Stochastic Fluctuation of RC Frame Structures in Seismic Region

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ABSTRACT :

The variation of the structure response stems from the randomness involved both in the loads and in the structural parameters. Although the load randomness is deemed as the dominant one by many researchers, the debate on the contribution of the two sources to the variation of the structure response is still going on. In order to help to clarify this question, based on the Taylor expansion of nonlinear stochastic function, the feasibility of stochastic fluctuation is verified. Introducing the global sensitivity index, which considers the relative contribution of each variable to the total variance of the target functions, and adopting the probability density evolution method (PDEM), the contribution of random variables to target quantities be explained rationally and quantitatively, and the importance of load randomness to second statistical moments of the responses is verified. Finally, based on the above conclusions, stochastic fluctuation of the response of a 9-story 3-bay RC frame designed by fortification intensity 6 is analyzed, and it can be concluded from the analysis results that the relative stochastic fluctuation is obvious, but different for different responses, varies with the nonlinear development process.

KEYWORDS: Stochastic Fluctuation, Probability Density Evolution Method, Global Sensitivity Index

1. INTRODUCTION

Based on the concept of theory of reflection (Li 1996), the randomness derives from uncontrollability. And the responses of structure depend on both structural parameters and loads. Obviously, the randomness of structural responses is influenced by both structural parameters and loads. And, theoretically, the most rational analysis approach for stochastic system should be involved in both random sources. However, it is more complex than the one involved in only random parameters or only random loads. On the one hand, there must be more random variables for the system with both random parameters and random loads; on the other hand, it is more complex in algorithm when both structural parameters and loads are random.

In order to simplify the analysis of stochastic systems, there are two approaches, the one is replacing the complex systems with systems involved in only random parameters, and the other is reducing it into systems only with random loads. But which one is more rational? Although the randomness arising from load is deemed as the dominant one by many researchers, the debate on the contribution of the two sources to the variation of the structure response is still going on.

In this paper, based on the Taylor expansion of nonlinear stochastic function, the feasibility of stochastic fluctuation of the second order statistical moments is verified. And taking the RC frame in seismic region as example, the stochastic fluctuation of responses for structure under equivalent static load is analyzed, then introducing the global sensitivity, the contribution of variables to the stochastic fluctuation of responses is illustrated quantitative.

2. STOCHASTIC FLUCTUATION ANALYSIS BASED ON TAYLOR EXPANSION

2.1 introduction of stochastic fluctuation

Fluctuation, which mainly uses in the field of stochastic physics, stochastic mechanics and non-equilibrium statistical mechanics, is the deviation between the value of physical quantity and its macro mean value. Obviously, fluctuation is a random variable, so is also called as stochastic fluctuation. Up to now, the stochastic fluctuation usually describes by the variance of deviation quantitatively, what is more, relative fluctuation, which is equal to the standard variance divided by sample value, is introduced to describe the relatively variety of stochastic fluctuation(Hu 1994; Kadanoff 2000). Clear, relative fluctuation is nearly the coefficient of variation (COV) for deviation. So, for simplicity, in this paper the COV is adopted to describe the stochastic fluctuation.

2.2 Theoretical analysis of stochastic fluctuation based on Taylor expansion

Without loss of generality, the physical equation for structures under static load follows

$$X = G(\zeta, \boldsymbol{\Theta}, F) \tag{1}$$

where X is one of responses, such as sectional moment, shear force; is the vector of deterministic structural parameters; $\boldsymbol{\Theta} = [\boldsymbol{\Theta}_1, \cdots, \boldsymbol{\Theta}_m]'$ is the vector of random structural parameters and $F = [F_1, \cdots, F_n]'$ is the vector of random loads. For stating conveniently, define the joint random vector Ξ which reads

$$\Xi = [\boldsymbol{\Theta}, F]' = [\boldsymbol{\Theta}_1, \cdots, \boldsymbol{\Theta}_m, F_1, \cdots, F_n]' = [\Xi_1, \cdots, \Xi_m, \Xi_{m+1}, \cdots, \Xi_{m+n}]' \tag{2}$$

