CHARACTERISTIC EARTHQUAKE MAGNITUDE: MATHEMATICAL VERSUS EMPIRICAL MODELS

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ABSTRACT :

In the literature concerning the characteristic hypothesis, one basic question is widely discussed: is it possible to justify (by statistical tests) favouring the characteristic magnitude model for the interpretation of available catalogues? No generally accepted answer has been given now a days. In a previous paper (Grandori et al., 2008) we analyzed a different question, perhaps more useful from the engineering point of view: is it possible to judge (on the basis of statistical tests) which one of two competing magnitude models is more reliable (all other things being equal) for the evaluation of a specific hazard quantity at a given site?
In that paper we described a method which can give an answer to this question, and we studied the controversy surrounding the comparison between “characteristic-type” magnitude models and the classic doubly truncated exponential model. We found that in many cases a characteristic magnitude model is more reliable than the exponential model.
In the present paper we recall the main features of the method and we apply it to the comparison between a mathematical model F_M and an empirical (non parametric) distribution F*. The aim is to find an empirical F* which is more reliable than F_M, thanks to the substantial reduction of possible errors due to the use of a wrong model F_M.
We do not give a general method for the construction of such F*, nor we maintain that it exists in all cases. We simply show how, in a study case, we found the way to construct a very satisfactory F*.

KEYWORDS: Magnitude distribution, credibility of the model, comparison between competing models.

1. INTRODUCTION

In the frame of probabilistic seismic hazard analysis, applied to a given site X, one problem is the choice of an appropriate mathematical model of the magnitude-frequency law for the events that can strike site X. The comparison between the reliability of competing models can be based on “the agreement with the current fault segmentation concepts, observations and mechanics-based earthquake simulations” (Wu, Cornell and Winterstein, 1995). However, as regards discriminant statistical tests, it is generally recognized that the seismic record in most seismic zones is too short for a meaningful statistical comparison: “all the relations proposed in the literature appear consistent with the available seismicity catalogs” (Araya and Der Kiuregan, 1998), even if different relations may lead, all other things being equal, to important differences in the final results of the seismic hazard analysis.
In a recent paper (Grandori et al., 2008) we considered the comparison between two competing magnitude models on the basis of the following criterion: instead of asking which one of the two models explains better the data of the available catalog, we ask which one is more reliable for the estimation of a specific target quantity, A, related to the seismic hazard at the given site X. In the same paper it is shown how the criterion works when the comparison is between two mathematical distribution models. The aim of the present paper is to study, with the same criterion, the competition between a mathematical model and a non parametric empirical distribution F*, free (by its nature) from modeling errors.

2. THE METHOD

We call F^0 the unknown true magnitude distribution $F_M(m;\vartheta) = P(M \le m)$ and we assume that: 1) it is independent of the space and time coordinates of the events; 2) the available catalog is a random sample S° drawn from F°.
We compare the competing models "all other things being equal". Precisely, we assume that the models are applied to a test-site for which all other elements that contribute to the estimation of A are known and independent of the magnitude distribution F_M. As a consequence, if a magnitude distribution $\overline{F}_j$ is given (both form and parameters), then a known procedure Z, applied to $\overline{F}_j$, yield the quantity A_j:

$$A_j = Z(\overline{F}_j)\,. \tag{2.1}$$

In particular, obviously, $A^0 = Z(F^0)$ is the true value of A.
The fundamental tool for the achievement of the comparison is the evaluation, for each model F_i, of the foreseeable errors in estimating A, under a given hypothetical "true" distribution F^0. The distribution of such errors will be described by: 1) the mean value $\hat{A}_m$ of 1000 independent estimations obtained from 1000 random samples S^0 drawn from F° with the same size as the available catalog; 2) the standard deviation σ of the 1000 estimates of A_i; 3) an indicator Δ_i^0 that we call the "credibility" of the model F_i with respect to F^0:

$$\Delta_i^0 = P\left\{\left|\hat{A}_i - A^0\right| < kA^0\right\}, \tag{2.2}$$

where $\hat{A}_i$ is the estimator of A with the model F_i, and the parameter k defines a conventional limit.
The selection of the form F_i is affected with the epistemic uncertainty, while the statistical uncertainty, due to the randomness of the sample S^0, concerns the estimation of the parameters. Δ_i^0 is a synthetic index that accounts for both these uncertainties and is connected, through the parameter k, with a level of error which is considered meaningful in the estimation of A.

