NATURAL VIBRATION ANALYSIS OF CONTINUOUS HORIZONTALLY CURVED GIRDER BRIDGES USING DQEM *

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ABSTRACT :
The differential quadrature element method (DQEM) is applied to computation of the eigenvalues of small amplitude natural vibration for continuous horizontally curved girder bridges with rigid piers in this paper. The DQEM uses the differential quadrature technique to discretize the governing differential eigenvalue equations defined on each member, the transition conditions defined on the inner-element boundary of two adjacent members and the end conditions of the continuous curved girder. Natural frequencies are calculated for a two-equal-span, continuous, curved, uniform girder bridge, and are compared with existing exact solutions by another method. Finally, parametric results for the effects of section gyration radius and flexure-torsion stiffness ratio on five different end conditions, to the out-of-plane natural frequencies of two-equal-span, continuous, curved girder bridges, are presented in dimensionless form.

KEYWORDS:
natural vibration analysis, continuous curved girder bridge, differential quadrature element method

1. INTRODUCTION

With the rapid development of vehicle traffic and city construction, the number of curved girder bridges is rising rapidly. There is huge difference between the curved girder bridges and normal rectilinear girder bridges in dynamic characteristics, because of space curvature of the bridges. The torsional motions will be produced in the superstructures of the curved girder bridges accompanied with the bending responses under excitations of earthquake ground motion horizontally or vertically. Owing to their importance in structural design, the dynamic behavior of curved girders has been the subject of a large number of investigations. The total extent of the work in this field of natural vibration of curved girders is now too great to review in detail, but part of those papers that set the present work in context are described. Shore and Chaudhuri (1972) studied the free vibration of horizontally curved beams using closed-form solutions of the equations of motion. Snyder and Wilson (1992) calculated the free vibration frequencies of continuous horizontally curved beams using a nonexplicit closed-form solution of the partial differential equations of motion. Kang, Bert and Striz (1996) analyzed the natural vibration characteristics of single-span, horizontally curved beams with warping using the differential quadrature method (DQM). Howson and Jemah (1999) obtained the exact out-of-plane frequencies of curved Timoshenko beams using dynamic stiffness matrix, and discussed the effects of shear deflection and rotary inertia.

The differential quadrature element method (DQEM) is developed on the basis of DQM (Chen 2005, 2008). This method absorbs advantages of both the traditional DQM and the finite element method (FEM). For the inner-element boundary conditions, encountered in continuous curved girders, which have been difficult problems for the traditional DQM, the DQEM can solve them effectively. The DQEM adopts the differential quadrature to discretize the governing differential equations defined on each element, the transition conditions defined on the inner-element boundary of two adjacent elements and the end conditions of the curved continuous girder. Then unite all the discrete equations and give solution to them.

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The objective of this paper is to provide an approach of the free vibration behaviors for the continuous horizontally curved girder bridges with rigid piers. Natural frequencies are calculated for a two-equal-span, continuous, curved, uniform girder bridge, and the calculated frequencies of the bridge are compared with the existing exact solution. Parametric results of the two-equal-span, continuous, curved girder bridges are also presented in dimensionless form.

2. DIFFERENTIAL QUADRATURE

The idea of differential quadrature, introduced by Bellman and Casti (1971), is to approximate the values of the derivative at each gird point as weighted linear sums of the function values at all sampling grid points within the domain under consideration, i.e.

$$\left.\frac{d^m}{dx^m} y(x)\right|_{x=x_i} = \sum_{j=1}^{N} A_{ij}^{(m)} y(x_j) \qquad \text{for} \quad i=1, 2, \ldots, N \tag{2.1}$$

Equation (2.1) relates the mth-order derivative of the function $y(x)$ at a sampling grid point $x=x_i$ to the function value $y_j=y(x_j)$ at $x=x_j$, with $A_{ij}^{(m)}$ and N denoting the corresponding weighting coefficient and number of discrete points within the domain, respectively. The weighting coefficients can be determined such that Equation (2.1) is satisfied exactly for m linearly independent test functions. These functions can be polynomials, trigonometric functions, or spline functions. If the Lagrange polynomials are chosen, the weighting coefficients for the first-order derivative can be calculated by

$$A_{ij}^{(1)} = \prod_{\substack{k=1 \\ k\neq i,j}}^{N} (x_i - x_k) \Big/ \prod_{\substack{k=1 \\ k\neq j}}^{N} (x_j - x_k) \qquad \text{for} \qquad i \neq j \tag{2.2}$$