Taking the Taylor expansion of $G(\zeta, \boldsymbol{\Theta}, F)$ at $\Xi = \Xi_0 = [\Xi_{1,0}, \cdots, \Xi_{m,0}, \Xi_{m+1,0}, \cdots, \Xi_{m+n,0}]'$, where Ξ_0 is the mean value of Ξ, $G(\zeta, \boldsymbol{\Theta}, F)$ can be expanded as follows:

$$\begin{aligned} G(\zeta, \boldsymbol{\Theta}, F) &= G(\zeta, \Xi) = G(\zeta, \Xi_0) + \sum_{i=1}^{m+n} \frac{\partial G}{\partial \Xi_i}\Big|_{\Xi=\Xi_0} \cdot (\Xi_i - \Xi_{i,0}) \\ &+ \frac{1}{2} \sum_{i1=1}^{m+n} \sum_{i2=1}^{m+n} \left[\frac{\partial^2 G}{\partial \Xi_{i1} \partial \Xi_{i2}}\Big|_{\Xi=\Xi_0} \cdot (\Xi_{i1} - \Xi_{i1,0})(\Xi_{i2} - \Xi_{i2,0}) \right] + \cdots \end{aligned} \tag{3}$$

Then the variance of X can be calculated by:

$$\begin{aligned} D[X] &= \sum_{i=1}^{m+n} \left(\frac{\partial G}{\partial \Xi_i}\Big|_{\Xi=\Xi_0} \right)^2 \cdot D[\Xi_i] + 2 \sum_{i1=1}^{m+n-1} \sum_{i2=i1+1}^{m+n} \left(\frac{\partial G}{\partial \Xi_{i1}} \cdot \frac{\partial G}{\partial \Xi_{i2}}\Big|_{\Xi=\Xi_0} \right) \cdot \mathrm{Cov}[\Xi_{i1}, \Xi_{i2}] + \\ & \sum_{i3=1}^{m+n} \sum_{i4=1}^{m+n} \sum_{i1=1}^{m+n} \sum_{i2=1}^{m+n} \left(\frac{1}{4} \frac{\partial^2 G}{\partial \Xi_{i1} \partial \Xi_{i2}} \cdot \frac{\partial^2 G}{\partial \Xi_{i3} \partial \Xi_{i4}}\Big|_{\Xi=\Xi_0} \right) \cdot \mathrm{Cov}\Big[(\Xi_{i1} - \Xi_{i1,0})(\Xi_{i2} - \Xi_{i2,0}), \\ & (\Xi_{i3} - \Xi_{i3,0})(\Xi_{i4} - \Xi_{i4,0}) \Big] + \sum_{i3=1}^{m+n} \sum_{i1=1}^{m+n} \sum_{i2=1}^{m+n} \left(\frac{\partial G}{\partial \Xi_{i3}} \cdot \frac{\partial^2 G}{\partial \Xi_{i1} \partial \Xi_{i2}}\Big|_{\Xi=\Xi_0} \right) \\ & \cdot \mathrm{Cov}\Big[(\Xi_{i3} - \Xi_{i3,0}), (\Xi_{i1} - \Xi_{i1,0})(\Xi_{i2} - \Xi_{i2,0}) \Big] + \cdots \end{aligned} \tag{4}$$

where $D[\cdot]$ is variance and $\mathrm{Cov}(\cdot,\cdot)$ denotes the covariance.