3. THE COMPARISON BETWEEN MATHEMATICAL MODELS

Let F_1 and F_2 be the mathematical forms of two competing models. A preliminary "basic approach" proceeds through the following four steps.
The first step is the analysis of the errors of F_1 under the hypothesis that the true magnitude distribution has the same mathematical form as F_1 (i.e. F_1 is the right model). The results of this analysis give a measure of the statistical uncertainty connected with the use of the model F_1. A second step regards again the errors of F_1, but under the alternative hypothesis that the model F_1 is wrong (in particular because the truth has the mathematical form of the competing model F_2). This second experiment is representative of the robustness of the model F_1.
The third and fourth steps, with analogous procedure, give an idea of the statistical uncertainty and the robustness of the model F_2.
The results of the basic approach open interesting statistical perspectives, as shown for instance by the application to the following case.
The test-site X has the features that are plausible for a site located in a seismic zone of Southern Italy. The events are uniformly distributed over the zone and follow a Poisson process. The rate of occurrence is 0.13 events per year. The number of events in the catalog is n=40, and the lower cutoff of the catalog is m_0=4.
The target quantity A is the peak ground acceleration (PGA) with 500-year return period at the site X: A=a(500). We assume that for the estimation of a(500) an error larger than 20% is meaningful from the engineering point of view; i.e. we assume k=0.2.

The first model, F_E, is the classic doubly truncated exponential distribution derived from the Gutenberg and Richter relation.
The second model, F_C, is a mixture between the exponential and a linear distribution defined in the two fixed non overlapping ranges $[m_0\ m_E]$ and $[m_E\ m_1]$, a hybrid model in which the relative frequency of strong "characteristic" earthquakes is given by the parameter p, that is the weight of the linear distribution component. By hypothesis m_1-m_E=0.5.
The corresponding probability densities f_E and f_C are shown in the Figures 1 and 2.

