PRINCIPLE OF STATIONARY TOTAL POTENTIAL ENERGY IN STRUCTURAL DYNAMICS: THEORY AND APPLICATIONS

Ying Wen[1], Qing-yuan Zeng[2]

[1]*Ph.D. candidate, Dept. of Bridge Engineering, Central South University, Changsha, China*
[2]*Professor, Dept. of Bridge Engineering, Central South University, Changsha, China*
Email: zhwen@mail.csu.edu.cn

ABSTRACT:

In this work we derive the principle of stationary total potential energy in structural dynamics firstly presented by the second of the authors and demonstrate its application to the formulation of structural dynamic problems. The analysis procedure is performed in terms of pre-defined potential energy of the vibrating structure. The implementation implies that the entire structural system plus the load system can be regarded as conservative in this study. It will be of interest to note that this principle is an extension of the principle of stationary potential energy in statics to account for dynamic effects. In practice, the proposed theory is used in conjunction with a new matrix assembly technique called the "set-in-right-position" rule also proposed by us to obtain finite element formulations for vibrations of engineering structures. Such an approach enables another way of efficient formulating the equations of motion of a complete finite element model. Finally, several illustrative examples are presented to testify the approach and to discuss the applicability of the present formulation.

KEYWORDS: Equations of motion; Finite element formulation; Principle of stationary total potential energy in structural dynamics; "Set-in-right-position" rule

1. INTRODUCTION

In structural vibration response analysis, the equations of motion describing the essential feature of the vibration must be appropriately formulated prior to determining the time-histories of related structural variables. Clough and Penzien (2003) in their well-known monograph summarized the role of dynamic formulation as follows: the formulation of the equations of motion of a dynamic system is possibly the most important, and sometimes the most difficult, phase of the entire analysis procedure.

Historically, there are many ways to derive the equations of motion for a vibrating body. Among them, the most common approaches include: (i) Direct equilibration using d'Alembert's principle, (ii) Principle of virtual displacements, (iii) Lagrange's equations of motion, and (iv) Hamilton's principle (or the so called variational approach). For a thorough understanding of the fundamental concepts associated with each of these methods, one can see the textbook by Clough and Penzien (2003), Meirovitch (1986), and Thomson (2000). Here, we would like to make a brief review about their characteristics. A direct vectorial equilibrium conditions satisfied everywhere within structural domain may be difficult to obtain for distributed parameter systems by means of d'Alembert's principle. The principle of virtual displacements actually reflects the whole equilibrium of the

system. In this regard, it is more efficient than the direct equilibrium relationships. Due to its efficiency in describing dynamic equilibrium, this principle serves as the basis for the following two energy formulation and the present method. Lagrange's equations of motion and Hamilton's principle are both recognized as the energy approach because the kinetic and potential energy terms are utilized. Lagrange' equations essentially represents a set of differential relations of motion and provide a competitive tool for modeling simple mechanical systems, but these equations fail to use together with the finite element method to establishing finite element equations for large scale system. In this case, the advantage of the finite element method in modeling complex system can not be exerted, see e.g. Zeng and Guo (1999). On the other hand, Hamilton's principle leads to a complicated integration of time-variations of the difference in kinetic and potential energies of the system and the work done by the nonconservative forces over any time range from t_1 to t_2 so that it is also less convenient for obtaining the system's equations of motion. The aforementioned methods are completely equivalent and give rise to identical equations of motion. The difference lies in that they have their own advantages in the study of special classes of problems. In other words, we can say that their applicability is restricted according to the nature of the dynamic system under consideration, see e.g. Clough and Penzien (2003).

In this paper, a methodologically new and physically intuitive formulation is proposed for general purpose deriving the equations of motion of dynamic systems. This formulation is based on the d'Alembert's principle and the principle of virtual displacements, but is used more conveniently than the two approaches as we shall show later. We further extend the concept of potential energy within the context of virtual displacements. In the present formulation, all forces induced during vibration, taking account of inertia effects using d'Alembert's principle are assigned the corresponding potential energy function, or taken as conservative. In this sense, the influence of inertial and applied forces upon the dynamic behavior of the system is represented by their potential energy functions, whereas these vectorial quantities are not explicitly involved in the formulation. This is also an advantage of the presented approach. We show that with such an approach a unified framework for formulating the equations of motion can be obtained. This framework is also known as the principle of stationary total potential energy in structural dynamics.

