FUNDAMENTAL FREQUENCIES AND CRITICAL CIRCUMFERENTIAL MODES OF FLUID-TANK SYSTEMS

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ABSTRACT :

In this work explicit expressions are presented to obtain the fundamental frequency and the circumferential critical n^* wave number for the longitudinal critical wave number m=1 for clamped-free vertical cylindrical tanks partially filled with water. The hydrostatic pressure is taken into count, free surface motion is neglected and dynamic pressure is considered like an added virtual mass. The solution is based upon an improved Flügge's shells theory that is solved by means of the use of covariants and contravariants modals forms. The liquid is assumed as non-viscous and incompressible, and the coupling between the deformable shell and the liquid is taken into count. The solution for the liquid velocity potential satisfies Laplace equation and the relevant boundary condition. A regression model is used in order to fit mathematical results previously computed to obtain two explicit expressions with excellent approximation compared with experimental data. The equations are proposed for the case of steel tanks for a direct use in seismic analysis and design of storage tanks.

KEYWORDS: Fundamental frequencies, Cylindrical tanks, Free vibration, Theory of shells.

1. INTRODUCTION

The dynamic features of cylindrical tanks are modified due to the presence of: hydrostatic pressure and inertial forces produced by the fluid; and the behavior of the free surface. These effects introduce serious complications in the seismic behavior´s analysis of the fluid-tank system. Cylindrical shells, partially filled or empty, show vibration modes for different longitudinal m and circumferential n modes (Figure 1) that are related with their respective natural frequencies ω_{mn}. In this work we propose a methodology and compare with another results available in current literature to determine the fundamental frequency and the critical longitudinal numbers m and circumferential n for clamped-free (CF) vertical cylindrical tanks partially filled with water.

There are many works related with the natural frequencies' assessment for CF cylindrical shells and partially filled with water. Chiba et al. (1984 and 1985) consider the effects of the hydrostatic pressure through the nonlinear equations of Donnell's shells theory which are solved by the Galerkin method; Koga and Tsushima (1990) consider the hydrodynamic pressure like a virtual mass and neglect hydrostatic pressure; Mazúch et al. (1996) uses finite element analysis and Lakis (1997) uses Sander's shells theory with finite element analysis and considers the free surface of fluid to solve the problem and calculate natural frequencies.

To solve this problem some authors use different shells theories sometimes including the contribution of the fluid characteristics like: hydrostatic pressure, added virtual mass or free surface to determine natural frequencies. Other authors relate the critical circumferential modes n^* with fundamental frequency ω^* only for empty tanks (Arango et al. 1989, Urrutia 1989, El Mously 2003). However, there are not explicit expressions to calculate de minimum frequencies or critical circumferential modes for clamped-free tanks partially filled with fluid.