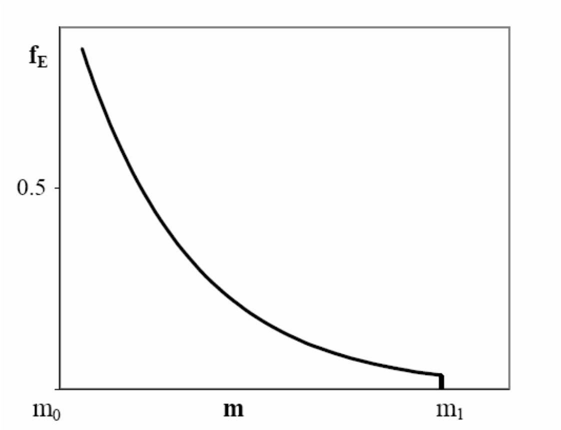

Figure 1 Exponential model m_0=4, m_1=7, b=0.9

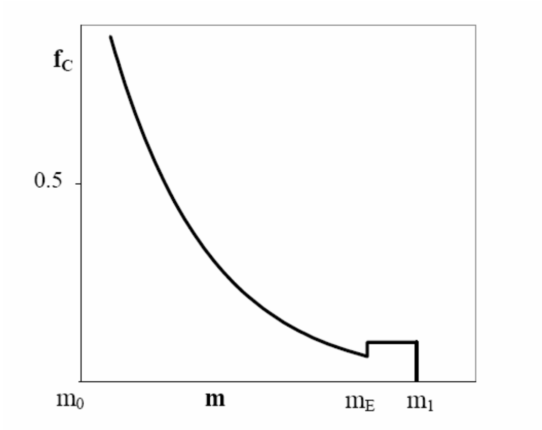

Figure 2 Hybrid model m_0=4, m_1=7, b=0.9,p=0.05
p = area between m_E and m_1

The numerical computations have been carried out by a systematic use of the Montecarlo method for the production of the random samples S^0, and by the maximum likelihood (ML) method for the estimation of parameters.
First step of the basic approach: the hypothetical truth F^0 is a truncated exponential distribution with the parameters $m_1^0 = 7$, $b^0 = 0.9$, shortly indicated exp(7,0.9). The selected model F_1 is exactly a truncated exponential model exp(m_1,b), whose parameters have to be estimated from each one of the 1000 random samples S°. However, the ML estimator of m_1 is biased (Pisarenko et al., 1996), on the other hand, at present there is no generally accepted method for estimating m_1 (Kijko, 2004). At this point, we introduce a simplifying hypothesis, the influence of which will be discussed later in detail: we assume that m_1 has been correctly estimated (e.g. on the basis of geological elements); i.e. the estimate $\hat{m}_i$ coincides with the true value m_1^0. Thus the model becomes exp(7,b) and from each sample S° only the b-value has to be estimated; given m_1 the ML estimator of b is unbiased.
The results of the numerical computations are shown in Table 3.1, first row. The mean value $\hat{A}_m$ coincides with A°; the dispersion of the estimates leads to a credibility Δ=0.68.
Second step. Let us see what happens with the same model if the truth is different; for instance a hybrid distribution with the same parameters $m_1^0 = 7$ and $b^0 = 0.9$ as in the previous case, but with a 5% of the events concentrated between magnitude 6.5 and 7: shortly, hybr(7,0.9,0.05). Still keeping the hypothesis that the true m_1 is known, the results (Table 3.1, second row) show how great may be the influence of the epistemic uncertainty on the reliability of the truncated exponential model: the value of A is remarkably underestimated and the credibility becomes very small.
Third and fourth steps. The behavior of the hybrid model (Table 3.2) is quite different: it gives good estimate of A, with credibility 0.6. if it is the right model; and behaves very well even if the truth is a truncated exponential distribution.
The high effectiveness of the hybrid model derives from the fact that, on the one hand, the parameter p is strictly connected with the relative frequency of strong earthquakes and, on the other hand, the quantity a(500) is mainly governed by such events.

Table 3.1. Results obtained when the truncated exponential is the right model (first row) and when it is a wrong model (second row).

model	truth	A^0	$\hat{A}_m$	σ	Δ
exp(7,b)	exp(7,0.9)	0.19	0.19	0.037	0.68
	hybr.(7,0.9,0.05)	0.38	0.24	0.050	0.10

Table 3.2. Results obtained when the hybrid is the right model (first row) and when it is a wrong model (second row).

model	truth	A^0	$\hat{A}_m$	σ	Δ
hybr.(7,b,p)	hybr.(7,0.9,0.05)	0.38	0.36	0.091	0.60
	exp(7,0.9)	0.19	0.21	0.051	0.59

In conclusion, the results of the above described basic approach (which is one of the examples contained in Grandori et al., 2008) would suggest the hypothesis that F_C is more reliable than F_E for the estimation of a(500) at site X. However, before accepting this hypothesis, two further explorations are needed.

First, the results of the comparison must be in favour of F_C for a plausible range of the "true "parameters m_1^0, b^0. Second, if another model F_D is considered to be plausible, the two models F_C and F_E must be compared also under the hypothesis that the truth has the form F_D.
If all the above mentioned tests are in favour of F_C, it can be concluded that a purely statistical scenario strongly supports the hypothesis that F_C is more reliable than F_E for the estimation of a(500) at site X.
Note that, being the comparison based on 1000 random samples S° (1000 possible catalogs) drawn from each hypothetical true distribution, the result of the competition does not depend on the data contained in the really available catalog, the role of which deserves a few comments.
The catalog cooperates with geological and geophysical knowledge by suggesting plausible models. Moreover, once a model has been chosen, the catalog is essential for the estimation of the parameters.
The point is precisely the choice between competing models, given that "all the relations proposed in the literature appear consistent with the available seismicity catalogs". However, it should be noted that the result of the competition between two models depends on the hazard quantity that one wants to infer from the magnitude distribution. For instance it may happen that F_1 is more reliable than F_2 for the estimation of a(500), while F_2 is more reliable than F_1 for the estimation of a(50). This is the reason why it has been suggested (Grandori et al. 2003) to compare the competing models looking at their credibilities Δ_1^0 and Δ_2^0 in the estimation of the target quantity A.
It is true that the real distribution F^0 is not known, but in that paper it has been shown how, starting from the available catalog, it is possible to obtain, under some reasonable hypotheses, the probability that $\Delta_1^0 > \Delta_2^0$. In other words, thanks to the introduction of the credibility index, the catalog can lead to a discriminating symptom; i.e. it is not always true that the proposed models are equally consistent with the available seismic catalogs: it depends on the comparison criterion. Two ways are then open to the aim of obtaining more stringent results. One way is the method that has been summarized as yet, which is only based on the mathematical structure of the two models and on the features of the site, while it is independent of the available catalog. In a second way, on the contrary, the catalog becomes the main tool leading, through a non parametric procedure, to a substantial reduction of possible errors due to wrong mathematical modelling. This second way is illustrated in what follows.