If the components of Ξ are independent, Eq. (4) will be reduced into

$$\begin{aligned} D[X] &= \sum_{i=1}^{m+n} \left(\frac{\partial G}{\partial \Xi_i}\Big|_{\Xi=\Xi_0} \right)^2 \cdot D[\Xi_i] + \sum_{i3=1}^{m+n} \sum_{i4=1}^{m+n} \sum_{i1=1}^{m+n} \sum_{i2=1}^{m+n} \left\{ \left(\frac{1}{4} \frac{\partial^2 G}{\partial \Xi_{i1} \partial \Xi_{i2}} \cdot \frac{\partial^2 G}{\partial \Xi_{i3} \partial \Xi_{i4}}\Big|_{\Xi=\Xi_0} \right) \right. \\ & \left. \cdot \mathrm{Cov}\Big[(\Xi_{i1} - \Xi_{i1,0})(\Xi_{i2} - \Xi_{i2,0}), (\Xi_{i3} - \Xi_{i3,0})(\Xi_{i4} - \Xi_{i4,0}) \Big] \right\} \\ & + \sum_{i3=1}^{m+n} \sum_{i1=1}^{m+n} \sum_{i2=1}^{m+n} \left(\frac{\partial G}{\partial \Xi_{i3}} \cdot \frac{\partial^2 G}{\partial \Xi_{i1} \partial \Xi_{i2}}\Big|_{\Xi=\Xi_0} \right) \cdot \mathrm{Cov}\Big[(\Xi_{i3} - \Xi_{i3,0}), (\Xi_{i1} - \Xi_{i1,0})(\Xi_{i2} - \Xi_{i2,0}) \Big] + \cdots \end{aligned} \tag{5}$$

It can be found from Eq. (5) that stochastic fluctuation of responses depend on both the randomness of structural parameters and the one of loads. When X is replaced by the its linear terms, the variance of responses is the linearly weighted sum of the contribution of all random variables, where the weighted coefficient is the square of the partial derivative for X on Ξ_i. As known to all that responses of linear structures can be formulated by the linear function of loads only, but not the linear function of structural parameters, what's more, structural parameters are coupled with loads. Therefore, the linear approximation of X is exact for linear structures only with random loads, and for the other cases, the precise of linear approximation is not enough.

Besides the stochastic fluctuation of structures under entire loads, the one of structures during the loading process is also significant, and the former is just the part of the latter.

Assume the loads increase proportionally, i.e. $F = F_0\tau$, where F_0 is the nominal load parameter and τ is the load factor, and the nonlinear development equation of structures follows:

$$X = G(\zeta, \boldsymbol{\Theta}, F_0\tau) = G(\zeta, \boldsymbol{\Theta}, F_0, \tau) \tag{6}$$

Because τ is not a random variable, the formula for variance of X doesn't change and is still Eq. (5) in format except G varying with τ.

Taking linear structures with proportional loads as example, we investigate the development process of stochastic fluctuation. The responses can be calculated by (Zhong et. al 1989; Shen 1989)

$$\boldsymbol{\Delta} = \left[K(\boldsymbol{\Theta})\right]^{-1} F_0\tau \tag{7a}$$

$$\{\boldsymbol{X}\}^e = \left[K(\boldsymbol{\Theta})\right]^e \{\boldsymbol{\Delta}\}^e \tag{7b}$$

where $K(\cdot)$ is stiffness matrix, $\boldsymbol{\Delta}$ is the vector of displacement response, superscript 'e' denotes element, and X is the vector of internal forces. So the internal forces can reduces into

$$X = G(\boldsymbol{\Theta}) F_0\tau \tag{8}$$

By comparing Eq. (8) with Eq. (7a), it can be found that Eq. (8) is also fit for $\boldsymbol{\Delta}$. Then the relative stochastic fluctuation varying with τ follows

$$\delta[X] = \frac{\tau\sigma\left[G(\boldsymbol{\Theta})F_0\right]}{\tau\mu\left[G(\boldsymbol{\Theta})F_0\right]} = \frac{\sigma\left[G(\boldsymbol{\Theta})F_0\right]}{\mu\left[G(\boldsymbol{\Theta})F_0\right]} \tag{9}$$

Obviously, during the process of loading, the relatively stochastic fluctuation keeps constant. On the other hand, this conclusion is not true for nonlinear structures. In other word, stochastic fluctuation or relative stochastic fluctuation derives from randomness, while the irregularity of nonlinear development for relative stochastic fluctuation comes from the couple between randomness and nonlinearity.