The paper is organized as follows: in section 2 we systematically present the theory aspects of the principle of stationary total potential energy in structural dynamics. In section 3 we provide the analysis procedure of the "set-in-right-position" rule for evaluating system property matrices used in deriving the finite element equations of motion under the finite element procedure. Illustrative examples are presented in section 4 and conclusions are drawn in section 5.

2. BASIC THEORY

With the aid of d'Alembert's principle, the equations of motion governing any structural dynamic problems are allowed to be expressed as equations of dynamic equilibrium. The most general expression of such equilibrium relationships can be given as

$$\mathbf{f}_S + \mathbf{f}_I + \mathbf{f}_D = \mathbf{p}(t) \tag{2.1}$$

where $\mathbf{f}_S$ are the elastic forces, $\mathbf{f}_I$ the inertia forces, $\mathbf{f}_D$ the damping forces, and $\mathbf{p}(t)$ the externally applied forces. In accordance with d'Alembert's principle and the assumption of equivalent viscous damping mechanism, $\mathbf{f}_I$ and

$\mathbf{f}_D$ are respectively given by

$$\mathbf{f}_I = \int_V \rho \ddot{\mathbf{u}} dv, \quad \mathbf{f}_D = \int_V c\dot{\mathbf{u}} dv \tag{2.2}$$

In the following development, we start to apply the principle of virtual displacements. From this principle, the total work done by the equilibrium set of forces appeared in Eqn. 2.1 upon system virtual displacements and the associated virtual strains must vanish. Therefore, Eqn. 2.1 can be rewritten as

$$\int_V \delta\boldsymbol{\varepsilon}^T \boldsymbol{\sigma} dv = \int_V -\delta\mathbf{u}^T \rho \ddot{\mathbf{u}} dv + \int_V -\delta\mathbf{u}^T c\dot{\mathbf{u}} dv + \int_V \delta\mathbf{u}^T \mathbf{p}(t) dv \tag{2.3}$$

In Eqn. 2.3, the minus sign denotes that the direction of inertia and damping forces is opposite to that of virtual displacements. As was stated previously, the principle of virtual displacements is another way of describing the system's equilibrium. As a result, the occurrence of virtual displacements must not change all forces maintaining the dynamic equilibrium of the system, and hence the time variable. Bearing this in mind, the variation operator δ can be factored out on both sides of Eqn. 2.3, which leads to

$$\delta_\varepsilon (U_i + V_I + V_D + V_p) = \delta_\varepsilon \prod_d = 0 \tag{2.4}$$

where U_i is the strain energy stored in system during vibration. V_I, V_D, and V_p are the potential energy functions corresponding to inertia force, damping force, and external load, respectively. These functions are assumed to be of the following form:

$$V_I = -\int_V -\mathbf{u}^T \rho \ddot{\mathbf{u}} dv, \quad V_D = -\int_V -\mathbf{u}^T c\dot{\mathbf{u}} dv, \text{ and } \quad V_p = -\int_V -\mathbf{u}^T \mathbf{p}(t) dv \tag{2.5}$$

In the same manner, it seems reasonable to consider that on the context of virtual displacements for any type of force encountered in dynamic system there does exist a potential energy function like that in Eqn. 2.5 associated with that force. To this end, a general definition of potential energy of force component (excluding the elastic force) in dynamics can be drawn from Eqn. 2.5 as follows: *such a potential energy is set equal to the negative value of the work performed by the stationary force upon the corresponding displacement measured from the reference position with zero potential energy to the state at which that force is applied*, and can be expressed mathematically as

$$V_f = -W_f = -\mathbf{F}_t \cdot \mathbf{u}_t \tag{2.6}$$

where subscript t denotes that both forces and displacements are time-dependent. Note that specifying the zero position for potential energy is prerequisite to the present formulation, and in the above definition such position is assumed to be the initial configuration of vibration. In general, the choice of a location at which the value of the potential energy equals zero is completely arbitrary because here we only concern with the relative change in potential energy caused by vibration, and such difference is independent of the choice of the position of zero potential energy. In this case our choice is mainly based on a consideration of convenience.