4. A NON PARAMETRIC PROCEDURE

As we observed before, the robustness of the hybrid model, as far as the estimation of a(500) is concerned, depends mainly on the fact that the parameter p derived from each sample S^0 is strictly connected with the relative frequency of strong earthquakes in the sample. If we abandon mathematical models and from each sample we derive an empirical distribution F*, for instance the cumulative frequency polygon (CFP), we could expect to obtain a procedure with robustness similar to that of the hybrid model. Actually, the CFP follows by its nature the tail of the sample. Let us try with a very simple construction of the empirical distribution: the values of F* are derived from each sample S^0 for magnitude less than 5,6,6.5,7,7.5 (an example in Figure 3). The Table 4.1 shows the results obtained with this non parametric procedure, compared with those obtained from the mathematical models.

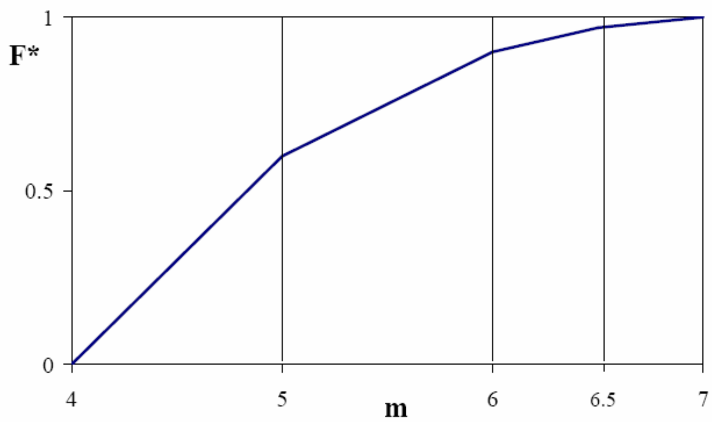

Figure 3 CFP of a sample S^0 drawn from F^0=exp(7,0.9)

Table 4.1 Comparison between the empirical F* and the mathematical models.

truth	A^0	model	A_m	Δ_R	Δ_w	Δ*	Δ^*/Δ_R
exp(7,0.9)	0.19	exp(7,b)	0.19	0.68			0.86
		hybr.(7,b,p)	0.21		0.59		
		F*	0.21			0.58	
hibr.(7,0.9,0.05)	0.38	hybr.(7,b,p)	0.36	0.60			0.98
		exp.(7,b)	0.24		0.10		
		F*	0.36			0.59	

With simplified symbols, Δ_R is the credibility of the right mathematical model (RMM), Δ_W is the credibility of the wrong mathematical model and Δ*is the credibility of the empirical F*.
As expected, the behavior of the non parametric procedure is similar to that of the hybrid model. The exponential model shows a slightly larger credibility when it is the right model, but it is by far loosing the competition if the truth is the hybrid distribution. The results of Table 4.1 are partially a remake and partially an extension of those published in a previous paper (Grandori et al., 2006). The fact that a simple empirical procedure may have a credibility Δ* not far from the credibility Δ_R of the RMM is interesting; and it is worth examining closely a few aspects of this comparison.
Let us consider in detail the comparison between the first and the third row of Table 4.1, i.e. between the RMM and the empirical F* (when the truth is a truncated exponential distribution). The comparison is affected with two main approximations.
First, we did not take into account the uncertainty in the estimation of m_1: the introduction of this uncertainty would decrease the credibility Δ_R. On the other hand, the very simple technique adopted for the construction of F* is open to improvements, that would lead to an increase of the credibility Δ*. It is important to analyze the quantitative influence of these two possible corrections on the ratio Δ^*/Δ_R.
It is true that there is no generally accepted method for estimating m_1. However, the applications of various methods described in the literature give some information about the uncertainty of such estimation. For instance, Pisarenko et al. (1996) in the case of Southern Italy, with 44 events M≥5 and max observed M=7.1, find for the estimate $\hat{m}_1$ a standard deviation (SD) of the order of 0.5. Kijko (2004) finds for Southern California with a non parametric procedure $\hat{m}_1 = 8.54 \pm 0.45$. In the same paper Kijko uses also synthetic data generated according to a G-R relation with m_0=6, m_1=8 and b=1. He finds that the bias of the estimate $\hat{m}_1$ is low: it does not exceed 0.1 unit of magnitude if the number of earthquakes in the catalogue is N=50. However the bias is larger if (m_1-m_0)>2: it may rich 0.3 units of magnitude if (m_1-m_0)=3, as in our site X.
In order to take into account in a synthetic approximate way the available information, given the conditions of site X, we do not assign now to $\hat{m}_1$ the true value m_1^0, but we assume the estimate $\hat{m}_1$ to be a random

variable: normally distributed with mean m_1^0=7 and SD=0.7 in the range $5.65 \le \hat{m}_1 \le 8.35$; being the residual probabilities uniformly distributed in the two extreme classes $5.55 \le \hat{m}_1 \le 5.65$ and $8.35 \le \hat{m}_1 \le 8.45$. With reference to the hypothetical truth $F^0 = \exp(7,0.9)$, the RMM will be now indicated $\exp(\hat{m}_1, b)$ and, in order to remember this kind of estimation for m_1, the relative credibility will be $\hat{\Delta}_R$.