3. GLOBAL SENSITIVITY INDEX

According to Eq. (1), the probabilistic structure of X depends on the one of Ξ, but the contribution of each Ξ_i to X is quite different. Up to now, researchers usually focus on the study of sensitivity index, which is the partial differential of X on Ξ_i. Obviously, the sensitivity index reflects the contribution partially. For example, the sensitivity index of X on Ξ_1 is bigger than the other random parameters, but the variance of Ξ_1 itself is much smaller, so the contribution Ξ_i to X is not essential according to Eq. (5).

Recently, Chen & Li (2008a) proposed a more rational sensitivity index, that is global sensitivity index. Its basic idea is to considers the relative contribution of Ξ_i to the total variance of X entirely.

Firstly, assume Ξ_i is a deterministic variable and the value of Ξ_i is equal to its mean value $\bar{\xi}_i$, then the COV of X reads

$$\delta\left[X|_{\Xi_{\bar{i}}}\right] = \sigma\left[X|_{\Xi_{\bar{i}}}\right] \Big/ E\left[X|_{\Xi_{\bar{i}}}\right] \tag{10}$$

where $\Xi_{\bar{i}} = \left[\Xi_1, \cdots, \Xi_{i-1}, \Xi_{i+1}, \Xi_{m+n}\right]$ means the subset of Ξ without the component Ξ_i, $\sigma[\cdot]$ is standard variance and $E[\cdot]$ is mean value. Secondly, suppose Ξ_i is a random variable and the COV of X reads

$$\delta\left[X|_{\Xi}\right]=\sigma\left[X|_{\Xi}\right]/E\left[X|_{\Xi}\right] \tag{11}$$

Obviously, the difference between Eqs. (10) and (11) lies on the contribution of the randomness for Ξ_i, so the global sensitivity index of COV can be defined as

$$S_{\delta,\Xi_i}=\left(\delta\left[X|_{\Xi}\right]-\delta\left[X|_{\Xi_{\bar{i}}}\right]\right)/\delta\left[X|_{\Xi}\right] \tag{12}$$

And the index can describe the contribution of Ξ_i to X entirely.

4. GENRTALIZED DENSITY EVOLUTION EQUATION OF NONLINEAR DEVELOPMENT PROCESS

The full range of nonlinear development process for structures under proportional loads can be formulated as Eq. (6). Therefore, for a given sample of Ξ, i.e. ξ, there must exist a formula as follows

$$p_{X|\Xi}\left(x,\tau|\xi\right)=\delta\left[x-G\left(\zeta,\boldsymbol{\Theta},F_0,\tau\right)\right]=\delta\left[x-G\left(\zeta,\xi,\tau\right)\right] \tag{13}$$

where $p_{X|\Xi}(\cdot)$ is the conditional probability density function (PDF) of X and $\delta[\cdot]$ the one-dimensional Dirac delta function. Actually, Eq. (13) can be regarded as the result of random event description of the principle of preservation of probability.

After differential operation and reduction, Eq. (13) becomes

$$\frac{\partial p_{X|\Xi}\left(x,\tau|\xi\right)}{\partial\tau}+\frac{\partial G\left(\zeta,\xi,\tau\right)}{\partial\tau}\cdot\frac{\partial p_{X|\Xi}\left(x,\tau|\xi\right)}{\partial x}=0 \tag{14}$$

According to the conditional PDF formula, Eq. (14) can be translated into

$$\frac{\partial p_{X,\Xi}\left(x,\xi,\tau\right)}{\partial\tau}+\frac{\partial G\left(\zeta,\xi,\tau\right)}{\partial\tau}\cdot\frac{\partial p_{X,\Xi}\left(x,\xi,\tau\right)}{\partial x}=0 \tag{15}$$

of which the initial condition reads

$$p_{X,\Xi}\left(x,\xi,\tau\right)\big|_{\tau=0}=\delta\left(x\right)p_{\Xi}\left(\xi\right) \tag{16}$$

Therefore, $p_X\left(x,\tau\right)$ can be obtained by

$$p_X\left(x,\tau\right)=\int p_{X,\Xi}\left(x,\xi,\tau\right)\mathrm{d}\xi \tag{17}$$

And the Eq. (15) is named as the GDEE of the nonlinear development process.