In practice, after evaluating the explicit expression of these potential energy functions and substituting them into Eqn. 2.4, we obtain the first variation of system total potential energy Π_d must be equal to zero. This is the statement of the principle of stationary total potential energy in structural dynamics. It is interesting to note that Eqn. 2.4 will reduce to the well-known principle of stationary total potential energy (see e.g. Bleich (1952)) when all time-related terms are dropped. The subscript ε in Eqn. 2.4 is used to stress that in order to reserve the nature of the principle of virtual displacements in deriving Eqn. 2.4 the variation of Π_d should only taken with respect to strain and displacement components, whereas all forces contributing to forcing the system in dynamic equilibrium should be invariant. This consideration is also based on the physical meaning of principle of virtual displacements (see e.g. Przemieniecki (1985)). Eqn. 2.4 is obtained as the system equations of motion. From its derivation, we know that this equation is simple in form and can be used without any restrictions.

3. THE "SET-IN-RIGHT-POSITION" PROCEDURE

The aim of this section is to demonstrate the application of the principle and the "set-in-right-position" rule to the formulation of equations of motion of continua that discretized by finite element techniques.

Suppose that the finite element assembly now has N independent displacement variables. Eqn. 2.4 derived for general continua is also applicable for this case, then we obtain

$$\delta u_1 \frac{\partial \Pi_d}{\partial u_1} + \delta u_2 \frac{\partial \Pi_d}{\partial u_2} + \cdots + \delta u_N \frac{\partial \Pi_d}{\partial u_N} = 0 \tag{3.1}$$

From the arbitrary character of system virtual displacements δu_i(i=1,2,…,N), each term in Eqn. 3.1 containing any one of these displacements must vanish. Thus, we have the following set of equations

$$\delta u_1 \frac{\partial \Pi_d}{\partial u_1} = 0, \delta u_2 \frac{\partial \Pi_d}{\partial u_2} = 0, \ldots, \delta u_N \frac{\partial \Pi_d}{\partial u_N} = 0 \tag{3.2}$$

By arranging these equations in i=1,2,…,N order, Eqn. 3.2 can be written in matrix form as

$$[\delta U]\left([M]\{\ddot{u}\} + [C]\{\dot{u}\} + [K]\{u\} - \{p(t)\}\right) = \{0\} \tag{3.3}$$

where $\{\ddot{u}\}, \{\dot{u}\}$ and $\{u\}$ are the acceleration, velocity and displacement vectors of the system, respectively; [M], [C], [K], and {p(t)} corresponds to the mass, damping, stiffness matrices and the applied load vector; [δU] is a diagonal matrix with the system virtual displacements being its diagonal elements.

Obviously, now the most essential step in explicitly obtaining Eqn. 3.3 is to determine [M], [C], [K] and {p(t)}. Consider Eqn. 3.3, from the rules for matrix multiplication, it is found that the subscript i in δu_i used to denote the row number of [δU] also indicates the ith row of all matrices and load vector, and the subscript j in $\ddot{u}_j, \dot{u}_j$ and u_j (to be the corresponding component of $\{\ddot{u}\}, \{\dot{u}\}$ and $\{u\}$ is used to designate the jth column of matrices [M], [C] and [K], respectively. As a result, in Eqn. 3.2 all terms that contain a common factor $\delta u_i \ddot{u}_j$ should be added together and their sum ought to be placed at the ith row and jth column of matrix [M]. In the same manner, for

terms involving $\delta u_i \dot{u}_j$ and $\delta u_i u_j$ should be separately superposed on the ith row and jth column of matrices [C] and [K]. Moreover, the terms only associated with δu_i are lumped at the ith row of vector {p(t)}. In this case, each term in Eqn. 3.2 apparently exhibits its exact position in the property matrices of the whole system so that the present procedure for forming system matrices is also named as the "set-in-right-position" rule.

It can be seen that this procedure distinguishes in nature from the direct stiffness method in which for individual element a connectivity array is used for analysis, since it is derived directly from the principle of stationary total potential energy in structural dynamics. According to Eqn. 2.4 and Eqn. 3.3, stiffness matrix [K] is derived from the first variation of strain energy; mass matrix [M] derived from the first variation of the potential energy of inertia force; damping matrix [C] derived from the first variation of the potential energy of damping force; and external load vector {p(t)} derived from the first variation of the potential energy of applied force. It follows that the proposed principle and the "set-in-right-position" rule allows us to formulate system property matrices through taking variation of system total potential energy. In other words, for any part of a dynamic system its contributions on the overall stiffness matrix, mass matrix, and damping matrix can be obtained by considering the total potential energy of that part. This character may result in a simple and efficient procedure for deriving the equations of motion of a complicated system.

