# PREDICTING INELASTIC TORSIONAL RESPONSE WITH THE INCLUSION OF DYNAMIC ROTATIONAL STIFFNESS

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**ABSTRACT:**

Inherent to the development of performance-based seismic design and assessment techniques, is the need to adequately predict the inelastic displacements of structures. To date, research has provided a range of prediction approaches based on 2 and 3-dimensional representations. While the 2-D response can often be adequately assessed for design using simplified hand-predictions, the 3-D cases have tended to rely on push-over techniques that do not capture the effects of the rotational inertia on the diaphragm twist that develops in asymmetric structures.

This paper presents the basis of a new approach for estimating, by hand calculation, the expected maximum torsional response of buildings with in-plan asymmetry. Fundamental to a prediction procedure is the quantification of the apparent twist restraint that is a result of the rotational mass inertia of the floor diaphragm. The derivation of this dynamic torsional restraint is presented here. For a series of simple structures subjected to sinusoidal pulse inputs, comparative results between recorded inelastic time-history and predicted response are presented. For realistic eccentricity cases the predictions are shown to provide sufficiently accurate estimates of response for use in design.

**KEYWORDS**: Inelastic torsion, energy balance, capacity design, yield displacement, displacement profile.

## 1. INTRODUCTION

The study of inelastic torsional response of buildings has grown in significance with the ongoing improvement in computation capabilities. The increased investigation into inelastic response has allowed a shift from directly considering increases in displacement and diaphragm rotations under elastic conditions [Kan and Chopra, 1977; Dempsey and Irvine, 1979; Trombetti and Conte, 2005; Sommer and Bachmann, 2005], to more researchers giving consideration to the ductility demands imposed on elements due to asymmetric in-plan response [Paulay, 1998; Castillo, 2004; Humar and Kumar, 1999; Beyer et al., 2008]. This shift in focus to ductility demand was largely driven by the demonstration by Paulay [1998] and Priestley et al. [2007] that the stiffness and the strength of lateral-resisting elements are interdependent. The significance of this being that stiffer (and stronger) elements in a ductile system were likely to be the limiting component because their ductility capacity is more likely to be reached, even though the more flexible element sustains the larger displacement [Castillo, 2004].

It has also been demonstrated [Humar and Kumar, 1999; Castillo, 2004] that the rotational mass inertia has a significant level of restraint against diaphragm rotations. This is in turn can lead to larger ductility demands on the stronger and stiffer element in a structure due to the reduced twist. Within a series of publications by Paulay [e.g. 1998], it was stated that in order for capacity design to be properly applied to an asymmetric structure, the torsion mechanism that leads to peak deformation (using the centre-of-mass as a reference for design or peak displacement) must be identified. Recently proposed approaches [Trombetti and Conte, 2005; Sommer and

Bachmann, 2005; Beyer et al., 2008; Au et al., 2008] are based on simple hand-calculated predictions of response. However, to follow the inelastic phases of an asymmetric structural response requires that a static approximation be made of the non-linear dynamic problem. In particular, as demonstrated by Castillo [2004], an allowance for the inertial restraint of the rotational mass must be included. The contents of this paper summarises a proposed method of quantifying the effective of torsional restraint provided by the rotational inertia. Full details and application examples are given in Pettinga et al. [2007].

## 2. OUTLINE OF THE INVESTIGATION

In this study, two-dimensional spring models to represent the single-mass structures that were assumed to be square in-plan and were considered either torsionally unrestrained or restrained according to the definitions of Paulay [1998]. Two seismic-resisting elements were present in the direction of the single-component ground motion, and one or two elements in the orthogonal direction giving the systems the unrestrained or restrained conditions respectively (Figure 1). The use of such simplified systems with a rigid diaphragm assumption was demonstrated by Castillo [2004] to adequately capture the trends of more complex models. The hysteretic behaviour of each spring was modeled as elasto-perfectly-plastic.