For general nonlinear structures, $\dfrac{\partial G\left(\zeta,\xi,\tau\right)}{\partial\tau}$ is usually unknown, so it is impossible to obtain the analytical solution of Eq. (15). Nonetheless, GDEE can be numerically solved by the following steps:

① Select discretized representative points in the domain of the random vector, and the strategy of selecting points can be a lattice grid, a tangent sphere point set (Chen & Li 2008b) or a number theoretical point set (Li & Chen 2007).

② For a given representative point, obtain $\dfrac{\partial G\left(\zeta,\xi,\tau\right)}{\partial\tau}$ from the full range nonlinear analysis of the deterministic structure.

③ Solve Eq. (15) under the corresponding initial condition to obtain the numerical solution.

④ Take numerical integration to get the numerical solution of $p_X\left(x,\tau\right)$.

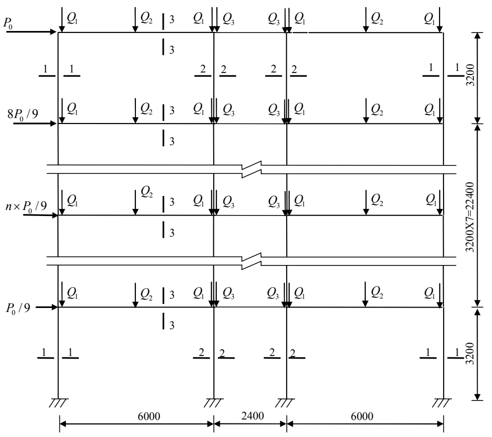
Based on $p_X\left(x,\tau\right)$, $E\left[X|_{\Xi_{\bar{i}}}\right]$ and $\sigma\left[X|_{\Xi_{\bar{i}}}\right]$ or $E\left[X|_{\Xi}\right]$ and $\sigma\left[X|_{\Xi}\right]$ can be calculated by one-dimensional integral. Clear, it is easy to obtain the nonlinear development process of global sensitivity index.

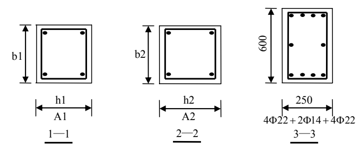
5. NUMERICAL CASE

Consider a three-bay nine-story RC frame designed by fortification intensity 6, of which the details are show in Fig. 1. Randomness is involved in loads, dimension of section and strength of materials. Q_1, Q_2 and Q_3 are perfect correlated and P_0, which is equivalent static load of earthquake action, is independent with them, and dimension of all sections are perfect correlated too. The statistical parameters for random variables follow: $\mu_{f_y} = 352\text{M}Pa$, $\mu_{f_c} = 25\text{M}Pa$, the mean value of dimension shows in Fig .1, $\mu_{Q_1} = 100\text{k}N$, $\mu_{Q_2} = 60\text{k}N$, $\mu_{Q_3} = 30\text{k}N$, and μ_{P_0} increases gradually, $\delta_{f_y} = 0.03$, $\delta_{f_c} = 0.09$, the COV of dimensions is 0.03, $\delta_{Q_1} = \delta_{Q_2} = \delta_{Q_3} = 0.09$ and $\delta_{P_0} = 0.2$.

According to different combinational cases of random variables which list in Tab. 1, the stochastic fluctuations and their nonlinear development process for several typical responses, such as the moment and shear force for the bottom section of the left column, the moment and shear force for the left section of the beam at the first floor and the displacement of the top left node, show in Fig. 2 to Fig. 6 (in which $r = \mu_{P_0}/100$).

Based on the foregoing results, the nonlinear development process for global sensitivity indexes of different responses on dimensions, material parameters and load illustrate in Fig. 7 to Fig. 11 respectively.





