MULTI-TIME-STEP IMPLICIT METHODS FOR NON-LINEAR REINFORCED CONCRETE STRUCTURES

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ABSTRACT :

The paper explores multi-domain strategies in order to reduce time costs as well as mitigate memory problems for simulations of the non linear behaviour of reinforced concrete structures under seismic loading. The multi-scale time approach proposed by **COMBESCURE AND GRAVOUIL [2001,2002,2003]** enables to divide the computation domain in several subdomains connected by interfaces and to choose for each subdomain the best appropriate time-integrator amongst the Newmark family (α and β time integration scheme parameters) with a suitable time-step. Each subdomain is first solved independently (free solution) and the global solution is then computed by taking into account the link relations on the interfaces (link corrections added to previous free solutions). The subdomain coupling is achieved by prescribing continuity of velocities on interfaces enabling the stability of the algorithm as proved by the authors for all following cases of coupling time-integration schemes: implicit/implicit, explicit/implicit, explicit/explicit. This work is here focused on coupling implicit schemes (mean acceleration scheme) with different time-steps depending on subdomains. First we present the multi-domain strategy for a structure decomposed in two subdomains: a linear implicit subdomain with a large time step is coupled with a non linear subdomain with a fine time step. The flowchart of the subdomain strategy is given. The subdomain strategy is applied to two structures modelled with multi-fibre Timoshenko beam elements: a reinforced concrete column and a simplified three-story one bay building designed according to older construction code without earthquake provisions. Energy dissipation at subdomain interfaces is scrutinised in order to assess the effect of the size of the large time-step as well as the ratio between the two time steps.

KEYWORDS: **multi-time-step method ; domain decomposition ; transient analysis ; reinforced concrete structures**

1. INTRODUCTION

Transient non linear analyses for large structures face with computation time costs and memory requirements for storing numerical data. In order to reduce time costs as well as mitigate memory problems, we seek to set up a multi-time step strategy depending on subdomains composing the whole structure. Time-space multi-scale algorithms have been proposed in literature **[GRAVOUIL AND COMBESCURE, 2001,2002]** for the resolution of transient problems: linear and non linear transient analyses have been carried out in a first time independently on each subdomain and then globally on the whole structure through the resolution of interface problems. The method enables to keep for each subdomain its own time-step and time-integration scheme (explicit/implicit scheme of the Newmark's family) as well as an appropriate mesh not necessary matching with its neighbour subdomains. The subdomain capabilities are particularly attractive when structure is submitted to actions with very different time scales. For instance, a structure submitted to an impact loading may experience very localised non linear effects with high frequency content associated with a fine time scale on a subdomain whereas the rest of the structure may be concerned by structural vibrations on a much larger time scale. One another great advantage is that, from a computational efficiency point of view, time-steps associated with subdomains can be very different allowing substantial computation time reductions with respect to an analysis whose time-step is set by the shortest finite element of the whole structure in explicit time integration.
In this paper, the subdomain method is applied to reinforced concrete structures modelled as multi-fibre Timoshenko beams. The main interest consists in coupling Newmark implicit scheme classically used for a structure under earthquake loading. The purpose is to gain numerical efficiency by taking into account large time-steps for linear parts of the structure and concentrating numerical efforts on non linear parts with a fine time step. The general subdomain method is presented in the first part of this paper. Then, we set up the algorithm coupling linear and non linear implicit schemes with different time steps. Finally, the method is validated by considering two reinforced concrete structures: a column and a simplified three-story structure. Specimens under consideration are derived from a full scale three story two-times-two bays irregular reinforced concrete structure pseudodynamically tested at the European Laboratory for Structural Assessment (ELSA) of the Joint Research Center (JRC) in Ispra, under the auspices of the EU project Seismic Performance Assessment and Rehabilitation (SPEAR) **[FARDIS, 2002]**.

2. SUBDOMAIN METHOD FOR COUPLING TIME INTEGRATION SCHEMES

The purpose of this fist paragraph is to highlight the key points of the subdomain method proposed by Gravouil and Combescure [1,2]. The Finite Element Method applied to a structure, associated with a domain noted Ω, gives the equilibrium equation:

$$M\ddot{U} + F_{\text{int}}(U) = F_{ext} \tag{2.1}$$

where M is the symmetric, definite, positive mass Matrix, F_{ext} the external forces and $F_{\text{int}}(U)$ the internal forces, $\ddot{U}$ the acceleration.
The domain Ω is decomposed into several subdomains, designed by the indice $k = 1,\ldots,s$. The equilibrium equation can be rewritten for each subdomain k:

$$M^k\ddot{U}^k + F_{\text{int}}^k = F_{ext}^k + F_{link}^k \tag{2.2}$$

with F_{link}^k represents the interface loads applied to the subdomain k under consideration (interface loads), which can be linked to the kinematic constraints on interfaces of the k subdomain. The kinematic constraint connecting all subdomains together is written in a global form, for the whole domain:

$$\sum_{k=1}^{s} C^k\dot{U}^k = 0 \tag{2.3}$$

where C^k is the constraint matrix for a k subdomain. If we consider only matching interfaces, this constraint matrix is a Boolean one, depending if the node is located on interfaces (1 into the matrix) or not (0 otherwise).
The link force is then obtained by introducing Lagrange multipliers Λ:

$$F^k_{link} = C^{k^T}\Lambda \qquad (2.4)$$

where the vector Λ gathers all Lagrange multipliers for the whole domain. The decomposition of the structure follows a Schur dual formulation: the equilibrium at interfaces is automatically satisfied through the use of Lagrange multipliers while kinematic continuity has to be prescribed. It is important to underline that the choice of kinematic constraint on velocities rather than on displacements and accelerations has been proven to be the best choice because it enables coupling between any Newmark scheme without altering the global stability of the time-integration method **[GRAVOUIL AND COMBESCURE, 2001,2002].**

The method proposed by the authors is based on the observation that the acceleration vector for the subdomain k can be expressed as the sum of two terms: accelerations $\ddot{U}^k_{free}$ without considering interface forces and accelerations $\ddot{U}^k_{link}$ by considering only interface loads.
The equilibrium equation can then be splitted into a free problem and a link problem as follows:

$$\begin{aligned} &M^k\ddot{U}^k_{free} + F^k_{\text{int}} = F^k_{ext} \\ &M^k\ddot{U}^k_{link} = F^k_{link} \qquad \textit{for } k = 1,\dots,s \\ &\ddot{U}^k = \ddot{U}^k_{free} + \ddot{U}^k_{link} \end{aligned} \qquad (2.5)$$

with the kinematic constraints on all interfaces: $\sum_{k=1}^{s} C^k\dot{U}^k = 0$.

The strategy of decomposition of the global quantities into a free part and a link part is the key point of the method proposed by the authors. The method is valid with any Newmark time-integration scheme, for linear as well as non-linear problems.
Each subdomain k is discretised in time with its own Newmark scheme defined by the two parameters β_k and γ_k (we can consider here implicit or explicit integration depending on subdomains) and its own time step Δt_k.
The time-integration schemes of the Newmark family can be sketched as follows at a given each time step $\Delta t_k = [t_n; t_{n+1}]$. The scheme begins with the computation of predictor quantities obtained directly from the quantities at the time t_n, given by:

$$\begin{aligned} {}^p\dot{U}^k &= \dot{U}^k_n + \Delta t_k\left(1-\gamma_k\right)\ddot{U}^k_n \\ {}^pU^k &= U^k_n + \Delta t_k\dot{U}^k_n + \frac{\Delta t_k^2}{2}\left(1-\beta_k\right)\ddot{U}^k_n \end{aligned} \qquad (2.6)$$

and after computing the new accelerations from the equilibrium equation at time t_{n+1}, the final quantities are obtained by:

$$\begin{aligned} \dot{U}^k_{n+1} &= ({}^p\dot{U}^k) + \gamma_k\Delta t_k\ddot{U}^k_n \\ U^k_{n+1} &= ({}^pU^k) + \beta_k\Delta t_k^2\ddot{U}^k_n \end{aligned} \qquad (2.7)$$

In the following, we shall omit the time symbols n and n+1. In order to simplify the presentation, we assume in this section an elastic behaviour: $F^k_{\text{int}} = K^kU^k$. In addition, in the sake of clarity, we assume that all subdomains have the same time step: $\Delta t_k = \Delta t \;\; \forall k$. The case of multi-time step will be exposed in the following section.
For a given subdomain k, the equilibrium equation dicretised in time and space on a given time-step Δt is given by:

$$(M^k + \beta_k\Delta t^2K^k)\ddot{U}^k = F^k_{ext} - K^k\,({}^pU^k) \qquad (2.8)$$

where ${}^pU^k$ is the displacement predicted by the chosen Newmark scheme.