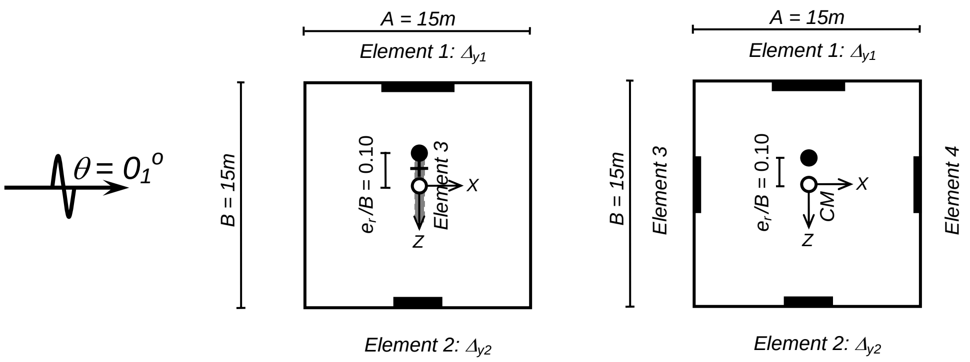


Figure 1. Stiffness eccentric 2D models (a) torsionally unrestrained (b) torsionally restrained

## 3. QUANTIFYING THE DYNAMIC TORSIONAL RESTRAINT

If a simplified approach is to be capable of reproducing the differential displacement demands of a range of systems, particularly those without twist restraint from orthogonal element pairs, the dynamic inertial restraint needs to be accounted for through some value that easily fits into a simple calculation. In a static sense the rotation of a system can be found from:

$$\varphi = \frac{V_E e_v}{K_{\varphi,static}} = \frac{\tau}{K_{\varphi,static}} \tag{3.1}$$

where $V_E$ is the system base-shear (or story shear) in the seismic-resisting elements, $e_v$ the associated strength eccentricity, $K_{\varphi,static}$ the static rotational stiffness of the system (i.e. from the instantaneous element stiffness values) and $\tau$ the torque applied as a result of the eccentricity between the centre-of-strength and centre-of-mass.

Based on Eq.(3.1) an allowance for the dynamic restraint can be made by adjusting the diaphragm rotation value itself, or by increasing the rotational stiffness. The method outlined here evaluates the rotational stiffness provided to the system, from here-on referred to as the *Dynamic Rotational Stiffness* ($K_{\varphi,dynamic}$) or *DRS*.

To quantify the relative level of torsional restraint provided by the seismic lateral resisting system, the following parameter is used:

$$\rho = \frac{\sqrt{r_{v,x}^2 + r_{v,y}^2}}{r_m} = \frac{r_v}{r_m} \tag{3.2}$$

where $r_{v,x}$ and $r_{v,y}$ are the radii of gyration of strength for the X- and Y-direction lateral resisting elements relative to the centre-of-mass (assumed at the geometric-centre of the rigid diaphragm), and $r_m$ is the mass radius of gyration of the system. Therefore $r_v$ is the equivalent system radius of gyration of strength. The term $\rho$ is used as a reference measure of the torsional restraint provided by the lateral resisting elements.

### 3.1 *Derivation for $K_{\varphi,dynamic}$ and a Dynamic Adjustment in Predicting Torsional Response*

Starting with the elastic coupled equation of motion in which the rotational component of the ground motion is assumed negligible and with stiffness eccentricity present about the y-axis only (with reference to Figure 2):

$$\begin{pmatrix} \ddot{u}_x \\ r_m \ddot{u}_\varphi \end{pmatrix} + \omega_x^2 \begin{bmatrix} 1 & \bar{e}_{ry} \\ \bar{e}_{ry} & \left( \left( \frac{r_k}{r_m} \right)^2 + \bar{e}_{ry}^2 \right) \end{bmatrix} \begin{pmatrix} u_x \\ r_m u_\varphi \end{pmatrix} = -\begin{pmatrix} 1 \\ 0 \end{pmatrix} \ddot{u}_{xo} \tag{3.3a}$$

$$\ddot{u}_y + \omega_y^2 u_y = -\ddot{u}_y \tag{3.3b}$$

$$\text{where } \omega_x^2 = \frac{K_{xs}}{M};\ \omega_y^2 = \frac{K_{ys}}{M};\ \omega_\varphi^2 = \frac{K_\varphi}{I_M};\ \bar{e}_{ry} = \frac{e_{ry}}{r_m}$$