$$A_{ii}^{(1)} = \sum_{\substack{k=1 \\ k\neq i}}^{N} A_{ik}^{(1)} \qquad \text{for} \qquad i = j \tag{2.3}$$

where the domain of independent variable x is usually changed into [0,1] by regularization transformation. Then the weighting coefficients for the mth-order derivative can be determined by

$$A^{(m)} = [A^{(1)}]^m \tag{2.4}$$

The accuracy of the quadrature solutions is dictated by the choice of the locations of the sampling grid points. Equal spacing is a natural and convenient choice. However, non-uniform grid points have been demonstrated to enhance the accuracy of the results. An unequal form of common use can be expressed as

$$x_i = \frac{1-\cos[(i-1)\pi/(N-1)]}{2} \tag{2.5}$$

where the domain of x is [0,1].

3. GOVERNING DIFFERENTIAL EQUATIONS

This paper deals with the continuous horizontally curved girder bridges with rigid piers, constant radius and uniform sections considering coincidence of torsion center and gravity center. Based on these assumptions, the bridge could be treated just as a horizontally curved girder with decoupling of the out-of-plane and in-plane motion. For convenience, only the out-of-plane motion is considered here.

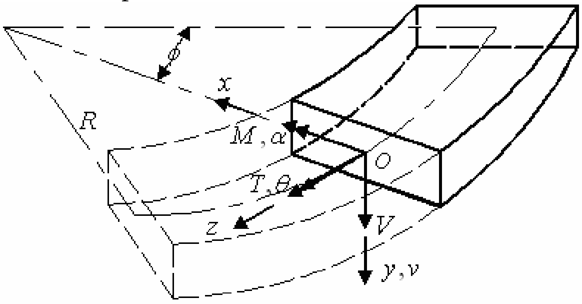

Figure1 Coordinate of the curved girder

Considering the coordinate system of the curved girder shown in Figure 1, in which R, ϕ, and O are radius, angle from the beginning section to a generic section, and gravity center of the generic section, respectively; V, M, and T are shear force parallel to axis y, bending moment about the radial axis x, and torsional moment about the centroidal axis z, respectively; and u, α, and θ are vertical displacement of the gravity center, bending slope, and twist angle, respectively. Considering the effect of shear deflections, the three inner forces V, M, and T can be written in terms of u, α, and θ as

$$V = \frac{kGA}{R}\left(\frac{dv}{d\phi} + R\alpha\right),\quad M = \frac{EI}{R}\left(\frac{d\alpha}{d\phi} + \theta\right),\quad T = \frac{GJ}{R}\left(\frac{d\theta}{d\phi} - \alpha\right) \tag{3.1}$$

where k is the shear correction factor; E and G are modulus of elasticity and shear modulus, respectively; A, I, and J are the area, second moment of area about axis x, and Saint-Venant torsion constant, respectively.

Let ρ and ω denote the mass density and natural frequency, respectively. Based on the assumption that the cross-sectional shape is constant and doubly symmetric (Vlasov 1971), and for reference in the sequel, the differential eigenvalue equations of the curved girders considering shear deformation are expressed in dimensionless form as

$$s^2 v'' + R\Phi s^2 \alpha' + \Phi^2 \lambda^2 v = 0 \tag{3.2}$$

$$-\Phi s^2 v' + R\alpha'' - R\Phi^2\left(\mu + s^2\right)\alpha + R\Phi(1+\mu)\theta' + R\Phi^2\gamma^2\lambda^2\alpha = 0 \tag{3.3}$$

$$-\Phi(1+\mu)\alpha' + \mu\theta'' - \Phi^2\theta + \eta\Phi^2\gamma^2\lambda^2\theta = 0 \tag{3.4}$$

where each prime ($'=d/d\xi$) denotes one differentiation with respect to the dimensionless coordinate $\xi = \phi/\Phi$; Φ =opening angle of the member; $s^2=kGAR^2/(EI)$; $\gamma^2=I/(AR^2)$; $\mu=GJ/(EI)$; $\eta=I_d/d$; $\lambda^2=\rho AR^4\omega^2/(EI)$; and I_d = polar second moment of area of the cross section.