We considered 27 classes of $\hat{m}_1$, width 0.1 unit, each class being identified by its center ($\hat{m}_1 = 7$ means $6.95 \le \hat{m}_1 \le 7.05$). The credibility of the model $\exp(\hat{m}_1, b)$ with respect to the truth $F^0 = \exp(7,0.9)$ is given by

$$\hat{\Delta}_R = \sum_{-14}^{15} p_i \hat{\Delta}_R^i \qquad (4.1)$$

where p_i is the probability that $\hat{m}_1$ is in the class i, and $\hat{\Delta}_R^i$ is the corresponding credibility of the model $\exp(\hat{m}_1, b)$.

The values of $\hat{\Delta}_R^i$ in the field i>0 (that is $\hat{m}_1 \ge 7$) are simply obtained by assuming for each class "given $m_1 = \hat{m}_1$" (instead of "given m_1 = 7" as in Table 3.1). As shown in Table 4.2, the credibility $\hat{\Delta}_R^i$ decreases regularly when m_1 increases, even if the variations are not dramatic.

Table 4.2 Hypothetical truth F^0 = exp(7,0.9). p_i=prob. that $\hat{m}_i$ is in class i. $\hat{\Delta}_R^i$ = credibility of the RMM if $\hat{m}_i$ is in class i.

i	class	p_i	$\hat{\Delta}_R^i$	$p_i \times \hat{\Delta}_R^i$
1	7.0	0.0558	0.675	0.0376
2	7.1	0.0553	0.666	0.0369
3	7.2	0.0536	0.648	0.0348
4	7.3	0.0511	0.637	0.0325
5	7.4	0.0478	0.629	0.0301
6	7.5	0.0437	0.621	0.0271
7	7.6	0.0392	0.616	0.0242
8	7.7	0.0345	0.613	0.0211
9	7.8	0.0299	0.609	0.0182
10	7.9	0.0252	0.606	0.0153
11	8.0	0.0210	0.603	0.0127
12	8.1	0.0171	0.602	0.0102
13	8.2	0.0136	0.601	0.0082
14	8.3	0.0107	0.600	0.0064
15	8.4	0.0294	0.600	0.0176
			$\sum =$	0.3329

In the field i<0 (that is $\hat{m}_1 \le 7$), it is difficult to catch the credibilities $\hat{\Delta}_R^i$, due to the fact that many samples drawn from F^0 will have maximum observed magnitude larger than $\hat{m}_1$. In order to overcome this difficulty, we accept the approximate hypothesis that the distribution of the credibilities $\hat{\Delta}_R^i$ in the field $\hat{m}_1 \le 7$ is symmetrical to that of the field $\hat{m}_1 \geq 7$. So the credibility of the RMM becomes:

$$\hat{\Delta}_R = 0.3329 \times 2 - 0.0376 = 0.63 \quad (4.2)$$

i.e. the uncertainty in the estimation of m_1 reduces the credibility of the RMM from 0.68 (Table 3.1) to 0.63: a reduction of the order of 7%.

As to the empirical F*, its construction may be changed in many ways (see for instance Grandori et al., 2004) Here we propose a correction that fulfils two main conditions: first, to keep a high simplicity and, second, to increase as much as possible the credibility of the empirical procedure. Starting from the simple F* of Figure 3, the corrected distribution F_h^* is given by:

$$F_h^* = F^* + h(1 - F^*)\frac{m - m_0}{m}. \quad (4.2)$$

The construction of F_h^* is very simple and, what is more important, the factor h can be adjusted in order to obtain a better performance of the empirical procedure. For instance, in the case of hypothetical truth F^0 = exp(7,0.9), the Figure 4 shows the influence of the factor h on the results obtained with the empirical F_h^*. With h=1.25 the expected estimate of A (A_m^{*h})coincides with A^0 and the credibility of the empirical F_h^* is even larger than the credibility of the RMM ($\Delta_h^* / \hat{\Delta}_R = 1.04$).

