sum_{i=1}^{N_{\mathrm{C}}}\left[\mathbf{A}_{\mathrm{V}i}\mathbf{T}\dot{\mathbf{q}}_i(t) + \mathbf{B}_{\mathrm{V}i}\mathbf{T}\mathbf{q}_i(t)\right] + \sum_{i=1}^{N_{\mathrm{P}}}\mathbf{A}_{\mathrm{V}i}^{\mathrm{P}}\mathbf{T}\mathbf{q}_i^{\mathrm{P}}(t) \tag{37}$$

where $\mathbf{A}_{\mathrm{V}i} = \mathbf{B}_{\mathrm{D}i} - 2\xi_i\omega_i\mathbf{A}_{\mathrm{D}i}$, $\mathbf{B}_{\mathrm{V}i} = -\omega_i^2\mathbf{A}_{\mathrm{D}i}$ and $\mathbf{A}_{\mathrm{V}i}^{\mathrm{P}} = -\omega_i^P\mathbf{A}_{\mathrm{D}i}^{\mathrm{P}}$.

Noted that Eqs. (32) and (37) are equivalent. However, Eq. (37) is preferred since the two additional modal responses $\ddot{\mathbf{q}}_i(t)$ and $\dot{\mathbf{q}}_i^{\mathrm{P}}(t)$ do not appear in the expression and it has a consistent expression with the displacement vector $\mathbf{u}(t,\theta)$ shown by Eq. (31). It is worthwhile to point out that Eq. (37) can also be derived directly from Eq. (17) following the similar procedure for deriving Eq. (31). The final results including the coefficient vectors, $\mathbf{A}_{\mathrm{V}i}$, $\mathbf{B}_{\mathrm{V}i}$ and $\mathbf{A}_{\mathrm{V}i}^{\mathrm{P}}$ are exactly same as Eq. (37), which is understandable because Eq. (19) is derived based on Eqs. (16) and (17). However, the currently adopted approach can bring out an extra important property of the generally damped structural system that is shown by Eq. (36).

### *3.3 Absolute acceleration response*

To derive the absolute acceleration vector, let us start from the relative acceleration, which can be obtained by taking derivative of the velocity vector $\dot{\mathbf{u}}(t,\theta)$ with respect to time. As a result, we have

$$\ddot{\mathbf{u}}(t,\theta) = \sum_{i=1}^{N_{\mathrm{C}}}\left[\mathbf{A}_{\mathrm{V}i}\mathbf{T}\ddot{\mathbf{q}}_i(t) + \mathbf{B}_{\mathrm{V}i}\mathbf{T}\dot{\mathbf{q}}_i(t)\right] + \sum_{i=1}^{N_{\mathrm{P}}}\mathbf{A}_{\mathrm{V}i}^{\mathrm{P}}\mathbf{T}\dot{\mathbf{q}}_i^{\mathrm{P}}(t) \quad \in \mathbb{R}^N \tag{38}$$

Substituting Eq. (33) and (34) into Eq. (38) leads to

$$\ddot{\mathbf{u}}(t,\theta)=\sum_{i=1}^{N_{\mathrm{C}}}\left[\mathbf{A}_{\mathrm{A}i}\mathbf{T}\dot{\mathbf{q}}_i(t)+\mathbf{B}_{\mathrm{A}i}\mathbf{T}\mathbf{q}_i(t)\right]+\sum_{i=1}^{N_{\mathrm{P}}}\mathbf{A}_{\mathrm{A}i}^{\mathrm{P}}\mathbf{T}\mathbf{q}_i^{\mathrm{P}}(t)-\left[\sum_{i=1}^{N_{\mathrm{C}}}\mathbf{A}_{\mathrm{V}i}+\sum_{i=1}^{N_{\mathrm{P}}}\mathbf{A}_{\mathrm{V}i}^{\mathrm{P}}\right]\mathbf{T}\ddot{\mathbf{u}}_{\mathrm{g}}(t) \tag{39}$$

where $\mathbf{A}_{\mathrm{A}i}=\mathbf{B}_{\mathrm{V}i}-2\xi_i\omega_i\mathbf{A}_{\mathrm{V}i}$, $\mathbf{B}_{\mathrm{A}i}=-\omega_i^2\mathbf{A}_{\mathrm{V}i}$ and $\mathbf{A}_{\mathrm{A}i}^{\mathrm{P}}=-\omega_i^{\mathrm{P}}\mathbf{A}_{\mathrm{V}i}^{\mathrm{P}}$.
Making use of the modal expansion of the inverse of the mass $\mathbf{M}^{-1}$ shown by Eq. (23), the last term in Eq. (39) becomes

$$\begin{aligned}\left[\sum_{i=1}^{N_{\mathrm{C}}}\mathbf{A}_{\mathrm{V}i}+\sum_{i=1}^{N_{\mathrm{P}}}\mathbf{A}_{\mathrm{V}i}^{\mathrm{P}}\right]\mathbf{T}\ddot{\mathbf{u}}_{\mathrm{g}}(t)&=\left[-2\sum_{i=1}^{N_{\mathrm{C}}}\left(\xi_i\omega_i\mathbf{R}_i^{\mathrm{R}}+\omega_{\mathrm{d}i}\mathbf{R}_i^{\mathrm{I}}\right)-\sum_{i=1}^{N_{\mathrm{P}}}\omega_i^{\mathrm{P}}\mathbf{R}_i^{\mathrm{P}}\right]\mathbf{MJT}\ddot{\mathbf{u}}_{\mathrm{g}}(t)\\&=\mathbf{M}^{-1}\mathbf{MJT}\ddot{\mathbf{u}}_{\mathrm{g}}(t)=\mathbf{JT}\ddot{\mathbf{u}}_{\mathrm{g}}(t)\end{aligned} \tag{40}$$