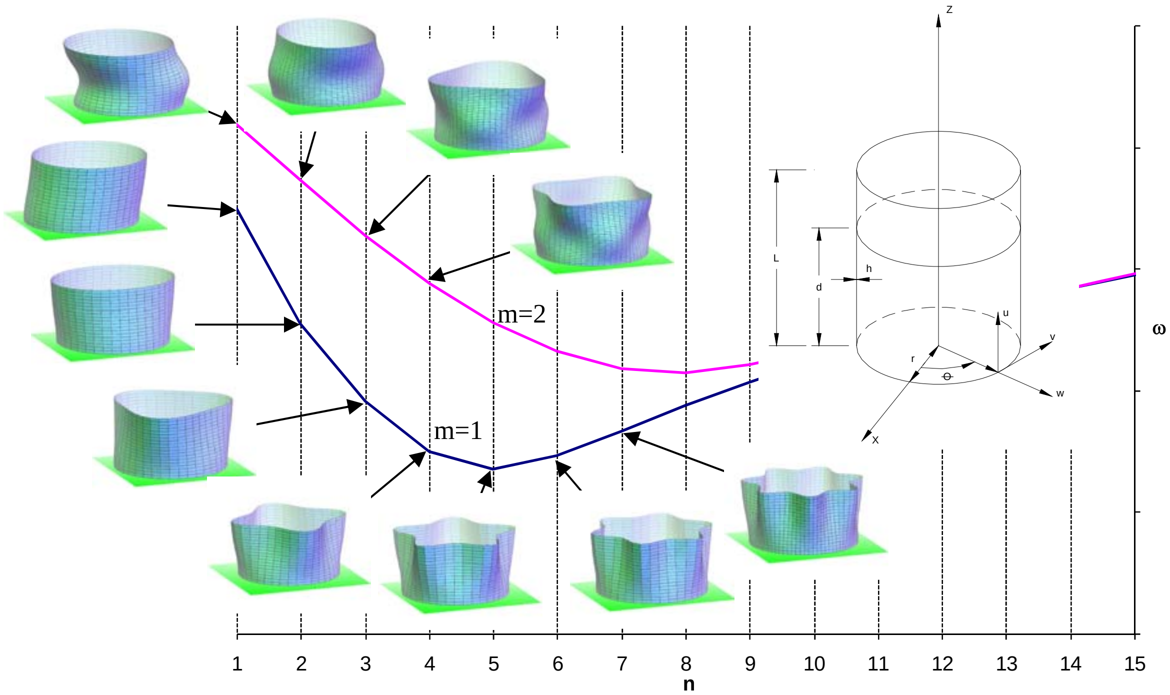

Figure 1 Natural frequencies and modal forms. Longitudinal “m” and circumferential “n” modes

Urrutia (1989) shows the invariance of the universal parameter $(\omega_{mn}\ r)^2$ and this opens up a way to find the different relationships between all the parameters (for empty tanks): geometrics (longitude L, radio r and thickness h) with critical parameters ω^*, m and n^*, which are the fundamental frequency and critical circumferential modes.

The fluid, governed by Navier-Stokes equations, is considered, like an incompressible and irrotational fluid. The influence of the fluid in the shells motion equations is included in the dynamic pressure equation and is introduced like an “added virtual mass”. The dynamic pressure equation, that provided the fluid on the cylindrical wall, neglected the free surface variation.

The main idea of this work is to present two expressions, obtained by a regression model, which can be use to make the analysis and pre design of the vertical cylindrical tanks considering the fundamental frequency and the longitudinal and circumferential critical waves. Besides, we provide an applied methodology to shell theory that results in a simple mathematical model that allows to obtain the natural frequencies of a CF vertical cylindrical tanks partially filled with water.

2. FLUID-TANK SYSTEM. MATHEMATICAL MODEL

The variables to determine the natural frequencies for the longitudinal mode *m* and circumferential mode *n* of a cylindrical shell, where the walls of the shell are subjected to initial stresses by the hydrostatic pressure of the fluid are: the radio *r*, the height *L*, the constant thickness *h*, and the height of flow *d* in a partially filled tank (Figure 1).

For the initial conditions of the shell it is necessary to consider the dynamics equations which include initial stresses on the shell wall (hydrostatic pressure). The terms that consider the initial stresses are those that in elastic stability are called parametric terms (Urrutia, 1984). For the vibration analysis it is necessary to consider dynamic pressure like a virtual added mass.

Finally, the mathematical model (MM) for the fluid-tank system (FTS), expressed in matrix notation, is

$$\begin{bmatrix} L_{11}+\overset{*}{L}_{11}+\omega^2\tau^2 & L_{12} & L_{13}+\overset{*}{L}_{13} \\ L_{21} & L_{22}+\overset{*}{L}_{22}+\omega^2\tau^2 & L_{23}+\overset{*}{L}_{23} \\ L_{31}+\overset{*}{L}_{31} & L_{32}+\overset{*}{L}_{32} & L_{33}+\overset{*}{L}_{33}-\omega^2\tau^2\gamma \end{bmatrix}\begin{bmatrix} u \\ v \\ w \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{2.1}$$

2.1. *Dynamics equations of shell*

When a cylindrical tank is not submitted to some kind of external pressure, the terms L_{ij} are those of the equations of Flügge (1973). However, when the tank is partially filled, the parametric terms due to the hydrostatic pressure (Urrutia, 1984) appear as follows