The global problem discretised in time can be written as follows:

$$\begin{bmatrix} M^1+\beta_1\Delta t^2 K^1 & 0 & 0 & -C^{1T} \\ 0 & \ddots & 0 & \vdots \\ 0 & 0 & M^s+\beta_s\Delta t^2 K^1 & -C^{sT} \\ -\gamma_1\Delta t C^1 & \cdots & -\gamma_s\Delta t C^s & 0 \end{bmatrix} \begin{bmatrix} \ddot{U}^1 \\ \vdots \\ \ddot{U}^s \\ \Lambda \end{bmatrix} = \begin{bmatrix} F_{ext}{}^1 - K^1\left({}^pU^1\right) \\ \vdots \\ F_{ext}{}^s - K^s\left({}^pU^s\right) \\ \sum_{k=1}^{s} C^k({}^p\dot{U}^k) \end{bmatrix} \quad (2.9)$$

The last line of the previous system is obtained by rewriting the kinematic condition. Indeed, the Newmark scheme gives:

$$\dot{U}^k = ({}^p\dot{U}^k) + \gamma_k \Delta t \ddot{U}^k$$

And after rewriting the kinematic constraints, we obtain:

$$-\sum_{k=1}^{s} C^k \dot{U}^k = -\sum_{k=1}^{s} C^k({}^p\dot{U}^k) - \sum_{k=1}^{s}\gamma_k \Delta t C^k \ddot{U}^k = 0 \quad (2.10)$$

The whole problem can decomposed into two problems:

1. A free problem:

$$\begin{bmatrix} M^1+\beta_1\Delta t^2 K^1 & 0 & 0 & 0 \\ 0 & \ddots & 0 & \vdots \\ 0 & 0 & M^s+\beta_s\Delta t^2 K^1 & 0 \\ 0 & \cdots & 0 & 0 \end{bmatrix} \begin{bmatrix} \ddot{U}_{free}{}^1 \\ \vdots \\ \ddot{U}_{free}{}^s \\ 0 \end{bmatrix} = \begin{bmatrix} F_{ext}{}^1 - K^1({}^pU^1) \\ \vdots \\ F_{ext}{}^s - K^s({}^pU^s) \\ 0 \end{bmatrix} \quad (2.11)$$

2. A link problem (obtained after condensation on the last line of the system):

$$\begin{bmatrix} M^1+\beta_1\Delta t^2 K^1 & 0 & 0 & -C^{1T} \\ 0 & \ddots & 0 & \vdots \\ 0 & 0 & M^s+\beta_s\Delta t^2 K^1 & -C^{sT} \\ 0 & \cdots & 0 & H \end{bmatrix} \begin{bmatrix} \ddot{U}_{link}{}^1 \\ \vdots \\ \ddot{U}_{link}{}^s \\ \Lambda \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ B \end{bmatrix} \quad (2.12)$$

where H, which is called the interface operator, is a condensed matrix on interfaces through constraint matrices C^k defined by:

$$H = \sum_{k=1}^{s} \gamma_k \Delta t\, C^k (M^k + \beta_k \Delta t^2 K^k)^{-1} C^{kT} \quad (2.13)$$

and B is a vector given by :

$$B = -\sum_{k=1}^{s} C^k \dot{U}^k \quad (2.14)$$

After computing the Lagrange multipliers Λ, we can compute the interface forces on a given subdomain k $F_{link}^k = C^{kT}\Lambda$, derive the link accelerations $\ddot{U}_{link}^k$ and complete the global solution by summing free part and link part.

3. COUPLING TWO IMPLICIT SCHEMES WITH DIFFERENT TIME-STEPS FOR NON LINEAR PROBLEMS

In the case of implicit non-linear subdomains, the method requires iterations on the implicit subdomain so as to comply with the equilibrium at each time of the Newmark scheme. We present here the method enabling to couple a linear Implicit subdomain A (with a large time-step Δt_A = macro-time scale) and a non linear Implicit subdomain B (with a fine time-step Δt_B =micro-time scale). We assume that we have $\Delta t_A = m\Delta t_B$.

