New Approach for the Construction of Inelastic Stiffness Matrix for Dynamic Analysis

A. Shooshtari[1] and R. Khajavi[2]

[1] *Assistant Professor, Dept. of Civil Engineering, Ferdowsi University of Mashhad, Iran*
[2] *Ph.D. Student, Dept. of Civil Engineering, Ferdowsi University of Mashhad, Iran*

ABSTRACT:

There seems to be many situations where an inelastic analysis needs to be performed on a frame structure. In the last decades, the subject has received much interest from many researchers, especially for seismic applications. Different models have been proposed for the frame element to represent its inelastic behavior, among which, the one component models with inelastic springs are widely used. In such models, each frame member is represented by an elastic element, with several nonlinear springs, mostly located at the ends of the element. These models suffer several deficiencies, mentioned in the context. To overcome such deficiencies, a new approach is suggested for the construction of inelastic stiffness matrix of the frame element. In the proposed method, some virtual inelastic springs are attached to the elastic element. Such springs model plasticity in the strain states beyond the rigid body modes of the element. The stiffness matrix of the frame element, obtained through this procedure, is compared with the conventional ones for its priority.

KEYWORDS:
Frame element, Inelasticity, Nonlinear analysis, Plastic hinge, Strain state, Virtual spring

1. INTRODUCTION

New advances in computing technology have persuaded structural engineers to adopt more efficient methodologies for the design of building frames, especially against the huge earthquake loadings. Such sophisticated design approaches are based upon complicated analyses which account for the inelastic behavior of materials as well. Instances to name are the two high-level methods of Nonlinear static (Pushover) and dynamic (Time history) analyses, that have put aside simplifying assumptions needed in conventional analysis and design procedures.

Several different approaches have been developed to model the elastic-plastic behavior of frame elements, among which, the lumped plastic hinge method is regarded as being of most use and popularity, certainly due to its simplicity. To implement the method, zero-length plastic hinges are integrated in series to the rotational dofs of beam nodes, mainly at the two ends of the element [1].

In this research, a new method is proposed to create the elastic-plastic stiffness matrix of a beam element, benefiting the concept of lumped plastic hinge approach. It is based on the knowledge of different strain states, the element is capable of modeling. At the first pace to comment formulation, we shall look through the concept of strain modes in glances. Next section will present an acquaintance with the current elastic-plastic stiffness matrices, along with a review of their deficiencies and imperfections. Then, the proposed method is introduced and applied to construct the elastic-plastic stiffness matrix of the Euler-Bernoulli beam element, while assessed and validated in comparison with its counterparts.

2. FINITE ELEMENT STRAIN STATES

In finite element method, analysts are seeking for an unknown field of displacement, stress, or strain over the structure domain. The field is assumed to be a linear combination of some basic sub-fields. In displacement formulation of FEM, Element deformation is considered as a superposition of a set of basic deformations, known as standard basis. From a mathematical point of view, these basic deformations are actually those named as shape functions. Such functions display how the element deforms, when subjected to a unit displacement of a dof, whilst all others are constrained against movement. The displacement function is presumed to be a linear combination of shape functions, with the coefficients turn out to be the nodal displacements:

$$u = \boldsymbol{N}\boldsymbol{D} \tag{2.1}$$

Here, $\boldsymbol{N}$ is the matrix of shape functions and $\boldsymbol{D}$ is the vector of nodal displacements. This interpolation results in a stiffness relation as:

$$\boldsymbol{K}\boldsymbol{D} = \boldsymbol{P} \tag{2.2}$$

Where $\boldsymbol{K}$ and $\boldsymbol{P}$ are stiffness matrix and nodal force vector in element level, respectively.

The standard basis is just one among an infinite number of bases that can be considered for the element deformation space. This basis is of practical use and interest due to its coefficients being interpreted as nodal displacements. The element deformation can be represented by another useful basic functions, honored with the title of strain-state basis:

$$u = \boldsymbol{N}_q \boldsymbol{q} \tag{2.3}$$

$$\boldsymbol{N}_q = \lfloor N_{q1} \;\; N_{q2} \;\; \cdots \;\; N_{qn} \rfloor \tag{2.4}$$