The following boundary conditions are taken for simply supported ends: no vertical deflection, no bending moment, and no torsional rotation; and for free ends: no shear force, no torsional moment and no bending moment. For clamped ends, v, α, and θ equal zero. The boundary conditions for simply supported, free, and clamped ends are, in dimensionless form respectively

$$v = \alpha' = \theta = 0 \tag{3.5}$$

$$v' + R\Phi\alpha = \alpha' + \Phi\theta = \theta' - \Phi\alpha = 0 \tag{3.6}$$

$$v = \alpha = \theta = 0 \tag{3.7}$$

4. DQEM FORMULATION OF CONTINUOUS CURVED GIRDERS

For simplicity of statement, this section takes an n-span continuous curved girder, with all supports simply supported, to illustrate the analysis process of the out-of-plane natural vibration of continuous curved girder bridges with rigid piers using the DQEM. Taking each span of the girder as an independent element, the number of all the elements is n. Let Φ^e denote central angle the eth element. Let ϕ^e be the coordinate variable of the local coordinate system with the origin located at node 1 of the element, and $\phi^e \in [0, \Phi^e]$. Let ξ equal ϕ^e / Φ^e, so the range of ξ is [0,1]. Then we have

$$\frac{d^k y^e}{d(\phi^e)^k} = \frac{1}{(\Phi^e)^k}\frac{d^k y^e}{d\xi^k} \tag{4.1}$$

Introducing Equations (2.1) and (4.1) to Equations (3.2)-(3.4), one may have

$$s^2 \sum_{j=1}^{N^e} A_{ij}^{e(2)} v_j + R\Phi^e s^2 \sum_{j=1}^{N^e} A_{ij}^{e(1)} \alpha_j + (\Phi^e)^2 \lambda^2 v_i^e = 0 \tag{4.2}$$

$$-\Phi^e s^2 \sum_{j=1}^{N^e} A_{ij}^{e(1)} v_j + R\sum_{j=1}^{N^e} A_{ij}^{e(2)} \alpha_j - R(\Phi^e)^2\left(\mu + s^2\right)\alpha_i + R\Phi^e(1+\mu)\sum_{j=1}^{N^e} A_{ij}^{e(1)}\theta_j + R(\Phi^e)^2 \gamma^2 \lambda^2 \alpha_i = 0 \tag{4.3}$$

$$-\Phi^e(1+\mu)\sum_{j=1}^{N^e} A_{ij}^{e(1)}\alpha_j + \mu\sum_{j=1}^{N^e} A_{ij}^{e(2)}\theta_j - (\Phi^e)^2\theta_i + \eta(\Phi^e)^2\gamma^2\lambda^2\theta_i = 0 \tag{4.4}$$

for i = 2, ..., N^e-1, where N^e denotes the number of the points on the eth element.

Using Equation (2.1), the conditions of the two ends of the continuous curved girder can be expressed as

$$v_r^m = 0\,,\quad \sum_{j=1}^{N^m} A_{rj}^{m(1)}\alpha_j = 0\,,\quad \theta_r^m = 0 \tag{4.5}$$

where m=1 while r=1, and m=n while r=N^n.

Using Equation (2.1), the transition conditions of the inner-element boundary of two adjacent members i and i+1 can be written as

$$v_{N^i}^i = 0\,,\quad v_1^{i+1} = 0\,,\quad \theta_{N^i}^i = 0\,,\quad \theta_1^{i+1} = 0\,,\quad \alpha_{N^i}^i = \alpha_1^{i+1}\,,\quad \frac{1}{\Phi^i}\sum_{j=1}^{N^i} A_{N^i,j}^{i(1)}\alpha_j = \frac{1}{\Phi^{i+1}}\sum_{j=1}^{N^{i+1}} A_{1,j}^{i+1(1)}\alpha_j \tag{4.6}$$

After uniting Equations (4.2)-(4.6) and giving solution to them, one can obtain the values of natural frequency parameter λ.

5. EXAMPLE AND COMPARISON

An example of a two-equal-span continuous curved girder is presented to illustrate the application of the DQEM. As shown in Figure 2, all of the three supports are simple. The geometrical and physical conditions of the two spans, i.e. AB and BC, are identical, and the opening angle of each span is $\pi/2$, the rotary inertia γ=1/23.39, shear correction factor k=0.83, and Poisson's ratio =0.3. Then the natural frequency parameter λ is calculated using the

DQEM formulation presented in Section 4. The results are compared with the analytical solutions as shown in Table 1. For simplicity of analysis, the number of sampling grid points of each member is chosen the same.