$\omega_x$, $\omega_y$ and $\omega_\varphi$ are the translational and rotational frequencies of the undamped system along the degrees of freedom of the diaphragm, $K$ is either the translational or rotational stiffness for a given axis, $u_x$ is the centre-of-mass displacement, $u_\varphi$ the diaphragm rotation (recorded at the centre-of-mass), $e_{ry}$ the y-axis stiffness eccentricity from the centre-of-mass, $r_k$ the radius of gyration of stiffness and $r_m$ the mass radius of gyration.

Considering the rotational component of Eq.(3.3a):

$$r_m \ddot{u}_\varphi + K_x e_{ry} u_x + K_x u_\varphi \left( r_k^2 + e_{ry}^2 \right) = 0 \tag{3.4}$$

If the energy balance of the rotational component of Eq.(3.3a) is now considered during the response-history:

$$I_M \int_0^t \ddot{u}_\varphi \dot{u}_\varphi dt + \int_0^t K_x \left( e_{ry} u_x + u_\varphi \left( r_k^2 + e_{ry}^2 \right) \right) \cdot \dot{u}_\varphi dt = 0 \tag{3.5}$$

By differential relations Eq.(3.5) can be transformed to:

$$I_M \int_0^{\dot{u}_\varphi} \dot{u}_\varphi d\dot{u}_\varphi + \int_0^{u_\varphi} K_x \left( e_{ry} u_x + u_\varphi \left( r_k^2 + e_{ry}^2 \right) \right) du_\varphi = 0 \tag{3.6}$$

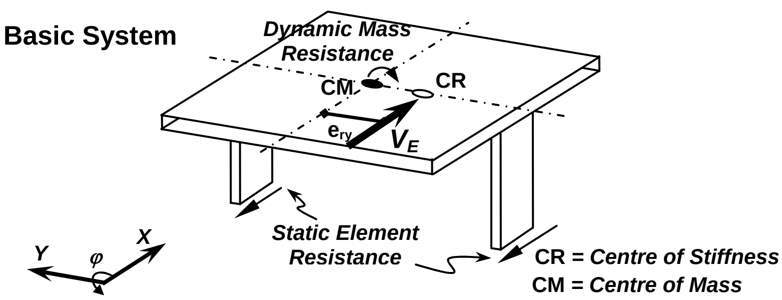


Figure 2. Breakdown of torsional resistance contributions

If Eq.(3.6) is considered as acting between two time steps *i-1* and *i*:

$$I_M \int_{\dot{u}_{\varphi,i-1}}^{\dot{u}_{\varphi,i}} \dot{u}_\varphi d\dot{u}_\varphi + \int_{u_{\varphi,i-1}}^{u_{\varphi,i}} K_x\left(e_{ry}u_x + u_\varphi\left(r_k^2 + e_{ry}^2\right)\right)du_\varphi = 0 \tag{3.7}$$

Evaluating the integrals gives:

$$\left[\tfrac{1}{2} I_M \dot{u}_\varphi^2\right]_{\dot{u}_{\varphi,i-1}}^{\dot{u}_{\varphi,i}} + \left[K_x e_{ry} u_x u_\varphi\right]_{u_{\varphi,i-1}}^{u_{\varphi,i}} + \left[K_x\left(r_k^2 + e_{ry}^2\right)\frac{u_\varphi^2}{2}\right]_{u_{\varphi,i-1}}^{u_{\varphi,i}} = 0 \tag{3.8}$$