The subdomain A is assumed to remain linear elastic. Non linear effects are taken into account for the subdomain B. Newmark parameters are $\gamma_A, \beta_A, \gamma_B, \beta_B$ for subdomains A and B. Non varying time steps Δt_A and Δt_B are assumed.
The non linear constitutive laws for the subdomain B are correspond to the concrete and steel material behaviour in the cross section of the multifiber Timoshenko beam **[GUEDES ET AL., 1994]**.
We have to rewrite the discrete in time and space equilibrium equation for the subdomain B:

$$(M^B + \beta_B \Delta t_B^2)\ddot{U}^B_{free,j} = F^B_{ext,j} - F^B_{\mathrm{int},j} + F^B_{link,j} \tag{3.1}$$

Internal forces are obtained from incremental strains, non linear stresses and internal variables compatible with constitutive laws. Newton iterations at equilibrium are carried out until the convergence is attained, corresponding to a residual small with respect to the maximum of the internal and external forces. At a given iteration I of the Newton scheme, the residual can be expressed as follows:

$$\left(R_j^B\right)^i = F^B_{ext,j} - \left(F^B_{\mathrm{int},j}\right)^i + M^B\left(\ddot{U}^B_j\right)^i + \left(F^B_{link,j}\right)^i \tag{3.2}$$

and the correction on accelerations is:

$$\left(\Delta\ddot{U}^B_{free,j}\right)^i = A\left(R_j^B\right)^i \quad with \quad A = K^B_{\tan} + \beta_B \Delta t_B^2 M^B \tag{3.3}$$

In a strictly Newton-Raphson method, the tangent matrix $K^B_{\tan}$ has to be used. Nonetheless, due to convergence difficulties when constitutive material laws exhibit high softnening behaviour, we prefer to use here the simple elastic stiffness matrix K^B.

The subdomain method briefly presented above is denoted the GC method in the literature **[GRAVOUIL AND COMBESCURE, 2001,2002,2003]**. This method is known to dissipate at interfaces when an implicit time-integration scheme is involved at the macro-time scale. This numerical dissipation at interfaces is generated by the linear interpolation of free velocities of the implicit domain A with the macro-time scale before solving the interface problem on the subdomain B at the micro-time scale. This dissipation appears in linear and non linear problems. For non linear problems concerned in this study, we have thus to check the global equilibrium balance so as to assess the amount of energy dissipated at interfaces. The expression of the discrete energy balance on a macro-time scale $\Delta t_A = [t_0; t_m]$ has been proposed by **[HUGUES AND BELYTSCHKO, 1995]**. This formulation has been extended to a structure decomposed in subdomains. The dissipated energy on a time-step $\Delta t_A = [t_0; t_m]$ for a structure decomposed into two subdomains is given by:

$$\Delta W_{\mathrm{interface}} = -\Delta W^A_{ext} - \Delta W^B_{ext} + \Delta W^A_{kin} + \Delta W^B_{kin} + \Delta W^A_{\mathrm{int}} + \Delta W^B_{\mathrm{int}} \tag{3.4}$$

where the terms in the right side of the equation are related to the variation of the external, kinetic and internal energies in subdomains A and B.
It is obvious that the interface energy must be negative otherwise energy creation at the interfaces will destroy the stability of the approach. When the interface energy is negative, some numerical damping appears at subdomain interfaces. Nonetheless, the stability is maintained. The best case is when this interface energy is null. This is the case for a unique time-step $\Delta t_A = \Delta t_B$ as it has been proven by the authors with any time-integration Newmark scheme as long as velocity continuity is prescribed at interfaces; the interface energy is then equal to zero and the discrete energy balance is preserved. The subdomain method is not dissipative in this case. This is not the case for different time steps in the case of an implicit subdomain at the macro-time scale. Explicit/Implicit coupling cases with the GC method investigated in the literature show that dissipated energy at interfaces of subdomain should be carefully observed so as not to jeopardise the global results. For instance, an Explicit/Implicit simulation concerning an impact on the containement vessel of a nuclear reactor exhibits altered solutions when the time-step ratio between subdomains becomes too high. In this paper, we will check the amount of dissipated energy against the quality of the global results for Implicit/Implicit simulations depending on the time-step ratio and the size of the macro-time scale.

3. APPLICATIONS

The coupling between implicit time integration schemes with different time-steps is applied to two reinforced concrete structures under earthquake loading. The first example deals with a reinforced concrete column and the second one with a three story one bay structure. The structural and earthquake characteristics chosen in this study are derived from the SPEAR structure pseudodynamically tested at the European Laboratory for Structural Assessment of the Joint Research Center (JRC/ELSA). The SPEAR structure was a full scale structure of 3 story 2×2 bays designed according to older construction code in Greece in the early 70's. So the structural configuration was chosen to be typical of non-earthquake-resistant construction in order to investigate different retrofitting strategies. Details of this tested structure can be found elsewhere **[FARDIS, 2002]**.