4. ANALYSIS EXAMPLES

4.1. Example 1 A physical pendulum supported on a moving mass

Let us consider a physical pendulum to be supported by a moving mass as depicted in Fig. 1. The mass is linked by a massless spring of stiffness k to the base, which undergoes certain specified displacement denoted by $u_o(t)$. In Fig. 1 for such a pendulum the distance from the hanging point to its gravity center is L, and the moment of inertia about the gravity center is $m\rho^2$. If the horizontal displacement x_1 of the mass from the free position of spring and the rotation angle x_2 of the pendulum to the vertical axis is selected as the system general coordinates, the governing equations of motion can be obtained from the principle of stationary total potential energy as:

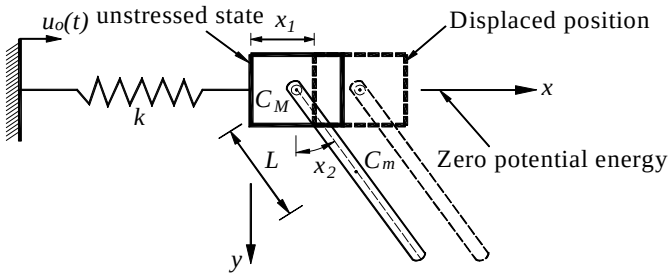

Figure 1 Physical pendulum supported on a moving mass

In this case, the position of gravity center of pendulum can be expressed in terms of general coordinates as

$$x = L\sin(x_2) + x_1, \quad y = L\cos(x_2) \tag{4.1}$$

Therefore, the acceleration components of gravity center are found by taking differentiation of Eqn 4.1 with

respect to time twice. Hence, we have

$$\ddot{x} = L\cos(x_2)\ddot{x}_2 - L\sin(x_2)\dot{x}_2^{\,2} + \ddot{x}_1, \quad \ddot{y} = -L\sin(x_2)\ddot{x}_2 - L\cos(x_2)\dot{x}_2^{\,2} \tag{4.2}$$

If we take the zero potential energy at the horizontal plane with y=0, the total potential energy, composed of the potential energy of inertia force V_I, gravitational potential energy due to pendulum V_G, and the strain energy V_K stored in the spring, can be given as

$$\prod_d = \underbrace{-(-m\ddot{x}x - m\ddot{y}y - M\ddot{x}_1x_1 - m\rho^2\ddot{x}_2x_2)}_{V_I} + \underbrace{(-mgL\cos(x_2))}_{V_G} + \underbrace{1/2k(x_1-u_o)^2}_{V_K} \tag{4.3}$$

Substituting Eqs. 4.1-4.3 into Eqn. 2.4 leads to

$$\begin{aligned}\delta\prod_d &= m(L\cos(x_2)\ddot{x}_2 - L\sin(x_2)\dot{x}_2^{\,2} + \ddot{x}_1)\times(L\cos(x_2)\delta x_2 + \delta x_1) + \\ &\quad m(-L\sin(x_2)\ddot{x}_2 - L\cos(x_2)\dot{x}_2^{\,2})\times(-L\sin(x_2)\delta x_2) + M\ddot{x}_1\delta x_1 + m\rho^2\ddot{x}_2\delta x_2 + \\ &\quad mgL\sin(x_2)\delta x_2 + k(x_1-u_o)\delta x_1 \\ &= \delta x_1[(m+M)\ddot{x}_1 + mL\cos(x_2)\ddot{x}_2 - mL\sin(x_2)\dot{x}_2^{\,2} + k(x_1-u_o)] + \\ &\quad \delta x_2[(mL^2+m\rho^2)\ddot{x}_2 + mL\cos(x_2)\ddot{x}_1 + mgL\sin(x_2)] = 0\end{aligned} \tag{4.4}$$

Consider x_1 and x_2 is arbitrary and the assumption of small-amplitude oscillations is adopted, the equations of motion can be formulated as

$$\begin{cases}(m+M)\ddot{x}_1 + mL\ddot{x}_2 + k(x_1-u_o) = 0 \\ (mL^2+m\rho^2)\ddot{x}_2 + mL\ddot{x}_1 + mgLx_2 = 0\end{cases} \tag{4.5}$$