The details of column section

floor	h1	b1	A1	h2	b2	A2
1-3	400	400	8Φ18	450	400	8Φ20
4-6	350	350	4Φ18 +4Φ14	400	350	4Φ20 +4Φ14
7-9	300	300	4Φ18	350	300	4Φ20

Fig. 1 The details of RC frame

Tab. 1 Combinational cases of random variables

case	1	3	5	7
variables	$P_0, Q_1, Q_2, Q_3, f_c, f_y, h, b$	P_0, Q_1, Q_2, Q_3, h, b	P_0, Q_1, Q_2, Q_3	h, b
case	2	4	6	
variables	$P_0, Q_1, Q_2, Q_3, f_c, f_y$	f_c, f_y, h, b	f_c, f_y	

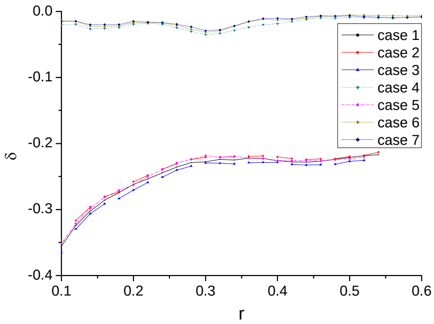


Fig. 2 Nonlinear development process of stochastic fluctuation for the moment of the bottom section of the left column

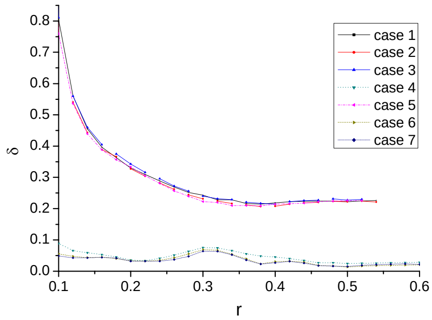


Fig. 3 Nonlinear development process of stochastic fluctuation for the shear force of the bottom section of the left column

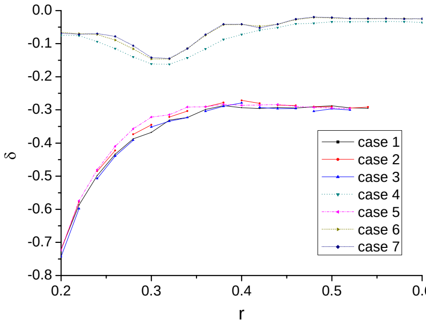


Fig. 4 Nonlinear development process of stochastic fluctuation for the moment of the left section of the beam at the first floor

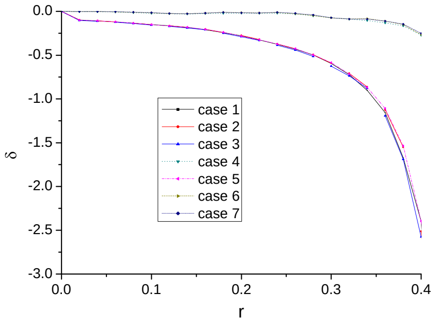


Fig. 5 Nonlinear development process of stochastic fluctuation for the shear force of the left section of the beam at the first floor

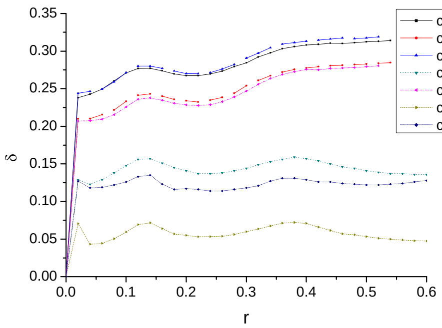


Fig. 6 Nonlinear development process of stochastic fluctuation for the displacement of the top left node

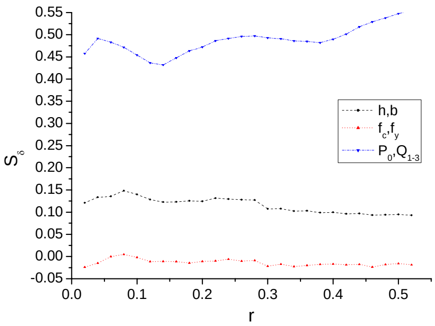


Fig. 7 Nonlinear development process for global sensitivity indexes of the displacement of the top left node