3.1. Modeling of reinforced concrete members

We use multifiber Timoshenko beam elements to model columns and beams composing the two structures under investigation. The finite element code CAST3M is employed **[VERPEAUX ET AL., 1998]**. The multifiber Timoshenko beam element is based on geometrical description of the cross section composed of concrete and steel fibers. This 3D-beam element assumes a unique Gauss section in order to avoid shear locking. So axial strain, curvature and shear strain is assumed to be constant on the element. Each fiber is associated with an uniaxial law representative of non linear concrete or steel behaviour. The uniaxial concrete law takes into account softening in tension and compression as well as cyclic specificities. The unixial steel law involves hardening, Baushinger effects and a simplified buckling treatment. More details of this beam element and uniaxial constitutive laws can be found in **GUEDES ET AL. [1994]**. For the first structure, the column is modeled by 11 multifiber Timoshenko beam elements with a 0.25m×0.25m cross-section. For the three story one bay structure, the cross-sections of columns are identical as previous and the beams are modeled by 10 multifiber Timoshenko beam elements with a 0.50m×0.25m cross-section ; the contributions of the slabs to the beam stiffness has been reflected by effective width of the T-section. The effective flange width is assumed to be the beam width plus 7% of the clear span of the beam on either side of the web as proposed by **FARDIS [2002]**. Consistent mass matrices are computed for the two structures and we have added concentrated mass at the top of columns. Concentrated masses are resumed above:
- RC column: M=2000 kg at the top node of the mesh (arbitrary mass chosen for the simulation).
- 3 story, one bay RC structure: M=2418 kg at each top node of columns (equal to the mass of a slab minus the self-weight of the beam with its slab flanges).

The concrete has a Young modulus of 25000 MPa, a compressive strength of 26.4 MPa and a tensile strength equal to 2.1 MPa. Due to the fact that the SPEAR structure exhibits very poor detailing, no confinement effect has been taken into account. For the steel, the measured elastic modulus, yield and failure stress were adopted: Young modulus of 200000 MPa, yield stress of 474 MPa and a strain-hardening range up to the maximum strength of 651.6 MPa for an ultimate strain equal to 0.28.

3.2. Multi-time decomposition analysis for a reinforced concrete column under earthquake loading

Non linear finite element analysis is carried out on the first structure submitted to the seismic signal used in the SPEAR tests, that is the Herceg-Novi earthquake recorded in Montenegro in 1979, scaled at 0.15g. The first 5 s of the signal is considered in the following validations of the subdomain method. After computing a classical non linear analysis without subdomain decomposition, a subdomain decomposition of the whole column is set up. 11 beam elements with the same size compose the whole structure: the non linear subdomain B includes 7 elements at the base of the column whereas the linear subdomain A corresponds to the rest of the structure (4 elements for the subdomain A). Implicit non linear time integration scheme on the subdomain B is coupled with an implicit linear scheme on the subdomain A. Different time steps for subdomains A and B are investigated for scrutinizing the numerical dissipation generated at the suboamin interface. The amount of dissipated energy (in absolute value) at the interface for two coupling cases is given in Fig. 1. For the first coupling case ($\Delta t_A = 5E-3s / \Delta t_B = 1E-3s$) the cumulated dissipated energy at the end of the simulation is equal to 0.8 J that is less than 1% of the maximum of the cumulated external energy during the simulation. Top displacement of the RC column shown in Fig. 2 is in very good agreement with respect to the reference results (non linear analysis without subdomain decomposition). It can be observed that a larger macro-time scale $\Delta t_A = 2E-2s$ with approximately the same time ratio than previously (4 in this case instead of 5) leads to a stronger dissipation and a visible stronger numerical damping on displacement peaks.

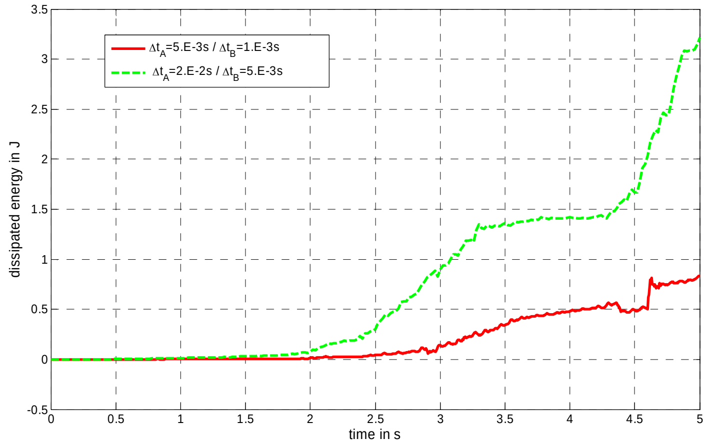

Figure 1 Dissipated energy (in absolute value) at the interface for two coupling cases (SD A = Implicit / SD B = Implicit non linear) with different time-steps