We want to consider the diaphragm response alone, therefore if the dynamic coupled system is now reduced so that only the free-body diagram of the diaphragm is considered, the equation of equilibrium can be re-written in the following form:

$$\left[\tfrac{1}{2} I_M \dot{u}_\varphi^2\right]_{\dot{u}_{\varphi,i-1}}^{\dot{u}_{\varphi,i}} + \left[K_x\left(r_k^2 + e_{ry}^2\right)\frac{u_\varphi^2}{2}\right]_{u_{\varphi,i-1}}^{u_{\varphi,i}} = +\left[K_x e_{ry} u_x u_\varphi\right]_{u_{\varphi,i-1}}^{u_{\varphi,i}} \tag{3.9}$$

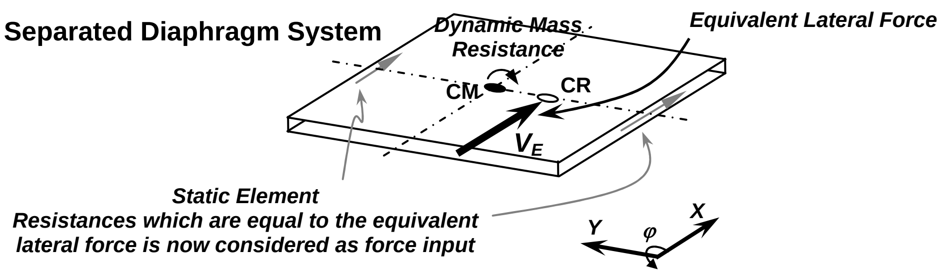


Figure 3. Breakdown of torsional resistance contributions considering only the diaphragm with lateral resisting element shears equal to the equivalent lateral force which results in an applied torque to the diaphragm.

The first term in Eq.(3.9) is the inertial resistance of the diaphragm mass to the developing rotation, the second term is the coupled rotational stiffness component, while the term on the right-hand side is the coupled translational component that is now considered to act as an applied torque to the diaphragm. This torque is due to the shear forces from the lateral resisting elements, the sum of which is equal to an equivalent lateral force applied at the diaphragm centre of strength (the stiffness eccentricity in Eq.(3.9) is assumed equal to the strength eccentricity). While the inertial term is easily calculated using results from a dynamic analysis where the angular velocity is known, it is not easily interpreted when using a static hand-calculation. With the intent to derive a value of the dynamic restraint (stiffness) without the need of an estimate for the angular velocity, the inertial term is considered an unknown to be found. Shifting the remaining element stiffness term to the right-hand side:

$$\left[\tfrac{1}{2} I_M \dot{u}_\varphi^2\right]_{\dot{u}_{\varphi,i-1}}^{\dot{u}_{\varphi,i}} = \left[K_x e_{ry} u_x u_\varphi\right]_{u_{\varphi,i-1}}^{u_{\varphi,i}} - \left[K_x\left(r_k^2 + e_{ry}^2\right)\frac{u_\varphi^2}{2}\right]_{u_{\varphi,i-1}}^{u_{\varphi,i}} \tag{3.10}$$

The left-hand term is the unknown work-done (*WD*) by the rotational mass in resisting the developing twist of the diaphragm, over a time-step *i-1* to *i*.

The units of each term in of Eq.(3.10) can be verified as being *Force·Length*. The angular work-done is given by:

$$WD = \tau \cdot u_\varphi \tag{3.11}$$

The rotational stiffness is now found from:

$$K_{\varphi} = \frac{\tau}{u_{\varphi}} = \frac{\tau}{WD/\tau} \tag{3.12}$$

Substituting the right-hand side of Eq.(3.10) into Eq.(3.12):

$$K_{\varphi,dynamic} = \frac{\tau_i^2}{\tau_i\left(u_{\varphi,i} - u_{\varphi,i-1}\right) - \frac{1}{2}K_x\left(r_k^2 + e_r^2\right)\left(u_{\varphi,i}^2 - u_{\varphi,i-1}^2\right)} \tag{3.13}$$