4.2. Example 2 A planar frame

Consider a one-story frame of Fig. 2 which is idealized for the purpose of illustration as an assemblage of elements (one element per each structural member). The present aim is to derive the free vibration equations of the idealized structure. Clough and Penzien (2003) discussed the same problem through applying the concept of structural properties influence coefficients and the method of superposition. The same material and geometry properties are adopted herein. Neglect axial deformation in the beam and columns. The system has three degrees of freedom shown in Fig. 3 to define the displaced position, which include lateral displacement v_1 of the beam and joint rotations v_2 and v_3 at nodal points 1 and 2, respectively. Assume that Hermitian polynomial functions are used to describe the deformation of each element. Since the system without damping of Fig. 3 vibrates freely, there are only elastic forces and inertia forces developed during vibrations. Correspondingly, the total potential energy of the system consists of elastic potential energy V_E, and the potential energy due to inertia forces V_I, and can be computed as

$$\begin{aligned}\prod_d = &\underbrace{\frac{1}{2}\int_0^L EI\left(\frac{\partial^2 w_1}{\partial x^2}\right)^2 dx + \frac{1}{2}\int_0^{2L} 4EI\left(\frac{\partial^2 w_2}{\partial x^2}\right)^2 dx + \frac{1}{2}\int_0^L EI\left(\frac{\partial^2 w_3}{\partial x^2}\right)^2 dx}_{V_E} + \\ &\underbrace{-\int_0^L(-m\frac{\partial^2 w_1}{\partial t^2})w_1 dx - \int_0^L(-m\frac{\partial^2 w_3}{\partial t^2})w_3 dx - \int_0^{2L}(-1.5m\frac{\partial^2 w_2}{\partial t^2})w_2 dx - (-1.5m\times 2L\ddot{v}_1)v_1}_{V_I}\end{aligned} \tag{4.6}$$

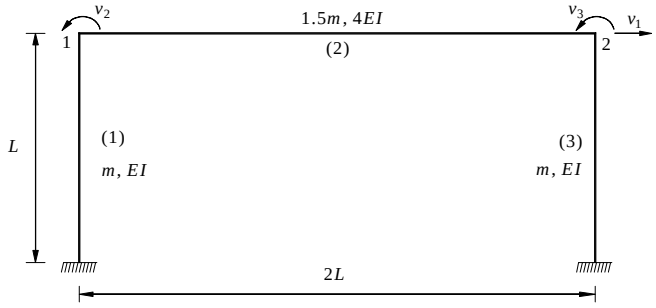

Figure 2 One-story building frame

where w_i (i=1,2,3) represents the small lateral displacement perpendicular to the original length of the element i. According to the cubic Hermitian FE formulation, these deformations are approximately given by

$$w_1=\left(3\left(\frac{x}{L}\right)^2-2\left(\frac{x}{L}\right)^3\right)v_1+\frac{x^2}{L}\left(1-\frac{x}{L}\right)v_2,\quad w_2=-x\left(1-\frac{x}{L}\right)^2v_2+\frac{x^2}{L}\left(1-\frac{x}{L}\right)v_3,\quad w_3=\left(3\left(\frac{x}{L}\right)^2-2\left(\frac{x}{L}\right)^3\right)v_1+\frac{x^2}{L}\left(1-\frac{x}{L}\right)v_3 \tag{4.7}$$

in which x denotes the local coordinate along the axis of the element. Substituting Eqn. 4.6 and Eqn. 4.7 into Eqn. 2.4, one can obtain the following expression for total potential energy of the system

$$\begin{aligned}\delta_\varepsilon\Pi_d=&\int_0^L EI\frac{\partial^2w_1}{\partial x^2}\delta\left(\frac{\partial^2w_1}{\partial x^2}\right)dx+\int_0^{2L}4EI\frac{\partial^2w_2}{\partial x^2}\delta\left(\frac{\partial^2w_2}{\partial x^2}\right)dx+\int_0^L EI\frac{\partial^2w_3}{\partial x^2}\delta\left(\frac{\partial^2w_3}{\partial x^2}\right)dx+\\&\int_0^L m\frac{\partial^2w_1}{\partial t^2}\delta w_1dx+\int_0^{2L}1.5m\frac{\partial^2w_2}{\partial t^2}\delta w_2dx+3mL\ddot{v}_1\delta v_1+\int_0^L m\frac{\partial^2w_3}{\partial t^2}\delta w_3dx\\=&\underbrace{\frac{12EI}{L^3}v_1\delta v_1+\frac{6EI}{L^2}v_2\delta v_1+\frac{6EI}{L^2}v_1\delta v_2+\frac{4EI}{L}v_2\delta v_2}_{V^e_{1K}}+\underbrace{\frac{8EI}{L}v_2\delta v_2+\frac{4EI}{L}v_3\delta v_2+\frac{4EI}{L}v_2\delta v_3+\frac{8EI}{L}v_3\delta v_3}_{V^e_{2K}}+\\&\underbrace{\frac{12EI}{L^3}v_1\delta v_1+\frac{6EI}{L^2}v_3\delta v_1+\frac{6EI}{L^2}v_1\delta v_3+\frac{4EI}{L}v_3\delta v_3}_{V^e_{3K}}+\underbrace{\frac{78mL}{210}\ddot{v}_1\delta v_1+\frac{11mL^2}{210}\ddot{v}_2\delta v_1+\frac{11mL^2}{210}\ddot{v}_1\delta v_2+\frac{2mL^3}{210}\ddot{v}_2\delta v_2}_{V^e_{1I}}+\\&\underbrace{\frac{24mL^3}{210}\ddot{v}_2\delta v_2+\frac{18mL^2}{210}\ddot{v}_3\delta v_2+\frac{18mL^2}{210}\ddot{v}_2\delta v_3+\frac{24mL^3}{210}\ddot{v}_3\delta v_3+3mL\ddot{v}_1\delta v_1}_{V^e_{2I}}+\\&\underbrace{\frac{78mL}{210}\ddot{v}_1\delta v_1+\frac{11mL^2}{210}\ddot{v}_3\delta v_1+\frac{11mL^2}{210}\ddot{v}_1\delta v_3+\frac{2mL^3}{210}\ddot{v}_3\delta v_3}_{V^e_{3I}}=0\end{aligned} \tag{4.8}$$