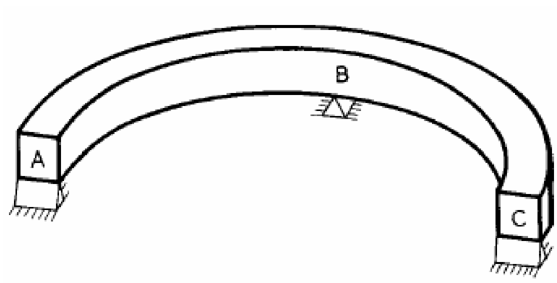

Figure 2 A two-equal-span continuous curved girder

Table 1 Comparison between the DQEM and exact solutions

Natural frequency number	Frequency parameter λ						
	Exact solution	N=5	N=9	N=13	N=17	N=20	N=30
1	2.967	2.883	2.965	2.965	2.965	2.965	2.965
2	5.394	5.545	5.39	5.39	5.39	5.39	5.39
3	14.24	—	14.26	14.23	14.23	14.23	14.23
4	17.89	—	17.91	17.87	17.87	17.87	17.87
5	31.29	—	31.42	31.26	31.26	31.26	31.26
6	35.57	—	35.59	35.54	35.54	35.54	35.54
7	52.43	51.27	54.12	52.36	52.39	52.39	52.39
8	56.82	54.37	60.01	56.75	56.77	56.77	56.77
16	158.8	—	—	—	159	158.6	158.6

From Table 1, it can be seen that there is a good agreement between the numerical and analytical results, and the accuracy of the numerical solution and the number of frequencies resolved increases with increasing N. For general engineering problems, 13 points on each element will be enough to satisfy the required precision, so the DQEM is of higher efficiency compared with the FEM for similar problems.

6. PARAMETRIC ANALYSIS

The model for the present parametric analysis is almost the same as the one in Section 5 except that the conditions of the two ends can change, that is, the middle support of the beam is simply supported for ever. The effects of section gyration radius and flexure-torsion stiffness ratio on five different ends conditions are presented in Figure 3-7, in dimensionless form. In this analysis, the sampling points are unequally spaced by Equation (2.5), and the number of sampling points on each member is 20.

From Figure 3-7, it can be seen that the free vibration frequencies of the girder become small with reducing the restraints of two ends; the effect of the flexure-torsion stiffness ratio can be neglected while μ>10; the effect of section gyration radius is small while γ<0.2 if the restraints of two ends are less, such as the case of having one simply supported and one free end, but the range reduces with increasing restraints of two ends.