The coefficients in this linear combination, arranged in vector $\boldsymbol{q}$, are not nodal displacements any longer, but represent the portion of corresponding basic functions in the total deformation of the element.
In the framework of this new basis, the stiffness relation reshapes into the form of:

$$\boldsymbol{K_q}\,\boldsymbol{q} = \boldsymbol{P_q} \tag{2.5}$$

This is a general transformation prototype that correlates the displacement and force spaces at the element level, according to the basis of $\boldsymbol{N_q}$. Eqn. 2.2. is a special form of this general formulation, reproduced in the basis of $\boldsymbol{N}$. An infinite number of relations, as Eqn. 2.5., could be imagined, all of which represent a unique transformation, though in different bases. All these matrices of $\boldsymbol{K_q}$ are mathematically said to be similar transformations, which are inter-related as:

$$\begin{aligned} \boldsymbol{K_q} &= \boldsymbol{G^T}\boldsymbol{K}\boldsymbol{G} \\ \boldsymbol{q} &= \boldsymbol{H}\boldsymbol{D} \\ \boldsymbol{P_q} &= \boldsymbol{G^T}\boldsymbol{P} \end{aligned} \tag{2.6}$$

$$\begin{aligned} \boldsymbol{K} &= \boldsymbol{H}^{\boldsymbol{T}} \boldsymbol{K_q} \boldsymbol{H} \\ \boldsymbol{D} &= \boldsymbol{G}\boldsymbol{q} \\ \boldsymbol{P} &= \boldsymbol{H}^{\boldsymbol{T}} \boldsymbol{P_q} \end{aligned} \tag{2.7}$$

Basic functions of $\boldsymbol{N_q}$ can be selected in such a way that no energy interaction occurs between different strain states of the element. This status of energy orthogonality restores a diagonal element stiffness matrix of $\boldsymbol{K_q}$, bestowed the name of *principal stiffness matrix*, with diagonal entries labeled as *eigenstiffnesses*:

$$\boldsymbol{K}_q = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix} \tag{2.8}$$

Each diagonal entry, or eigenstiffness, is a representative of its corresponding strain state capacity to store energy[2].

3. LUMPED PLASTIC HINGE MODELING

As mentioned before, amongst different methods for modeling the inelastic behavior of frame elements, the lumped plastic hinge approach has earned much interest and applicability for its simplicity and low expense. In this method, the inelastic behavior of the beam is represented by plastic hinges attached to special nodes along the element [3]. The hinges get activated as soon as the element enters the plastic range. Plastic hinges, to remark it clear, are non-activated infinite-stiffness springs, awakened when plasticity is triggered. They are typically defined as flexurally operated, though those of shear and slip performance are also applicable, particularly in concrete frames. In this paper, flexural springs are only referred.

To implement the method, plastic springs are constrained in series to the rotational dofs of the two end nodes of the element. Analytically, this means to add spring flexibilities to the elastic element flexibility matrix to get that of the whole system:

$$\boldsymbol{F}_p = \begin{bmatrix} \dfrac{1}{k_{pi}} & 0 \\ 0 & \dfrac{1}{k_{pj}} \end{bmatrix} \tag{3.1}$$

$$\boldsymbol{F}_S = \begin{bmatrix} \dfrac{4EI}{L} & \dfrac{2EI}{L} \\ \dfrac{2EI}{L} & \dfrac{4EI}{L} \end{bmatrix}^{-1} = \begin{bmatrix} \dfrac{L}{3EI} & \dfrac{-L}{6EI} \\ \dfrac{-L}{6EI} & \dfrac{L}{3EI} \end{bmatrix} \tag{3.2}$$

$$\boldsymbol{F}_{ep} = \begin{bmatrix} \frac{L}{3EI} + \frac{1}{k_{pi}} & \frac{-L}{6EI} \\ \frac{-L}{6EI} & \frac{L}{3EI} + \frac{1}{k_{pj}} \end{bmatrix} \tag{3.3}$$

The inverse of the last matrix denotes the stiffness matrix of the elastic element-plastic spring system as:

$$\boldsymbol{K}_{ep} = \begin{bmatrix} \frac{4EI\,k_i(3EI + Lk_j)}{4EI(3EI + Lk_j) + Lk_i(4EI + Lk_j)} & sym. \\ \frac{2EI\,L\,k_i k_j}{4EI(3EI + Lk_j) + Lk_i(4EI + Lk_j)} & \frac{4EI\,k_j(3EI + Lk_i)}{4EI(3EI + Lk_j) + Lk_i(4EI + Lk_j)} \end{bmatrix} = \begin{bmatrix} A & C \\ C & B \end{bmatrix} \tag{3.4}$$

The entries of this matrix, as is only defined for the rotational dofs, should be positioned in the complete stiffness matrix of the element:

$$\boldsymbol{K}_{ep} = \begin{bmatrix} \frac{12EI}{L^3} & & & sym. \\ \frac{6EI}{L^2} & A & & \\ \frac{-12EI}{L^3} & \frac{-6EI}{L^2} & \frac{12EI}{L^3} & \\ \frac{6EI}{L^2} & C & \frac{-6EI}{L^2} & B \end{bmatrix} \tag{3.5}$$