$$\begin{aligned} \overset{*}{L}_{11} &= \frac{(1-\upsilon^2)}{Eh}\left\{ r^2\breve{N}^{zz}\frac{\partial^2}{\partial z^2}+r^2\frac{\partial\breve{N}^{zz}}{\partial z}\frac{\partial}{\partial z}+r\breve{N}^{z\theta}\frac{\partial^2}{\partial z\partial\theta}+r\frac{\partial\breve{N}^{z\theta}}{\partial z}\frac{\partial}{\partial\theta}\right. \\ &\quad \left. +r\breve{N}^{\theta z}\frac{\partial^2}{\partial z\partial\theta}+r\frac{\partial\breve{N}^{\theta z}}{\partial\theta}\frac{\partial}{\partial z}+\breve{N}^{\theta\theta}\frac{\partial^2}{\partial\theta^2}+\frac{\partial\breve{N}^{\theta\theta}}{\partial\theta}\frac{\partial}{\partial\theta}\right\} & \overset{*}{L}_{31} &= \overset{*}{L}_{13} \\ \overset{*}{L}_{13} &= -\frac{r^2(1-\upsilon^2)}{Eh}\Pr\frac{\partial}{\partial z} & \overset{*}{L}_{32} &= \overset{*}{L}_{23} \\ \overset{*}{L}_{22} &= \overset{*}{L}_{11}-\frac{(1-\upsilon^2)}{Eh}\breve{N}^{\theta\theta}+\frac{r(1-\upsilon^2)}{Eh}\Pr & \overset{*}{L}_{33} &= -\overset{*}{L}_{22} \\ \overset{*}{L}_{23} &= \frac{(1-\upsilon^2)}{Eh}\left\{ r\breve{N}^{z\theta}\frac{\partial}{\partial z}+r\frac{\partial\breve{N}^{z\theta}}{\partial z}+2\breve{N}^{\theta\theta}\frac{\partial}{\partial\theta}+\frac{\partial\breve{N}^{\theta\theta}}{\partial\theta}+r\breve{N}^{\theta z}\frac{\partial}{\partial z}-r\Pr\frac{\partial}{\partial\theta}\right\} \end{aligned} \tag{2.2}$$

2.2. *Dynamic pressure*

For a rigid bottom tank the fluid is considered as non viscous and irrotacional, and the potential function that satisfies the Laplace's equation is

$$\phi=\sum_m\sum_n A(t)_{mn}I_n(\overline{m}_m\rho)\cos(\overline{m}_m z)\cos(n\theta) \tag{2.3}$$

where m is the longitudinal number mode and n is circumferential number mode and where the free surface motion is neglected.

The condition that considers coupling between the deformable shell and the liquid, is the boundary condition for the radial velocity

$$\left.\frac{\partial\phi}{\partial\rho}\right|_{z=d}=\frac{\partial w}{\partial t} \tag{2.4}$$

and we obtain the function

$$A(t)_{mn}=\frac{2i\omega e^{i\omega t}}{d\overline{m}_m I'_n(\overline{m}_m r)}\sum_k w_{kn}\int_0^d F_w(\lambda_k z)\cos(\overline{m}_m z)dz \tag{2.5}$$

By neglecting the effect of the free surface (sloshing), we accept that sloshing pressure is zero as follows

$$\rho_f \left.\frac{\partial \phi}{\partial t}\right|_{z=d} = 0 \tag{2.6}$$

Finally, the pressure equation on the shell wall where ρ= r, leads to the following conditions

$$P_d = \omega^2 \sum_m \sum_n \sum_k w_{kn}\ \rho_v \cos(n\theta)e^{i\omega t} \tag{2.7}$$

with

$$\rho_v = \rho_f \frac{2I_n(\overline{m}_m r)\cos(\overline{m}_m z)}{d\overline{m}_m I_n'(\overline{m}_m r)} \int_0^d F_w(\lambda_k z)\cos(\overline{m}_m z)dz \qquad\qquad \gamma = \left(1 + \frac{\rho_v}{\rho_c}\right) \tag{2.8}$$

where the quotient (ρ_v / ρ_c) defined by Kwak and Kim (1991), is the virtual added mass factor, ρ_v is the virtual mass of the fluid-tank system, ρ_f is the fluid density, and $I_n(\overline{m}_m r)$ is the modified function of Bessel of the first kind and first order *n*. Virtual mass ρ_v is different to those presented by Koga et al. (1990), who only presented results for natural frequencies when the tank is full or empty. The case of tanks partially filled with water is avoided.