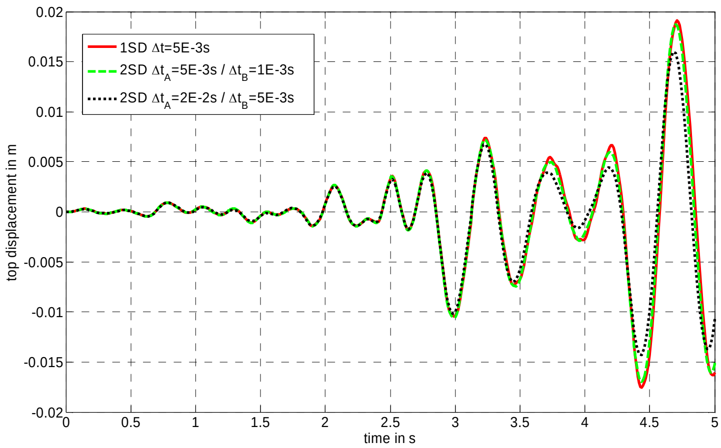

Figure 2 Top displacements for two coupling cases (SD A = Implicit / SD B = Implicit non linear) with different time-steps

3.3. Multi-time decomposition analysis for a 3 story, one bay reinforced concrete structure under earthquake loading

The previous earthquake has been again employed. The three story one bay RC structure has been derived from the SPEAR structure. The first 5 s of the signal is again considered in the following validations. The structure has been designed according to older construction code without earthquake provisions. Columns are slender and not strong enough to carry a large magnitude of bending caused by earthquake lateral forces due to earthquakes. They are also much more flexible than the beams. This weak column-strong beam leads to concentrate non linear degradations on the columns while preserving beams. A first simulation confirms that under the 0.15g Herceg-Novi earthquake, all non linearities are located in the columns. We propose the following subdomain decomposition: the non linear subdomain B is composed of columns and two beam elements at each floor (connected to the columns) whereas the other beam elements are included in the linear subdomain A. The displacements of the roof are compared for two cases of non linear Implicit/ linear Implicit coupling with different time step. Displacements are shown in Fig. 3 for two coupling cases compared with the reference results obtained without subdomain decomposition. Subdomain decomposition results are in excellent agreement with the reference results. In this case, we can conclude that numerical damping issued from the decomposition method remains sufficiently low for not altering the quality of the global results.

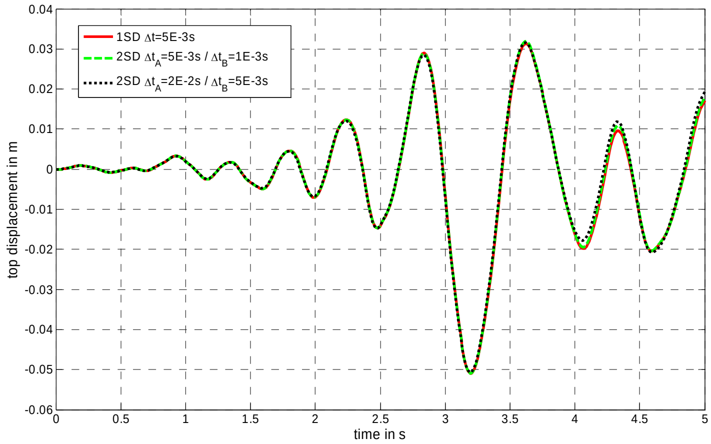

Figure 3 Top displacement for two coupling cases (SD A = Implicit / SD B = Implicit non linear)with different time-steps

3. CONCLUSION

Coupling of implicit schemes with different time steps depending on the subdomain has been investigated in this paper. The main goal is to reduce computation times and memory requirement by concentrating numerical efforts on non linear parts of the structure under earthquake loading; the linear part may then be dealt with a large time step. The GC method essentially used in the coupling Implicit/Explicit or Explit/Explicit Newmark schemes has been adopted for subdomain decomposition with Implicit schemes associated with different time-steps. The results are very consistent with reference results obtained without subdomain decomposition. Nonetheless, it has to be noted that energy dissipated at the interfaces may alter the quality of the global results for a long duration as a seismic excitation. Works are in progress to assess the methodology for a full scale reinforced structure.

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