The dynamic stiffness can be applied following the form of Eq.(3.1), thus giving:

$$\varphi = \frac{V_E e_v}{K_{\varphi,static} + K_{\varphi,dynamic}} = \frac{\tau}{K_{\varphi,static} + K_{\varphi,dynamic}} = \frac{\tau}{K_{\varphi,total}} \tag{3.14}$$

While it is generally found that the *Dynamic Rotational Stiffness* (*DRS*) during the torsional response is positive (and therefore additive to the static resistance), it is possible that certain conditions may produce negative values. Physically this can be interpreted as the rotational inertia acting to increase the diaphragm twist, a situation which is conceptually feasible, and appears to develop for some mass eccentric systems.

## 4. INCLUDING THE DYNAMIC ROTATIONAL STIFFNESS IN TORSIONAL RESPONSE PREDICTIONS

To verify that the inclusion of the DRS in a hand-calculation prediction of asymmetric response is appropriate, a series of numerical models representing simple structures with constant base-shear strength, uncoupled lateral stiffness and mass were analysed. These were either torsionally unrestrained or restrained based on the definitions promoted by Paulay [1998] and Castillo [2004]. The systems were subjected to a simple half-sine pulse (with full-wave period equal to the uncoupled lateral period of the systems). For each system the element displacements, shear forces and diaphragm rotation were recorded at each point associated to a change in phase (i.e. first and second element yield, and maximum CM displacement). Using these 'numerical' values, the *DRS* associated to the end of each phase was calculated and applied to the response predictions. While calculated for the end-of-phase state, it is assumed that the value of *DRS* is constant over the whole of that phase. Further details of the proposed approach to predicting the phased response of an asymmetric system are given by Pettinga et al. [2007].

The following plots show the instantaneous in-plan displacement profiles captured at the time of yield of each element from the non-linear response history analyses (shown by the filled circles and thin lines). Three profiles are shown for each combination of torsional restraint (as provided by the lateral resisting elements) and eccentricity. The displacements defining the element yield points and maximum CM displacement were taken from the time-history results and applied back into the prediction models in order to remove errors due to the inherently approximate nature of the simplified yield curvature equations that would be used for design. The prediction profiles are shown with empty circles and bold lines.

### 4.1 *Results of AIR Application to Study Systems subjected to Half-Sine Pulse without Dynamic Adjustment*

In Figure 4 a first set of estimated profiles without and with the DRS included in the prediction calculations are shown against the numerical response history results. These results are for one system only with a medium-high level of torsional restraint provided by the lateral resisting elements ($\rho$ = 1.50). Note that for a torsionally unrestrained model (i.e. $\rho \leq 1.0$), unless the DRS is allowed for, it is not possible to predict the diaphragm rotation for a perfectly-plastic response once one element of the lateral resisting element pair has yielded. In fact, as demonstrated by Au et al. [2008] the significance of the rotational inertia can be considered relatively minor for fully ductile systems with the level of torsional restraint considered in this figure. The comparison in prediction accuracy between plots (a) and (b) in Figure 4, for each stage of yield and peak displacement, clearly demonstrates that the inclusion of the dynamic rotational stiffness in the hand-calculation predictions leads to more accurate estimations of in-plan displacement profile for non-linear response particularly for partially inelastic system states (fully ductile states in both (a) and (b) are well predicted). As seen in Figure 4a the diaphragm twist is over-estimated if the additional restraint is not considered.