2.3. Covariant and contravariant functions method

There are many methods to uncoupling the system equations of the fluid tank system, in this work we use covariant and contravariant functions (Urrutia, 1992). The displacement field u, v, w are proposed to satisfy boundary conditions for the shell CF

$$\begin{aligned}
u &= \sum_m \sum_n u_{mn} F_u(\lambda_m z)\cos(n\theta)e^{i\omega t} \\
v &= \sum_m \sum_n v_{mn} F_v(\lambda_m z) sen(n\theta)e^{i\omega t} \\
w &= \sum_m \sum_n w_{mn} F_w(\lambda_m z)\cos(n\theta)e^{i\omega t}
\end{aligned} \tag{2.9}$$

Inner product of functions is used in the motion equations where contravariant functions *U*, *V*, *W* are orthogonal to covariant functions *u*, *v*, *w*, and should be satisfied

$$\langle u, U\rangle = \iint_S u\,U dS = u_{mn} \qquad \langle v, V\rangle = v_{mn} \qquad \langle w, W\rangle = w_{mn} \tag{2.10}$$

and the fluid-tank system becomes

$$\left\langle \left[\left(L_{11} + \overset{*}{L}_{11} + \omega^2\tau^2 \right) u + L_{12} v + \left(L_{13} + \overset{*}{L}_{13} \right) w \right], U \right\rangle = 0$$
$$\left\langle \left[L_{21} u + \left(L_{22} + \overset{*}{L}_{22} + \omega^2\tau^2 \right) v + \left(L_{23} + \overset{*}{L}_{23} \right) w \right], V \right\rangle = 0 \quad \Rightarrow \begin{bmatrix} l_{11} + \omega^2\tau^2 & l_{12} & l_{13} \\ l_{21} & l_{22} + \omega^2\tau^2 & l_{23} \\ l_{31} & l_{32} & l_{33} - \omega^2\tau^2\gamma \end{bmatrix} \begin{bmatrix} u_{mn} \\ v_{mn} \\ w_{mn} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$
$$\left\langle \left[\left(L_{31} + \overset{*}{L}_{31} \right) u + \left(L_{32} + \overset{*}{L}_{32} \right) v + \left(L_{33} + \overset{*}{L}_{33} - \omega^2\tau^2\gamma \right) w \right], W \right\rangle = 0 \tag{2.11}$$

This is a set of equations where the matrix determinant only depends on the value of the natural frequency ω of fluid-tank system.

3. EXPERIMENTAL DATA

With the purpose to compare the theoretical fundamental frequencies associated to its critical circumferential modes (from the mathematical model of this study) with the experimental data studies, we present the experimental studies from Mazúch et al. (1996) and Mistry et al. (1995) for steel cylinders with different geometric characteristics. Cylinders, C1 to C4, with fundamental frequencies, geometric and mechanical characteristics are presented in the Table 3.1.

Table 3.1 Experimental fundamental frequencies (Exp), m=1, steel

	C1		**C2**		**C3**		**C4**	
	Mazuch et al. 1996		**Mistry et al. 1995**		**Mistry et al. 1995**		**Mistry et al. 1995**	
L [mm]	231		280.1		325.5		398	
r [mm]	77.25		99.325		99.41		99.58	
h [mm]	1.5		0.65		0.82		1.16	
E [N/mm2]	2.05E5		2.05E5		2.05E5		2.05E5	
ρ_c [Ns2/mm4]	7.8E-9		7.75E-9		7.75E-9		7.75E-9	
d/L	**ω* [Hz]**	**n***	**ω* [Hz]**	**n***	**ω* [Hz]**	**n***	**ω* [Hz]**	**n***
0	616	3	---	---	---	---	---	---
0.5	---	---	276	4	---	---	---	---
0.697	522	3	---	---	---	---	---	---
0.7	---	---	---	---	213	3	---	---
0.8	---	---	---	---	---	---	190	3
1	388	3	---	---	---	---	---	---

4. THEORETICAL RESULTS

To determine the fundamental frequency ω^* of a cylindrical tank, is necessary to solve the fluid-tank system for the first critical longitudinal mode *(m=1)* and different circumferential waves *n*, being the minimum frequency the fundamental one that is associate to the critical circumferential mode *n**.

Real cylinders are analyzed and the absolute error between mathematical model results and experimental data are presented in Table 4.1. These differences show an excellent approximation for the water level dimensionless parameter *d/L*. The error in the theoretical result for the water level *d/L* is due mainly to the virtual mass given by Eqn. 2.8. This last equation presents a very good convergence in the mathematical model results. The excellent approximation that we get with the mathematical model allows inferring that the free surface terms in the fluid-tank system are not significant for the fundamental frequency assessments.