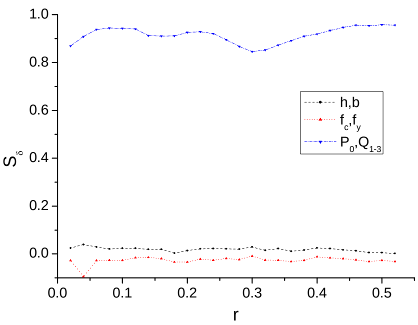


Fig. 8 Nonlinear development process for global sensitivity indexes of the moment of the bottom section of the left column

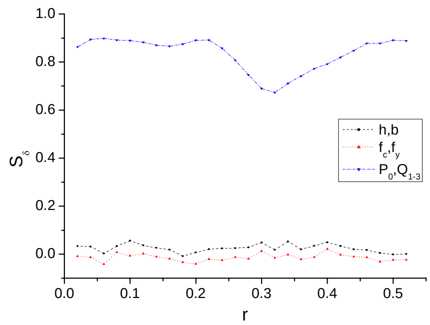


Fig. 9 Nonlinear development process for global sensitivity indexes of the shear force of the bottom section of the left column

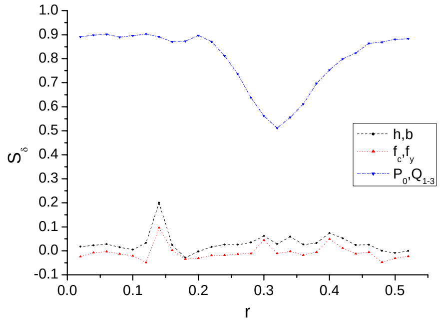


Fig. 10 Nonlinear development process for global sensitivity indexes of the moment of the left section of the beam at the first floor

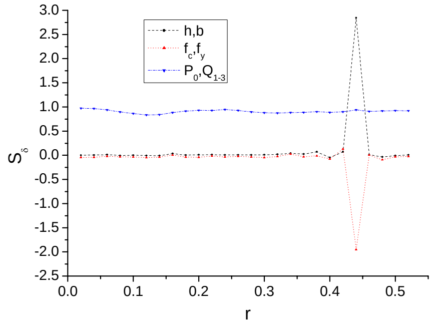


Fig. 11 Nonlinear development process for global sensitivity indexes of the shear force of the left section of the beam at the first floor

From the results mentioned above, the relative stochastic fluctuation is obvious, but different for different responses, on the other hand, it varies with the nonlinear development process. Among all of the random variables, the contribution of loads to the randomness of responses is essential, but the influence of dimensions and strength of materials is relative small. What is more, the global sensitivity index varies with the nonlinear development process.

It is worthy to point out that the phenomenon of stochastic fluctuation may be larger than the results obtained in this paper, because the complex reversed loads will speed up the structural damage while this behavior can't be described for proportional static loads.

6. CONLUSION

Based on the Taylor expansion of nonlinear stochastic function, the phenomenon of stochastic fluctuation in stochastic systems is verified, and the resources for stochastic fluctuation of responses are not only from the randomness of variables themselves, but also from some other factors, such as the behavior of structures. In order to quantify the importance of each random variable, introducing the global sensitivity index and adopting the probability density evolution method (PDEM), the nonlinear development process of relative stochastic fluctuation and the one of global sensitivity index is analyzed based on a 9-story 3-bay RC frame designed by fortification intensity 6 is analyzed, and it can be concluded from the analysis results that the relative stochastic fluctuation is obvious, but different for different responses. What is more, the contribution of loads to the randomness of responses is essential, but the influence of dimensions and strength of materials is relative small, so, in the sense of second statistical moments, it is more rational to consider the randomness of loads than the other variables.

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