SMOOTHED PARTICLE HYDRODYNAMICS METHOD IN MODELING OF STRUCTURAL ELEMENTS UNDER HIGH DYNAMIC LOADS

D. Asprone[1*], F. Auricchio[2], A. Reali[2], G. Sangalli[3], A. Prota[1], G. Manfredi[1]

[1] *Department of Structural Engineering , University of Naples "Federico II", Via Claudio 21, 80125 Napoli, Italy*
[2] *Department of Structural Mechanics - University of Pavia, Italy*
[3] *Department of Mathematics - University of Pavia, Italy*
**e-mail: d.asprone@unina.it*

ABSTRACT :

Nowadays, events like severe earthquakes or man-made malicious actions are often taken into account in structural design of critical infrastructures and consequently high dynamic loads are considered in structural analyses. In particular, it is aimed to reproduce large displacements fields, dynamic fracture mechanisms (fragmentations, etc.) and high stress concentrations. Classical numerical methods, like Finite Element Method (FEM), may be inadequate to model the mechanical behavior of structural elements under such actions. In fact, high deformation gradients and unforeseeable failure mechanisms can represent critical aspects for FEM methods. As a consequence, several meshless methods, originally developed for fluid-dynamics, have been recently investigated in order to adapt them to solid continuum mechanics.
Smoothed Particle Hydrodynamcs (SPH) method, belonging to meshless methods, is here described. Classical numerical formulations are presented and the basic idea of the SPH approach is described. Then, the attention is focused on the expressions used to approximate derivatives, since these formulations play a fundamental role in developing numerical framework to reproduce dynamic problems. Deficiencies and criticalities related to such a point are described and the most common improvements proposed in literature are summarized. Then, an original approach is presented, based on a direct control of the convergence error. Performances of the proposed expressions are outlined via numerical tests. In particular second order of convergence in treating second derivatives is outlined and numerical spectra are derived and described, comparing results from the proposed formulation with those from other SPH methods and from linear FEM.

KEYWORDS: Severe dynamic conditions, Numerical methods, Meshless methods, Smoothed Particle Hydrodynamics

1. INTRODUCTION

In computational mechanics an intense research activity is constantly conducted aiming to propose and improve numerical methods and procedures, to simulate particular mechanics conditions, such as high dynamic loading conditions, large displacements or rapid deformations. Indeed, these conditions can represent severe criticalities for classical numerical methods, such as Finite Element Methods (FEM) or Boundary Element Methods (BEM). Mesh distortion and stress concentration, in particular, represent two of the main issues that researchers try to overcome, introducing new numerical procedures. In case of seismic engineering, particularly complex analyses, especially for severe seismic excitations, can be subjected to such undesirable lacks of accuracy and consequently can need to be conducted via particular numerical procedures. Some of the most interesting numerical methods for severe dynamic conditions are the so-called meshless methods. This expression indicates a number of numerical methods that, using different approaches, do not need a mesh discretization of the continuum, as intended in the classical FEM approach. Alternatively, a number of points are defined, where all the variables and equations are evaluated. In this way, points, classically referred to as “particles”, can be considered as degrees of freedom, whose displacements do not introduce any mesh distortion and consequently numerical instabilities are avoided. On the contrary, meshless methods often suffer of lack of accuracy close to the boundary, which leads to the necessity of particular procedures, trying to enforce convergence of numerical solutions in every point of the domain.

2. SMOOTHED PARTICLE HYDRODYNAMICS METHOD

SPH (Smoothed Particle Hydrodynamics) method was first introduced by Lucy (1977) and Gingold and Monaghan (1977) to address astrophysics problems. Then the method was used in a number of applications, mainly in fluid-dynamics (Monaghan 1992); then it was successful introduced in mechanics and structural dynamics (Benz and Asphaug, 1995, Belytschko et al. 1996).
SPH acronym is commonly attributed in literature to a group of meshless methods, sharing the same basic approach. Indeed, in all SPH procedures, aiming to approximate a function $A(x)$, the following holding expression is considered:

$$A(x_i) = \int_D A(x)\delta_i(x)d\Omega \tag{2.1}$$

where $\delta_i(x)$ is the Dirac function and D is the domain of $A(x)$; hence, a first approximation is introduced, through the following expression:

$$A(x_i) \cong \int_D A(x)W_i(x)d\Omega \tag{2.2}$$

where $W_i(x)$, called kernel function, is substituted to approximate the Dirac function. Common kernel functions are Gaussian-like functions, as that depicted in Figure 1, whose expressions can be exactly that of Gaussian functions or also polynomial functions.