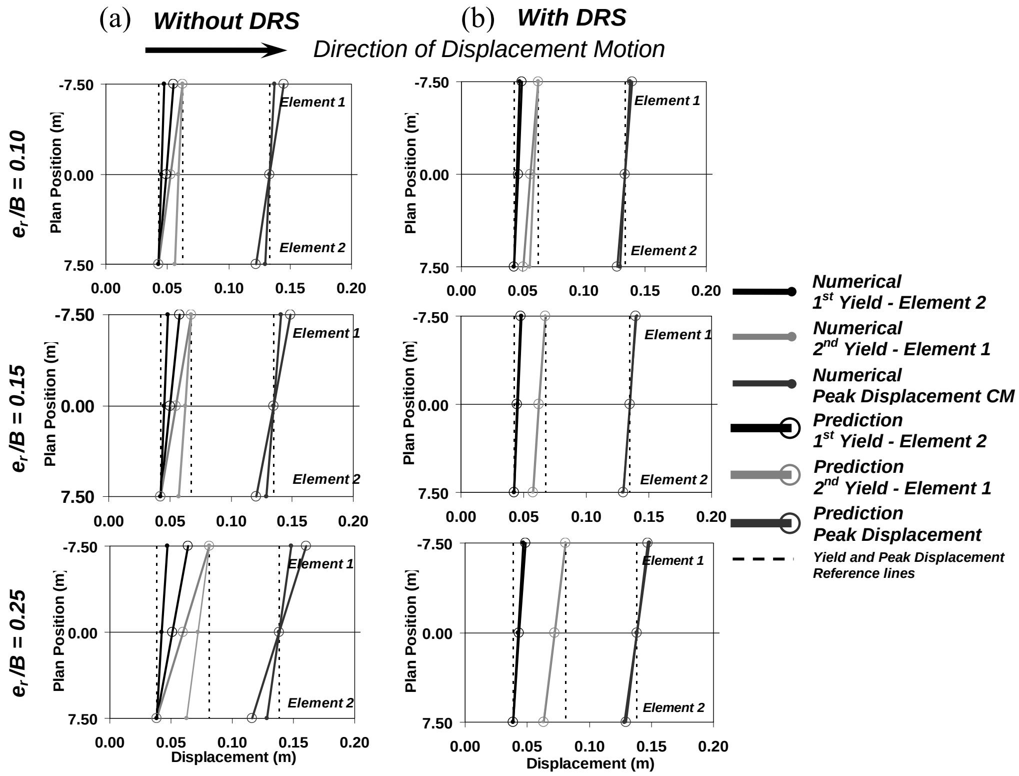


Figure 4. Comparison of numerical and predicted displacement profiles for torsionally restrained ($\rho$ = 1.50) stiffness eccentric systems (a) without and (b) with account of the dynamic torsional restraint (*DRS*) in the predictions. Note that the dashed lines are the individual element yield displacement and peak centre-of-mass displacement reference lines.

In Figure 5 an extreme case of a torsionally unrestrained system ($\rho$ = 0.50) is presented for (a) stiffness eccentric and (b) strength and stiffness eccentric conditions. As with Figure 4 comparisons between predicted in-plan displacement profile and numerical phased response are given. The *DRS* has been included in both cases for the predictions. For both forms of asymmetry, and all levels of eccentricity the predictions give a close match to the numerical results. As highlighted by Castillo [2004] systems with only stiffness eccentricity do not suffer significant twist effects, a finding reflected in the profiles of Figure 5a, where as the equivalent stiffness and strength eccentric models exhibit significant diaphragm rotations Figure 5b, particularly when fully ductile. It should be noted that the stiffness eccentric systems have well defined phased response from first to second yield (i.e. Phase 1 to Phase 2) due to the eccentricity being simply a function of the difference in yield displacement between Element 1 and 2. By comparison the strength and stiffness eccentric models do not have the developing phase definition because the eccentricities are a result of Element 2 having greater strength than Element 1.

Overall the inclusion of the *DRS* in a hand-calculation procedure significantly improves the accuracy of the predicted twist response of the simple 2D models considered. Not only are the diaphragm rotations captured, but the development of the torsional mechanism is correctly predicted, which was a primary requirement. In the majority of cases the percentage error in predicting individual element displacements is less than 15% which is considered satisfactory for design purposes.