The factor h has been adjusted looking at the hypothetical truth exp(7,0.9). However, with h=1.25, the empirical procedure F_h^* is very robust; Table 4.3 shows the results concerning different hypothetical truths. In all the considered cases, keeping h=1.25, we obtained $\Delta_h^* \cong \hat{\Delta}_R$.

The application of the non parametric procedure that we just described refers to the evaluation of a(500) at the site X; the results obtained cannot be simply extrapolated to other cases. They only show that, once a specific mathematical model F_i has been selected, it is worthwhile to explore the possible existence of a non parametric F* (our F_h^* is just an example) having credibility Δ* practically equal to the credibility $\hat{\Delta}_R^i$ of F_i (remember that $\hat{\Delta}_R^i$ is evaluated under the optimistic hypothesis that F_i is the right model). If such a F* exists, its adoption instead of the mathematical F_i reduces substantially possible errors due to epistemic uncertainty.

5. CONCLUSIONS

The comparison between two plausible competing magnitude models is often difficult because the available catalog is too short. This difficulty can be overtaken by comparing the foreseeable errors made by the two models (in the estimation of the target quantity A) under appropriate hypothetical "true" magnitude distributions. Following this method it is possible to obtain a statistical scenario which suggests rational decisions when facing the choice between two mathematical magnitude models.

In particular, we considered the controversy surrounding the comparison between the classic exponential model F_E and a "characteristic type" model F_C, applied to the estimation of a(500) at a given test site X. The results of a preliminary basic approach are clearly in favour of F_C. Further results are contained in our previous paper (Grandori et al., 2008). Once a mathematical model F_i has been selected (whatever method has been used for the selection) the non parametric procedure may lead to a further reduction of possible errors due to wrong mathematical modeling. This happens if an empirical F* succeeds in reaching a credibility Δ* practically equal to the credibility $\hat{\Delta}_R$ of the model F_i. In this case let us call A* the value of a(500) obtained from the empirical F* applied to the available catalog, and A^i the value corresponding to F_i. In spite of the fact

$\Delta^* \cong \hat{\Delta}_R$ it may be that A* and A^i differ notably from one another; this would be a symptom suggesting that F_i is a wrong model and that A* is more reliable than A^i.

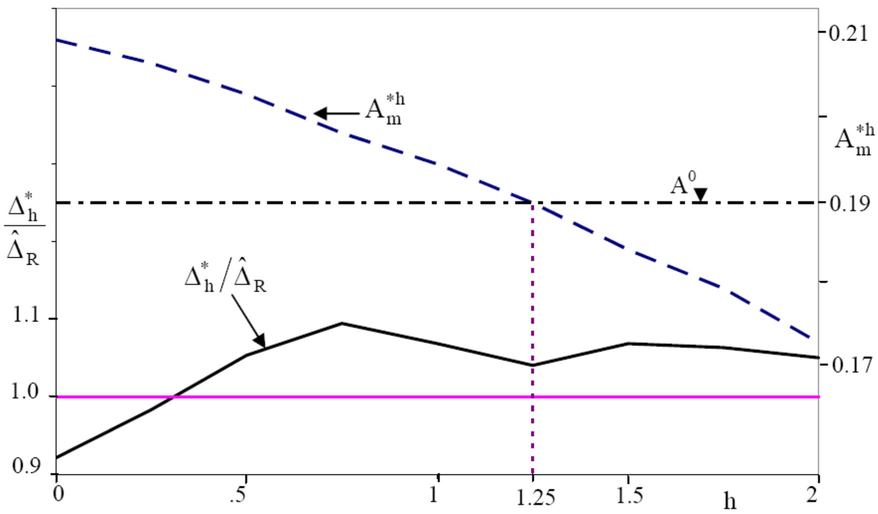

Figure 4 Performance of F_h^* as function of h in the case F^0 = exp(7,0.9)

Table 4.3 Performance of F_h^* (h=1.25) versus RMM

truth	$\Delta_h^* / \hat{\Delta}_R$	**truth**	$\Delta_h^* / \hat{\Delta}_R$
exp. (7,0.9)	1.04	hybr. (7,0.9,0.05)	1.11
exp. (7,1.3)	1.04	hybr. (7,1.3,0.05)	1.27
exp. (7.2,0.9)	1.02	hybr. (7.2,0.9,0.05)	1.17
exp. (7.2,1.3)	1.05	hybr. (7.2,1.3,0.05)	1.09

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