where V^e_{iK} (i=1,2,3) is the element strain energy, V^e_{iI} (i=1,2,3) is the element potential energy due to inertia force. After applying the "set-in-right-position" rule, the structural property matrices are listed below

$$[M]=\begin{array}{c}\\ \delta v_1\\ \delta v_2\\ \delta v_3\end{array}\begin{array}{c}\begin{array}{ccc}\ddot{v}_1 & \ddot{v}_2 & \ddot{v}_3\end{array}\\ \begin{bmatrix}\left[\frac{78mL}{210}\right]_1+[3mL]_2+\left[\frac{78mL}{210}\right]_3 & \left[\frac{11mL^2}{210}\right]_1 & \left[\frac{11mL^2}{210}\right]_3\\ \left[\frac{11mL^2}{210}\right]_1 & \left[\frac{2mL^3}{210}\right]_1+\left[\frac{24mL^3}{210}\right]_2 & \left[\frac{18mL^3}{210}\right]_2\\ \left[\frac{11mL^2}{210}\right]_3 & \left[\frac{18mL^3}{210}\right]_2 & \left[\frac{2mL^3}{210}\right]_3+\left[\frac{24mL^3}{210}\right]_2\end{bmatrix}\end{array} \tag{4.9}$$

$$[K] = \begin{array}{c} \\ \delta v_1 \\ \delta v_2 \\ \delta v_3 \end{array} \begin{array}{c} \begin{array}{ccc} v_1 & v_2 & v_3 \end{array} \\ \begin{bmatrix} \left[\frac{12EI}{L^3}\right]_1 + \left[\frac{12EI}{L^3}\right]_3 & \left[\frac{6EI}{L^2}\right]_1 & \left[\frac{6EI}{L^2}\right]_3 \\ \left[\frac{6EI}{L^2}\right]_1 & \left[\frac{4EI}{L}\right]_1 + \left[\frac{8EI}{L}\right]_2 & \left[\frac{4EI}{L}\right]_2 \\ \left[\frac{6EI}{L^2}\right]_3 & \left[\frac{4EI}{L}\right]_2 & \left[\frac{4EI}{L}\right]_3 + \left[\frac{8EI}{L}\right]_2 \end{bmatrix} \end{array} \tag{4.10}$$

It can be seen that the mass and stiffness matrices have been given in the same form as those evaluated by the direct stiffness method, but the derivation process is much more compact. From the results, the contributions of each structural member (beam or columns) to the inertia (mass) and elastic (stiffness) properties of the structure are clearly noted by associated member number marked as a right subscript.

5. CONCLUDING REMARKS

The principle of stationary total potential energy has been thoroughly examined. It has been derived from the well-known d'Alembert's principle and the principle of virtual displacements and presented in a form suitable for effective implementation. This principle has been introduced in such a manner that all time-varying forces and inertia force can be treated within the framework of conservative forces system. We note that one could also apply the principle and the "set-in-right-position" procedure to more complicated discretized continua. From the analysis process of the presented examples, it has been shown that the proposed method is simple and efficient for formulating the equations of motion of dynamic system.

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