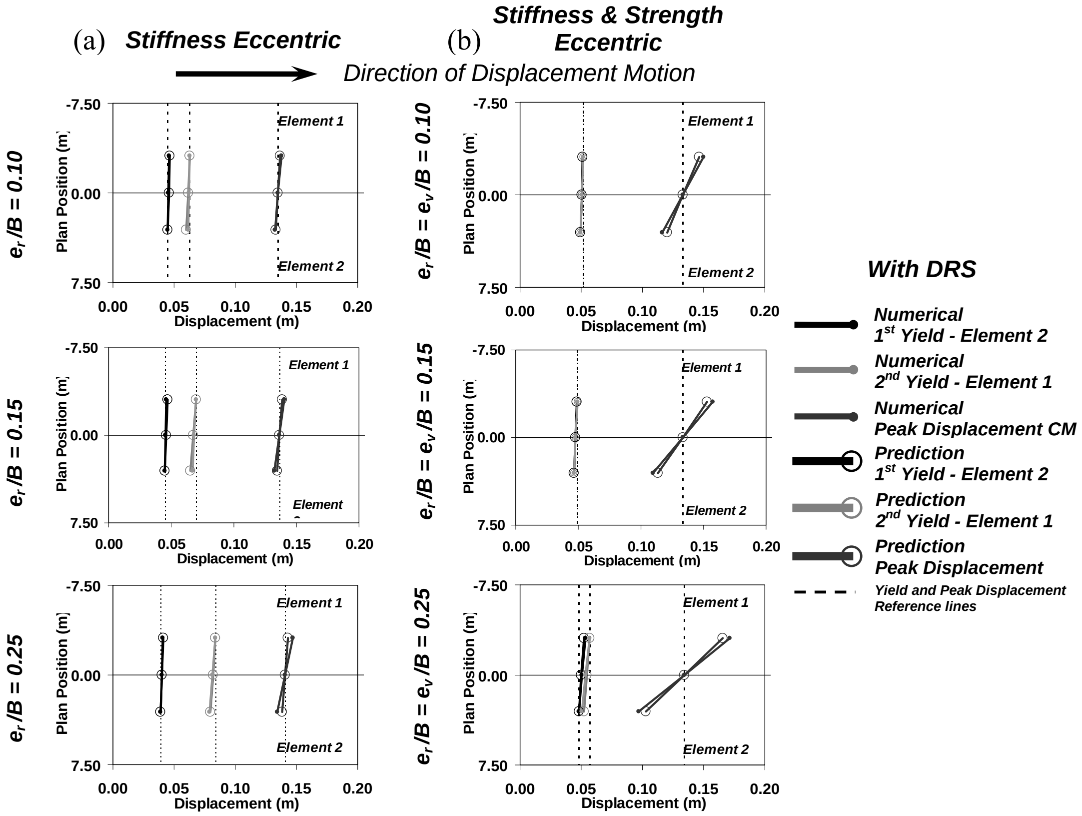


Figure 5. Comparison of numerical and predicted displacement profiles for torsionally unrestrained stiffness ($\rho$ = 0.50) (a) stiffness and (b) equal strength and stiffness eccentric systems. Allowance for the dynamic torsional restraint is included in the predictions. Note that the dashed lines are the individual element yield displacement and peak centre-of-mass displacement reference lines.

## 5. CONCLUSIONS

An approach to quantifying the dynamic inertial restraint to diaphragm rotations has been presented. A derivation of a so-called Dynamic Rotational Stiffness based on an energy balance during the response history of an undamped asymmetric 2D system, has been used to provide a numerical value of stiffness that acts in addition to the one provided by the lateral seismic resisting elements in a structure. The proposed allowance for this additional stiffness has been tested within a hand-prediction procedure for a range of simple 2D numerical models with varying torsional restraint and eccentric conditions. The results presented demonstrate that the allowance for the Dynamic Rotational Stiffness provides more accurate estimations of torsional response for inelastic asymmetric displacements, thus lending it to use with displacement-based seismic design.

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