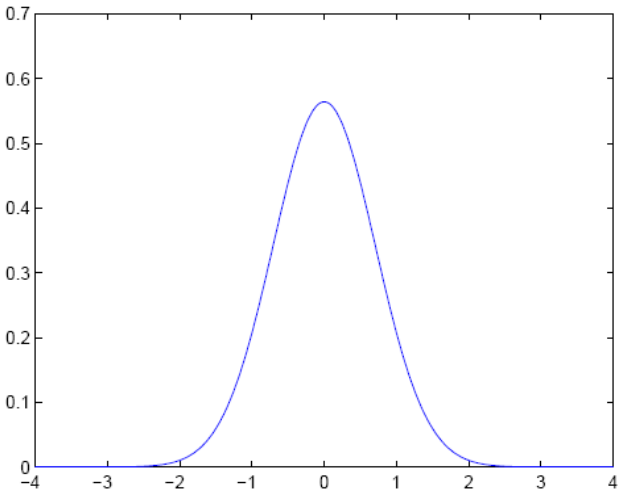

Figure 1 Kernel function

In order to correctly reproduce the Dirac and in order to guarantee the convergence of the approximation method, the following properties are commonly requested to kernel functions:

- Positive in its domain;
- Unitary area within its domain;
- more times derivable, with continuous derivatives;
- defined in a compact support.

Hence, a second approximation is introduced, substituting equation 2.2 with its discrete expression:

$$A(x_i) \cong \sum_{j=1}^{N} A(x_j) W_i(x_j) \Delta\Omega_j \qquad (2.3)$$

where a partition of D has been introduced through a finite number N of spaces $\Delta\Omega_j$, whose centroids are the so called “particles”:

$$\bigcup_{j=1}^{N} \Delta\Omega_j = D \qquad (2.4)$$

3. APPROXIMATION OF DERIVATIVES

In order to obtain an approximated expression of derivatives of function *A(x)*, a number of approaches has been developed and proposed. Classical one, applying Green formula and neglecting border contributions, expresses the first derivative of *A(x)* as:

$$\nabla A(x_i) \cong \int_D \left[A(x_i) - A(x) \right] \nabla W_i(x) d\Omega \qquad (3.1)$$

and in the discrete form:

$$\nabla A(x_i) \cong \sum_{j=1}^{N} \left[A(x_i) - A(x_j) \right] \nabla W_i(x_j) \Delta\Omega_j \qquad (3.2)$$

Further derivatives, according to classical approach, are then evaluated reiterating the procedure:

$$\nabla^2 A(x_i) \cong \sum_{j=1}^{N} \left[\nabla A(x_i) - \nabla A(x_j) \right] \nabla W_i(x_j) \Delta\Omega_j \tag{3.3}$$

In order to overcome lack of convergence, in particular occurring close to boundaries, different procedures have been proposed in literature. Starting from Taylor series expansion, Chen and Beraun (2000), proposed a generalized formulation leading to the following expressions for first and second derivatives:

$$\nabla A(x_i) = \frac{\sum_{j=1}^{N} \left[A(x_j) - A(x_i) \right] \nabla W_i(x_j) \Delta\Omega_j}{\sum_{j=1}^{N} (x_j - x_i) \nabla W_i(x_j) \Delta\Omega_j} \tag{3.4}$$

$$\nabla^2 A(x_i) = \frac{\sum_{j=1}^{N} \left[A(x_j) - A(x_i) \right] W_i(x_j) \Delta\Omega_j - \nabla A(x_i) \sum_{j=1}^{N} \left[x_j - x_i \right] W_i(x_j) \Delta\Omega_j}{\frac{1}{2} \sum_{j=1}^{N} (x_j - x_i)^2 W_i(x_j) \Delta\Omega_j} \tag{3.5}$$

Such a procedure, deriving from Taylor series expansion up to the first order, guarantees everywhere in the domain, even close to the boundary, a *h* order of convergence to the exact solution, where *h* represents the size of the particle discretization. In this case, no specific constraints are introduced for the kernel function. On the contrary, different approaches, also focusing on Taylor series expansion, aiming to manage the convergence error, indicate particular expressions for kernel functions. It is the case of Reproducing Kernel Particle Method (RKPM) from Liu and coworkers (1995) and Kulasegaram and Bonet (2000), where, forcing to zero *n*-th order momenti of kernel function, it is aimed to achieve higher order errors. However, boundary deficiencies in these cases are not completely eliminated and particular local procedures become necessary.
To overcome such problems and obtain a second order error in approximation of derivatives, authors are currently working on an original formulation, which starts from a basic idea of Liu et al. (2005) and Zhang and Batra (2004). In this procedure Taylor series expansion up to the second order is projected against a kernel function an its derivative, obtaining the following linear system, providing the first and the second derivatives of the function *A(x)*:

$$\begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} \begin{pmatrix} \nabla A(x_i) \\ \nabla^2 A(x_i) \end{pmatrix} = \begin{pmatrix} \int_D \left[A(x) - A(x_i) \right] W_i(x) dx \\ \int_D \left[A(x) - A(x_i) \right] \nabla W_i(x) dx \end{pmatrix} \tag{3.6}$$

where

$$B_{11} = \left[\int_D (x - x_i) W_i(x)\, dx \right]$$
$$B_{12} = \frac{1}{2} \left[\int_D (x - x_i)^2 W_i(x)\, dx \right]$$
$$B_{21} = \left[\int_D (x - x_i) \nabla W_i(x)\, dx \right] \qquad (3.7)$$
$$B_{22} = \frac{1}{2} \left[\int_D (x - x_i)^2 \nabla W_i(x)\, dx \right]$$