The above stiffness matrix suffers deficiencies, listed below:
1. The formulation, though capable of representing flexural hinges, is frustrated when shear hinges are the case.
2. It fails to render a correct form of stiffness matrix when one end is hinged.
3. The matrix ought to be able to characterize two rigid-body motions of translation and rotation, while it lets to only one zero eigenvalue, associated to translation.
4. The formulation can only model plasticity at the end nodes, and when a section in between gets hinged, it just fizzles out.
5. It is undoubtedly not an easy practice to determine the plastic spring coefficient, so that assumptions should be made to simplify the procedure.

Using the slope-deflection formulation, Chen modified the procedure to exhibit a sophisticated edition of beam element stiffness matrices, theoretically of great favor, yet, offended by cases 1, 4 and 5 of the above mentioned [4].

4. STRAIN STATE VIRTUAL HINGE MODELING

For the lumped plastic hinge approach, plastic springs are set at the two rotational dofs of the beam ends. However, in the proposed method, some virtual springs are appended in series to the strain modes of the element. The procedure is now implemented for the well-known Euler-Bernoulli beam element, with an elastic stiffness as below [5]:

$$\boldsymbol{K}_e = \begin{bmatrix} \frac{12EI}{L^3} & & & \\ \frac{6EI}{L^2} & \frac{4EI}{L} & & \\ \frac{-12EI}{L^3} & \frac{-6EI}{L^2} & \frac{12EI}{L^3} & \\ \frac{6EI}{L^2} & \frac{2EI}{L} & \frac{-6EI}{L^2} & \frac{4EI}{L} \end{bmatrix} \tag{4.1}$$

The element affords a stiffness equal to the magnitude of an eigenvalue, when is exposed to a deformation of associated eigenvector:

$$\begin{aligned} k_{e_1}^q &= 0 & &\begin{bmatrix} 1 & 0 & 1 & 0 \end{bmatrix} \\ k_{e_2}^q &= 0 & &\begin{bmatrix} -L & 1 & 0 & 1 \end{bmatrix} \\ k_{e_3}^q &= \frac{2EI}{L} & &\begin{bmatrix} 0 & -1 & 0 & 1 \end{bmatrix} \\ k_{e_4}^q &= \frac{24EI}{L^3} + \frac{6EI}{L} & &\begin{bmatrix} \frac{2}{L} & 1 & \frac{-2}{L} & 1 \end{bmatrix} \end{aligned} \tag{4.2}$$

Hence, the principal elastic stiffness matrix of the Euler-Bernoulli beam element is portrayed as:

$$\boldsymbol{K}_{\boldsymbol{e}}^{\boldsymbol{q}} = \begin{bmatrix} 0 & & & \\ & 0 & & \\ & & \frac{2EI}{L} & \\ & & & \frac{24EI}{L^3} + \frac{6EI}{L} \end{bmatrix} \tag{4.3}$$

Two zero eigenvalues stand for rigid-body modes, while two others represent constant and linear curvature states, as depicted in Figure1. A matrix layout of eigenvectors is a transformation tool to move between the two spaces of standard and strain modes:

$$G = \begin{bmatrix} 1 & -L & 0 & \frac{2}{L} \\ 0 & 1 & -1 & 1 \\ 1 & 0 & 0 & -\frac{2}{L} \\ 0 & 1 & 1 & 1 \end{bmatrix} \tag{4.4}$$

Now, virtual plastic springs are attached to the strain states of the element. As no energy is stored in the rigid body states, there is no need to define plastic springs for them. However, for the two non-zero stiffness states, virtual plastic springs are set, by adding eigenflexibilities of the elastic and plastic components:

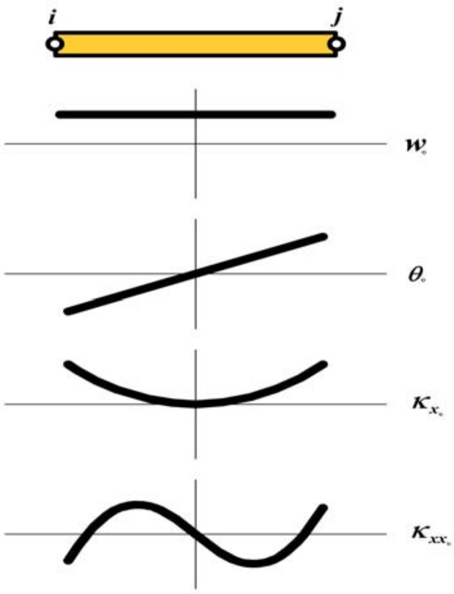

Figure 1. Strain states of Euler-Bernoulli beam element

$$f^{q}_{ep_3} = f^{q}_{e_3} + f^{q}_{p_3} = \frac{L}{2EI} + f^{q}_{p_3} = \frac{L}{2EA} + \frac{1}{k^{q}_{p_3}} \tag{4.5}$$