Table 4.1 Fundamental frequencies MM, m=1, steel

	C1		C2		C3		C4	
d/L	**ω* [Hz]**	**e [%]**	**ω* [Hz]**	**e [%]**	**ω* [Hz]**	**e [%]**	**ω* [Hz]**	**e [%]**
0	672.1	+9.11	---	---	---	---	---	---
0.5	---	---	282.6	+2.38	---	---	---	---
0.697	509.3	-2.43	---	---	---	---	---	---
0.7	---	---	---	---	208.7	-2.03	---	---
0.8	---	---	---	---	---	---	180.9	-4.79
1	396.2	+2.12	---	---	---	---	---	---

For a CF vertical cylindrical steel tank partially filled with water experimental data demonstrated that the critical circumferential mode *n** stay constant without importing the water level, and this is proven with the mathematical model, like it is shown for the cylinders C1 to C4 in Table 4.2 where the experimental number wave *n** it is shown with its integer value and theoretical *n** it is shown with her value between parenthesis (*n**).

Table 4.2 Critical circumferential mode MM, m=1, steel

	C1		C2		C3		C4	
d/L	**n***	**e [%]**	**n***	**e [%]**	**n***	**e [%]**	**n***	**e [%]**
0	3	0	(4)	---	(3)	---	(3)	---
0.5	(3)	---	4	0	(3)	---	(3)	---
0.697	3	0	(4)	---	(3)	---	(3)	---
0.7	(3)	---	(4)	---	3	0	(3)	---
0.8	(3)	---	(4)	---	(3)	---	3	0
1	3	0	(4)	---	(3)	---	(3)	---

5. REGRESSION MODEL

From the results of the mathematical model a parametric analysis for the steel cylinders is carried out, in the dimensionless parameters *r/h* and *L/r* from 50 to 1000 and from 1 to 10, respectively. In this intervals the mathematical pattern leads to excellent convergence in the calculation of the fundamental frequencies in the first way longitudinal mode m=1 and its corresponding critical circumferential wave number n^*.

The regression model (RM) that is proposed was based from Arango´s work et al. (1989) who presented two equations to calculated fundamental frequency and critical circumferential mode to simple supported horizontal empty cylindrical shells. Those expressions are now generalized to calculate fundamental frequencies from longitudinal mode m=1 and critical circumferential mode to CF vertical cylindrical steel tanks partially filled with water.

The mathematical model is calibrated by multiple regression analysis by a minimal square method that gives Eqns. 5.1 to determine fundamental frequency and critical circumferential mode, with an accuracy factor R^2=0.990 and R^2=0.908, respectively

$$(\omega^* r)^2 = \frac{1.8E}{\rho_c\left(1-\upsilon^2\right)\left[1+0.1\left(\frac{d}{L}\right)^{2.3}\left(\frac{L}{r}\right)^{0.5}\left(\frac{r}{h}\right)^{0.6}\right]}\left(\frac{L}{r}\right)^{-2}\left(\frac{r}{h}\right)^{-1} \qquad n^* \approx 0.0015\left(\frac{E}{\rho_c\left(1-\upsilon^2\right)}\right)^{1/4}\left(\frac{L}{r}\right)^{-0.6}\left(\frac{r}{h}\right)^{0.4} \tag{5.1}$$

where mechanical characteristics of steel are the elasticity module E, mass density ρ_c and Poisson module ν.

To compare how regression model, Eqns. 5.1, fits with experimental data we present Tables 5.1 and 5.2, and show fits the error between both models. The regression model for the fundamental frequencies has an excellent accuracy with experimental data because the maximum absolute error is 11.01%. Therefore we propose to use them to calculate fundamental frequency and circumferential mode to predesigned CF vertical cylindrical steel tanks partially filled with water comparing in a direct way against the amplitude frequencies accelerogram.