Hence, properly choosing kernel function, enforcing for each particle that specific momenti of the kernel function and of its derivative are null, the error is moved to h^2 order, even for particles close to the boundary.
In order to appreciate the enhancement introduced with this formulation, the method was used to solve in one dimension the following problem:

$$u'' = -f \qquad (3.8)$$

defined in the domain D [0;1], where u is un unknown function and f is equal to:

$$f = \sin(\pi x) \qquad (3.9)$$

Imposing Dirichlet and Neumann boundary conditions, Figure 2 is obtained; it depicts the relationship between the error of infinity norm, evaluated as the maximum distance between the numerical and the exact solution, and the particle number discretizing the domain.

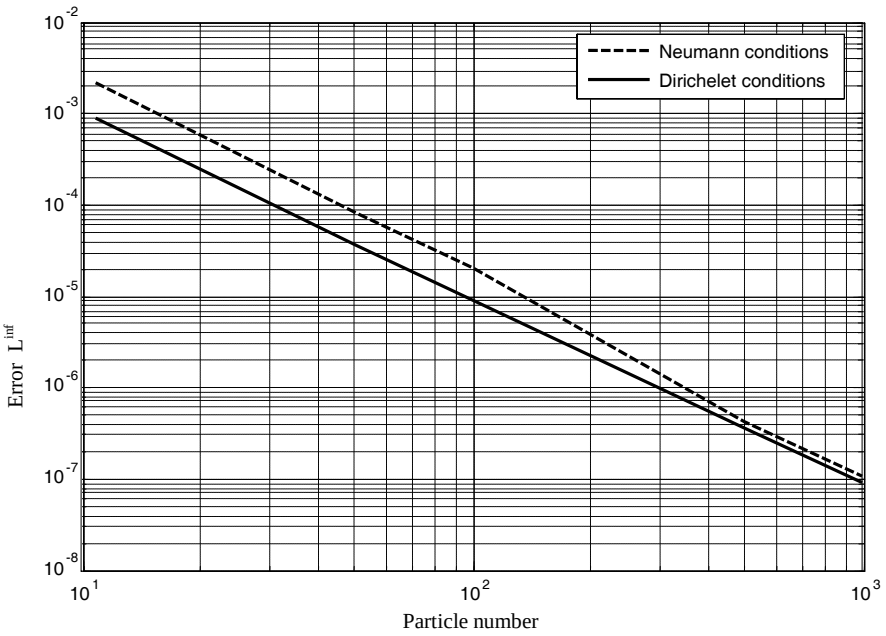

Figure 2 Error L^{inf} vs. particle number

It can be observed that, as it was expected, in both cases the numerical solution converges to the exact one with h^2 velocity.

4. NUMERICAL SPECTRA

In order to appreciate the capability of the investigated methods in reproducing high dynamic problems, it is useful to evaluate how higher vibration modes of a dynamic system are reproduced through the numerical approximations. A uniaxial elastic element is considered, of length *L*, longitudinal elastic stiffness *EA*, and longitudinal density ρ_L. Dirichlet boundary conditions are applied. The governing equation is

$$E_A \frac{\partial^2 x}{\partial X^2} + \rho_L \frac{\partial^2 x}{\partial t^2} = 0 \qquad (4.1)$$

where *X* and *x* are the initial and the actual configuration of the element, respectively. Introducing the celerity *c* as

$$c = \sqrt{\frac{E_A}{\rho_L}} \qquad (4.2)$$

equation (4.1) becomes

$$c^2 \frac{\partial^2 x}{\partial X^2} + \frac{\partial^2 x}{\partial t^2} = 0 \qquad (4.3)$$