Substituting the result of Eq. (40) into Eq. (39) and denoting the structural absolute acceleration vector as $\ddot{\mathbf{u}}_{\mathrm{A}}(t,\theta)$, which can be expressed as $\ddot{\mathbf{u}}_{\mathrm{A}}(t,\theta)=\ddot{\mathbf{u}}(t,\theta)+\mathbf{JT}\ddot{\mathbf{u}}_{\mathrm{g}}(t)$, one obtains

$$\ddot{\mathbf{u}}_{\mathrm{A}}(t,\theta)=\sum_{i=1}^{N_{\mathrm{C}}}\left[\mathbf{A}_{\mathrm{A}i}\mathbf{T}\dot{\mathbf{q}}_i(t)+\mathbf{B}_{\mathrm{A}i}\mathbf{T}\mathbf{q}_i(t)\right]+\sum_{i=1}^{N_{\mathrm{P}}}\mathbf{A}_{\mathrm{A}i}^{\mathrm{P}}\mathbf{T}\mathbf{q}_i^{\mathrm{P}}(t) \tag{41}$$

It is noted that the modal relative acceleration vector $\ddot{\mathbf{q}}_i(t)$ and the ground acceleration vector $\ddot{\mathbf{u}}_{\mathrm{g}}(t)$ are not included explicitly in the expression of the absolute acceleration vector $\ddot{\mathbf{u}}_{\mathrm{A}}(t,\theta)$.

### *3.4 A unified form for response expressions*

Comparing the expressions of the relative displacement, the relative velocity and the absolute acceleration vectors shown in Eqs. (31), (37) and (41), respectively, it may be observed that these three response quantities are the linear combination of the modal response $\mathbf{q}_i(t)$, $\dot{\mathbf{q}}_i(t)$ and $\mathbf{q}_i^{\mathrm{P}}(t)$. They only differ in their respective coefficient vectors. Thus, it is convenient to represent these three response vectors in a similar manner as follow.

$$\mathbf{u}_0(t,\theta)=\sum_{i=1}^{N_{\mathrm{C}}}\left[\mathbf{A}_{0i}\mathbf{T}\dot{\mathbf{q}}_i(t)+\mathbf{B}_{0i}\mathbf{T}\mathbf{q}_i(t)\right]+\sum_{i=1}^{N_{\mathrm{P}}}\mathbf{A}_{0i}^{\mathrm{P}}\mathbf{T}\mathbf{q}_i^{\mathrm{P}}(t) \tag{42}$$

In general, most response quantities, denoted as $r_0(t,\theta)$, are either deformation-related, such as bending moments, inter-story drifts, shear forces etc., velocity-related, such as the inter-story velocity or absolute acceleration-related , such as the floor acceleration. As a result, most response quantities within the structure can be expressed by a linear combination of the response vector $\mathbf{u}_0(t,\theta)$ through appropriate transformation. That is,

$$r_0(t,\theta)=\mathbf{d}^{\mathrm{T}}\mathbf{u}_0(t,\theta)=\sum_{i=1}^{N_{\mathrm{C}}}\left[\mathbf{d}^{\mathrm{T}}\mathbf{A}_{0i}\mathbf{T}\dot{\mathbf{q}}_i(t)+\mathbf{d}^{\mathrm{T}}\mathbf{B}_{0i}\mathbf{T}\mathbf{q}_i(t)\right]+\sum_{i=1}^{N_{\mathrm{P}}}\mathbf{d}^{\mathrm{T}}\mathbf{A}_{0i}^{\mathrm{P}}\mathbf{T}\mathbf{q}_i^{\mathrm{P}}(t) \tag{43}$$

where $\mathbf{d}$ is a transformation vector. This form is helpful in the formulation of a modal combination rule for response spectrum method.

## 4. SUMMARY AND CONCLUSION

A modal response history analysis approach for generally damped 3D linear structures subjected to multiple ground motion excitations is developed. In this approach, over-damped modes, if exist in the system, are taken into account for accurately evaluating the structural complete responses. In addition, the modal response combination formulation and the decoupled modal differential equations with first order and second order are expressed in real field and have clear physical interpretations. Also, a unified form capable of describing most response quantities of interest is established for multicomponent seismic analysis via the modal properties identified in this study. This unified form can be further used for the development of the response-spectrum-based analysis method for generally damped 3D linear structures, which has been performed by the authors (Song et al. 2008a and Chu 2008).

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