Table 5.1 Fundamental frequencies RM, m=1, steel

	C1		C2		C3		C4	
d/L	**ω* [Hz]**	**e [%]**	**ω* [Hz]**	**e [%]**	**ω* [Hz]**	**e [%]**	**ω* [Hz]**	**e [%]**
0	692.23	+11.01	---	---	---	---	---	---
0.5	---	---	276	-7.53	---	---	---	---
0.697	515.62	+1.24	---	---	---	---	---	---
0.7	---	---	---	---	213	+3.00	---	---
0.8	---	---	---	---	---	---	190	+0.57
1	410.73	+5.53	---	---	---	---	---	---

Tabla 5.2 Critical circumferential mode RM, m=1, steel

	C1		C2		C3		C4	
d/L	**n***	**e [%]**	**n* [Hz]**	**e [%]**	**n* [Hz]**	**e [%]**	**n* [Hz]**	**e [%]**
0	3	0	---	---	---	---	---	---
0.5	---	---	4	0	---	---	---	---
0.697	3	0	---	---	---	---	---	---
0.7	---	---	---	---	4	33.33	---	---
0.8	---	---	---	---	---	---	3	0
1	3	0	---	---	---	---	---	---

It would be necessary to call more experimental data to improve the regression model and verify if the equations of the regression model uniformly converge where mathematical model does not converge.

6. CONCLUSIONS

Mathematical model are presented to determine the natural frequencies of CF vertical cylindrical tanks partially filled with water obtained from the improved Flugge's equations uncoupled with the covariant and contravariant modal forms. The model includes initial stresses terms because of the hydrostatic pressure and dynamic pressure is consider like a virtual mass where free surface influence and the nonlinear thin shells theory effects are neglected.

From the approximation of the mathematical model to the fundamental frequency ω^* and the critical circumferential mode n^*, it can be observed that the proposal virtual mass equation presents an excellent approximation for the different water level d/L, so is possible to neglect the terms influenced by the free surface in the fluid-tank system equations.

Staring from experimental data and the mathematical model we propose an explicit expression from a regression model for the fundamental frequency ω^* and an expression for the critical circumferential mode n^*. The regression models given by Eqns. 5.1 present a good approximation with the literature experimental data. So it can be used to determine the fundamental frequencies ω^* and critical circumferential modes n^* with a simple an accurate calculation to predesigned CF vertical cylindrical steel tanks partially filled with water comparing in a direct way against the amplitude frequencies accelerogram.

Once we have more experimental data is necessary to include them for extend the study and confirm the precision of the mathematical model and regression model. With a higher amount of numerical simulations is possible to obtain a general equation from the regression model for the fundamental frequencies calculus for any kind of shell's material.

REFERENCES

Arango Bedoya L J and Urrutia Galicia J.L. (1989). Vibración libre en cascarones cilíndricos rigidizados. *Instituto de Ingeniería* **521**.

Chiba M., Yamaki N. and Tani J. (1984). Free vibration of a clamped-free circular cylindrical shell partially filled with liquid-part I: Experimental results. *Thin-Walled Structures* **2,** 265-284.

Chiba M., Yamaki N. and Tani J. (1985). Free vibration of a clamped-free circular cylindrical shell partially filled with liquid-part III: Experimental results. *Thin-Walled Structures*, **3,** 1-14.

El Mously M, (2003). Fundamental natural frequencies of thin cylindrical shells: a comparative study. *Journal of Sound and Vibration* **264,** 1167-1186.

Flügge W. (1973). Stresses in Shells, Springer-Verlag, New York, second edition, 222-225.

Koga T. and Tsushima M. (1990). Breathing vibrations of a liquid-filled circular cylindrical shell. *International Journal of Solids and Structures* **26,** 1005-1015.

Kwak M.K. and Kim K.C. (1991). Axisymmetric vibration of circular plates in contact with fluid. *Journal of Sound and Vibration* **146,** 381-389.

Lakis A.A. y Neagu S. (1997). Free surface effects on the dynamics of the cylindrical shells partially filled with liquid. *Journal of Sound and Vibration* **207,** 669-690.

Mazúch T., Horácek J., Trnka J. y Velesý J. (1996). Natural modes and frequencies of a thin clamped-free stell cylindrical storage tank partially filled with water: FEM and measurement. *Journal of Sound and Vibration* **193,** 669-690.

Mistry J. y Menezes J.C. (1995). Vibration of cylinders partially filled with liquid. Journal of Vibration and Acoustics **117,** 87-93.

Urrutia Galicia J.L. (1984), "Stability of fluid filled circular cylindrical shells", PhD Thesis, University of Waterloo, Ontario, Canada.

Urrutia Galicia J.L. (1989), "On the natural frequencies of thin simply supported cylindrical shells", Transactions of the CSME, **13: 1/2,** 35-40.

Urrutia Galicia J.L. (1992), "On the existence of covariant and contravariant modal forms of dynamic analysis", Transactions of the CSME, **16:2,** 201-217.