In a stationary problem the following equation holds:

$$\frac{\partial^2 x}{\partial t^2} = \omega^2 x \qquad (4.4)$$

where ω is the frequency of vibration of the element; hence, equation (4.3) can be rewritten as:

$$c^2 \frac{\partial^2 x}{\partial X^2} + \omega^2 x = 0 \qquad (4.5)$$

The solution of this differential equation is given by:

$$x = \sin(\alpha x) \qquad (4.6)$$

where

$$\alpha^2 = \frac{\omega^2}{c^2} \qquad (4.7)$$

Thus, imposing boundary conditions

$$\alpha = \frac{\pi n}{L} \qquad (4.8)$$

where *n* is a natural number. Substituting this expression in equation (4.7) it yields

$$\frac{\omega}{c} = \frac{\pi n}{L} \qquad (4.9)$$

Introducing computational methods, second derivative of actual position x with respect to the initial configuration X, can be expressed as a linear function of the actual position:

$$\frac{\partial^2 x}{\partial X^2} = kx \qquad (4.10)$$

where k is a matrix of coefficients depending on the numerical method. Substituting this expression into equation (4.4) it yields:

$$kx + \frac{\omega^2}{c^2} x = 0 \qquad (4.11)$$

Hence, eigenvalues of k matrix represent the terms ω^2 / c^2, evaluated through the numerical approach. Then, the ratio between the numerically evaluated vibration frequencies ω and the respective exact analytical values can be computed to obtain a spectrum of the numerical error in the evaluation of vibration frequencies, for each vibration mode.
Figure 3 depicts the obtained numerical spectra for some of the investigated methods, including two variants of the proposed formulation. Furthermore, in order to perform a comparison, also the spectrum of the FEM with linear shape functions is plotted.

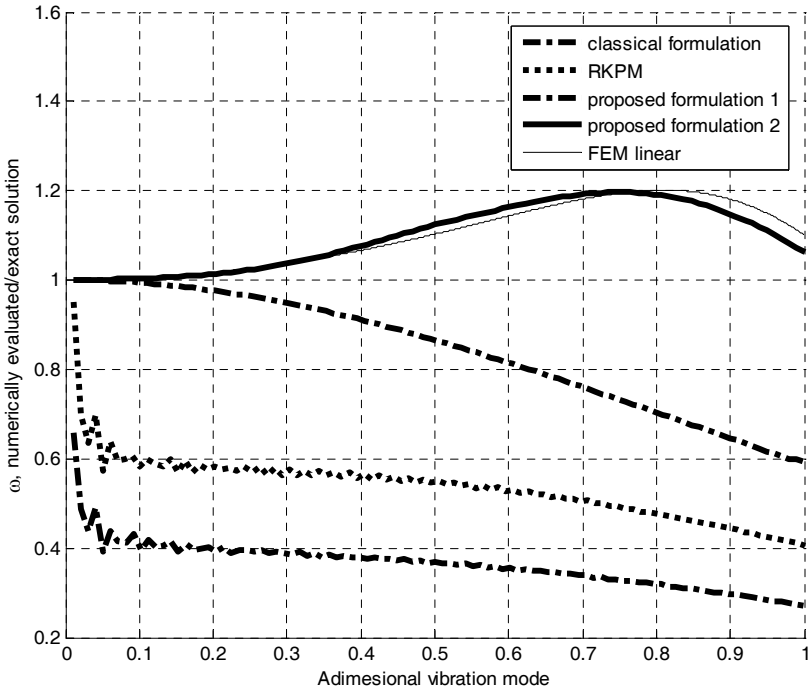

Figure 3 Numerical spectra

The proposed curves report the ratio between the numerically evaluated vibration frequencies and the corresponding exact values, for each of the vibration modes. It can be observed that classical formulations of the SPH method present a lack of accuracy in reproducing higher modes, since the spectra values are quite lower than unity. On the contrary spectra of the proposed formulation, presented in two variants depending on the choice of the kernel functions, are much closer to unity, appearing competitive with performance exhibited by classical linear FEM.

5. CONCLUSIONS

High dynamic problems represent a critical aspect of structural dynamics, which is currently attracting the efforts of many researchers of numerical community. In this field, meshless methods seem to be able to overcome the problems related to mesh distortions and numerical instabilities, which affect classical FEM, if employed under particular loading conditions.
The paper presented the basic idea of SPH method and summarized its basic formulations; in particular, the attention is focused on derivatives expressions, which play a fundamental role in deriving numerical framework to simulate dynamic structural systems. In this case, given the lack of accuracy occurring close to the boundary, different improvements have been proposed in literature and some of them are here cited. Then, the derivation of an original formulation due to authors is described. The approach here proposed appears to exhibit good results both in the direct approximation of derivatives, tested via a numerical problem, and in the derivation of the numerical spectrum. In particular, this last test reveals that existing SPH formulations lose completely accuracy in predicting higher vibration frequencies. On the contrary, the proposed formulation, presented in two variants, is able to better reproduce higher modes (especially in case of the second formulation), providing results comparable with those of FEM in case of linear shape functions, without the classical deficiencies characterizing FEM method in high dynamic analyses. Authors are currently working on differences between proposed formulations and other existing SPH procedures in order to investigate the numerical spectra, conducting further numerical tests, and extending the formulations to multidimensional cases.

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