$$f^{q}_{ep_4} = f^{q}_{e_4} + f^{q}_{p_4} = \frac{1}{\frac{24EI}{L^3} + \frac{6EI}{L}} + f^{q}_{p_4} = \frac{1}{\frac{24EI}{L^3} + \frac{6EI}{L}} + \frac{1}{k^{q}_{p_4}} \tag{4.6}$$

If these values are inversed and organized in an appropriate frame, the principal elastic-plastic stiffness matrix of the Euler-Bernoulli beam element is obtained:

$$\boldsymbol{K^{q}_{ep}} = \begin{bmatrix} 0 & & & \\ & 0 & & \\ & & \frac{2EI\,.\,k_3}{2EI + k_3 L} & k_{34} \\ & & k_{34} & \frac{1}{\frac{1}{\frac{24EI}{L^3} + \frac{6EI}{L}} + \frac{1}{k_4}} \end{bmatrix} \tag{4.7}$$

The non-diagonal entry, k_{34}, is also included to represent the interaction effect of constant and linear curvature states, as energy orthogonality may not happen in the plastic range, while is inevitable over the elastic domain. The elastic-plastic stiffness matrix of the element is now rendered by the use of the normalized inverse of ***G*** as a transformation:

$$K = G^{-T} K_{ep}^{q} G^{-1} \tag{4.8}$$

This new proposed stiffness matrix is of great rigor and robustness that overcomes what has defeated its ancestors:

1. The stiffness matrix can model shear hinge along the element by a selection of virtual spring coefficients as:

$$k_3 \to \infty\,,\ k_{34} \to 0\,,\ k_4 \to 0 \tag{4.9}$$

2. It can represent a correct form of stiffness matrix when ideally flexural hinges appear at each end of the element, assuming:

$$k_3 \to 6\,,\ k_{34} \to \frac{3\sqrt{5}}{2}\,,\ k_4 \to 10 \tag{4.10}$$

for a hinge at the first end, and

$$k_3 \to 6\,,\ k_{34} \to \frac{-3\sqrt{5}}{2}\,,\ k_4 \to 10 \tag{4.11}$$

if a hinge develops at the other end.

3. It has a correct number of two zero eigenvalues, to represent rigid body motions.

4. It is capable of modeling plasticity and hinge emerging procedure, occurring at any arbitrary section along the element. A case in point is a hinge appearing at the midspan of the girder, with an elastic-plastic stiffness matrix rendered by spring coefficients as:

$$k_3 \to 0\,,\ k_{34} \to 0\,,\ k_4 \to \infty \tag{4.12}$$

5. The method suggests a systematic procedure to determine the plastic spring coefficients, through the application of strain modes to the element, measuring the plastic deformations, and employment of appropriate transformations.

5. CONCLUSION

This research suggests a new method for the construction of elastic-plastic stiffness matrix of frame element, borrowing the conventional concept of lumped plastic hinges. The method has triumphed over the defects experienced by previous approaches, by placing plastic springs, virtually, in strain states, rather than the rotational dofs of the element. The method will make it possible to model different hinges of flexure, shear, and even slip performance, not only at the end nodes, but also at any arbitrary section along the element. Another appealing feature the method qualifies is its extensibility to other variety of finite elements, including plates and shells.

REFERENCES

[1] Elsiwat, J. (1993). Effect of anchorage slip and inelastic shear on seismic response of reinforced concrete frames, PhD Assertation, University of Ottawa, Canada.

[2] Khajavi, R. (2004). Partitioning the stiffness matrix and the principal stiffness matrix of finite element, MSc Assertation, Ferdowsi University of Mashhad, Iran.
[3] Nanakorn, P. (2004). A Two-dimensional Beam-column Finite Element with embedded rotational discontinuities. Computers and Structures 82, 753-765.
[4] Arfaei Shahidi zadeh, M. (2002). An investigation of frame stability including the nonlinear behavior of materials, MSc Assertation, Ferdowsi University of Mashhad, Iran.
[5] Rezaei Pajand, M. (1992). Theory of matrix analysis for structures, Ferdowsi University of Mashhad Press, Iran.