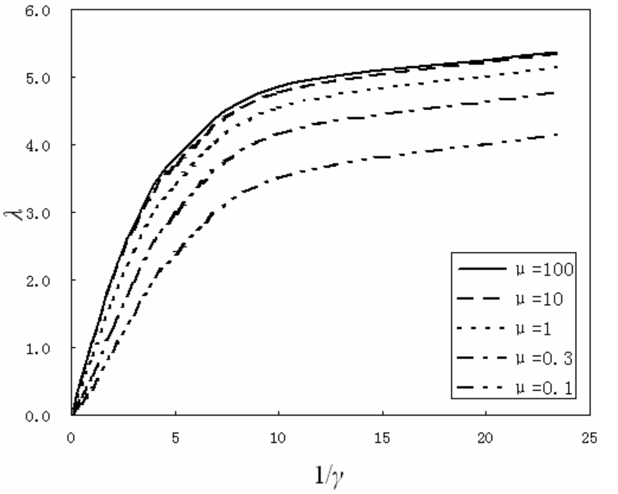

Fig. 3 Parametric results with two clamped ends

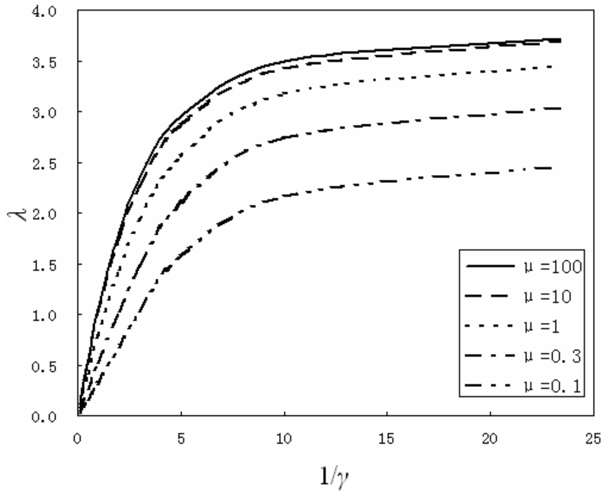

Fig. 4 Parametric results with one clamped and one simply supported end

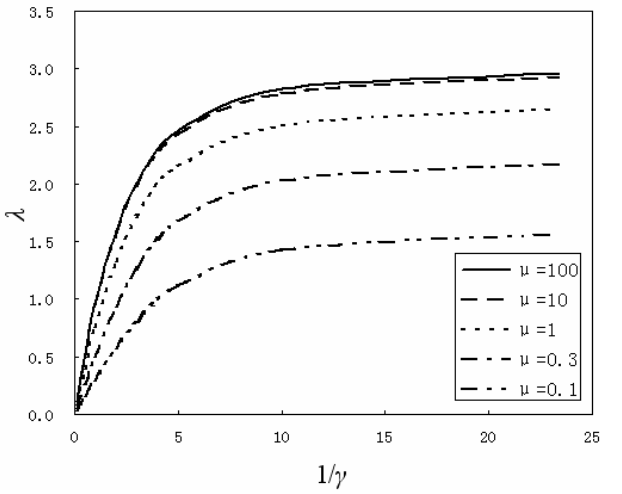

Fig. 5 Parametric results with two simply supported ends

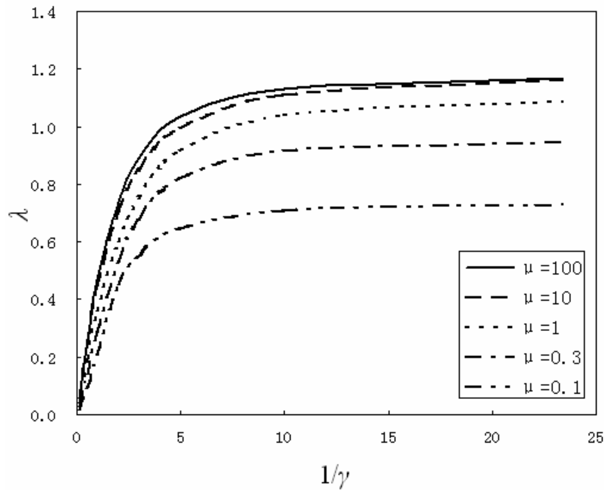

Fig. 6 Parametric results with one clamed and one free end

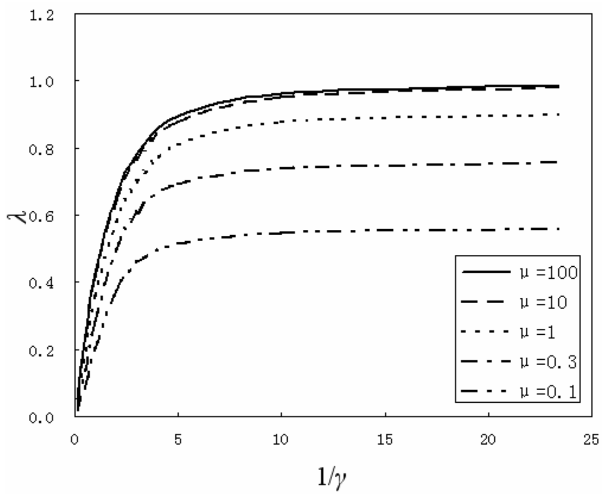

Fig. 7 Parametric results with one simply supported and one free end

7. CONCLUSIONS

Based on the assumption of rigid piers, the DQEM is used to calculate the out-of-plane natural vibration frequencies of continuous horizontally curved girder bridges considering the effect of shear deformation. Natural frequencies of a two-equal-span continuous curved girder are presented and compared with existing exact results. Comparison demonstrates high efficiency of the DQEM. Finally, the effects of section gyration radius and flexure-torsion stiffness ratio on different end conditions, to the out-of-plane natural vibration of two-equal-span continuous curved girders are analyzed using the DQEM. Results indicates that the free vibration frequencies of the girder becomes small with reducing the restraints of two ends; the effect of flexure-torsion stiffness ratio can be neglected while μ>10; the effect of section gyration radius is small while γ<0.2 if the restraints of two ends was less, but the range reduces with increasing the restraints of two ends.

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