Seismic Random Vibration Analysis of Complex Uncertain Dynamic Systems

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ABSTRACT :

The random uncertainties of the mass, damping and stiffness matrices in finite element models are modeled by random matrices, and a highly efficient pseudo excitation method for dynamic response analysis of nonparametric probability systems subjected to stationary random loading is developed. A numerical example shows that the dynamic responses of a conventional (i.e. the mean-valued) finite element model may be quite different from those based on the random matrix model. For precise manufacturing, the uncertainties of models can not be ignored and the proposed method will be quite useful in the analysis of such problems.

KEYWORDS: nonparametric probabilistic model, random vibration, pseudo excitation method

1. INTRODUCTION

Practical structures can not be built so accurately as they were designed. A perfect discription of an existing structure is also impossible. Therefore the mean-valued model of a practical structure, as is commonly used in the conventional finite element analyses, is no more than an approximation of the ideal or theoretical structure. In order to improve the reliability in the prediction of structural responses, using stochastic models would naturally be more reasonable (Capiez-Lernouta, et al, 2006; Ibrahim, 1987). However, because of the difficulty of the problem, such stochastic factors have previously been restricted to a limited level, e.g. the structural model only, or the excitation only. It would be quite attractive to set up a model based on both stochastic factors, establish a satisfactory mechanical model and find an effective numerical method (Soong, Dargush, 2005).

In general, the uncertainty of a practical structure may be due to two reasons: the uncertainty of the data and that of the model. The former means the uncertainty of some parameters of the structure such as the geometric sizes, Young's module or material density, which can be depicted by means of some proper probabilistic models. As for the latter, it is because not all details of the structure are known, some simplifications must be made in the modeling of the structure. Such simplifications are non-parametric, and can not be dealt with by means of methods for parametric uncertainty; instead the theory of random matrices has been considered a good choice. Based on this theory, Soize (2000, 2001, 2005) investigated the dynamic responses of such uncertainty models subjected to deterministic loads and achieved some progresses. Unfortunately, while the models used are all uncertain, all loads remain known and deterministic. The present paper avoids this shortcoming by extending these single-stochastic problems into double-stochastic ones, so that not only the model is uncertain, but also the loads (earthquakes) can be uncertain (i.e. random). Thus, for the method proposed, the uncertainties of the stiffness, damping and mass matrices are modeled using the theory of random matrices; while the uncertainty of the seismic loads is dealt with by means of the highly efficient Pseudo-Excitation Method (PEM) (Lin, Zhao, Zhang, 2001). Numerical examples show the effectiveness of the proposed method.

2. STATIONARY RANDOM VIBRATION OF COMPLEX UNCERTAIN DYNAMIC SYSTEMS

2.1. Probability Characteristics of Random Matrix

Let $\mathbf{A}$ be a symmetric positive-definite random matrix, and its probability density function $p_{\mathbf{A}}$ satisfies the following equations

$$\int p_{\mathrm{A}}(\mathbf{A})\tilde{\mathrm{d}}\mathrm{A}=1 \tag{2.1}$$

$$\int \mathbf{A}p_{\mathrm{A}}(\mathbf{A})\tilde{\mathrm{d}}\mathrm{A}=\bar{\mathbf{A}} \tag{2.2}$$

$$\int \ln(\det \mathbf{A})p_{\mathrm{A}}(\mathbf{A})\tilde{\mathrm{d}}\mathrm{A}=v\,, \quad |v|<+\infty \tag{2.3}$$

in which $\bar{\mathbf{A}}$ is the mean value of random matrix $\mathbf{A}$ and $\tilde{\mathrm{d}}\mathrm{A}$ is defined by

$$\tilde{\mathrm{d}}\mathrm{A}=2^{n(n-1)/4}\prod_{1\le i\le j\le n}\mathrm{d}\mathbf{A}_{ij} \tag{2.4}$$

Use the maximum entropy principle to construct the probability model of random matrix $[\mathrm{A}]$, the probability density function can be written as (Soize, 2000)

$$p_{\mathrm{A}}(\mathbf{A})=c_{\mathrm{A}}\times(\det \mathbf{A})^{\lambda_{\mathrm{A}}-1}\times\exp\left(-\frac{(n\text{-}1+2\lambda_{\mathrm{A}})}{2}\mathrm{tr}\left\{\bar{\mathbf{A}}^{-1},\mathbf{A}\right\}\right) \tag{2.5}$$

in which

$$c_{\mathrm{A}}=\frac{(2\pi)^{-n(n-1)/4}\left(\frac{n-1+2\lambda_{\mathrm{A}}}{2}\right)^{n(n-1+2\lambda_{\mathrm{A}})/2}}{\left\{\prod_{l=1}^{n}\Gamma\left(\frac{n-l+2\lambda_{\mathrm{A}}}{2}\right)\right\}\left(\det \bar{\mathbf{A}}\right)^{(n-1+2\lambda_{\mathrm{A}})/2}} \tag{2.6}$$

and $\Gamma(x)=\int_0^{+\infty}t^{x-1}e^{-x}\mathrm{d}t$ （$x>0$） is the gamma function.

The covariance of random matrix $\mathbf{A}$ is defined by

$$C_{jk,j'k'}=E\left[\left(\mathbf{A}_{jk}-\bar{\mathbf{A}}_{jk}\right)\left(\mathbf{A}_{j'k'}-\bar{\mathbf{A}}_{j'k'}\right)\right] \tag{2.7}$$

The characteristic function of random matrix Eq. (2.7) is written as (Soize, 2000)

$$C_{jk,j'k'}=\left\{\bar{A}_{j'k}\bar{A}_{jk'}+\bar{A}_{jj'}\bar{A}_{kk'}\right\}/\left(n-1+2\lambda_{A}\right) \tag{2.8}$$

When $j=j'$ and $k=k'$, Eq. (2.8) gives the variance

$$\sigma_{jk}^2=C_{jk,jk}=\left\{\bar{\mathbf{A}}_{jk}^2+\bar{\mathbf{A}}_{jj}\bar{\mathbf{A}}_{kk}\right\}/\left(n\text{-}1+2\lambda_{\mathrm{A}}\right) \tag{2.9}$$

Let δ_A be defined by

$$\delta_{\mathrm{A}}=\sqrt{E\left[\left\|\mathbf{A}-\bar{\mathbf{A}}\right\|_{\mathrm{F}}^2\right]/\left\|\bar{\mathbf{A}}\right\|_{\mathrm{F}}^2} \tag{2.10}$$

As $E\left[\left\|\mathbf{A}-\bar{\mathbf{A}}\right\|_{\mathrm{F}}^2\right]=\sum_j\sum_k\sigma_{jk}^2$, it is deduced that

$$\delta_{\mathrm{A}} = \left\{ \frac{1}{n\text{-}1+2\lambda_{\mathrm{A}}} \left(1 + \frac{\left(\mathrm{tr}\bar{\mathbf{A}}\right)^2}{\mathrm{tr}\left(\bar{\mathbf{A}}^2\right)} \right) \right\}^{1/2} \tag{2.11}$$

With n fixed, if $\lambda_A \to \infty$, then both δ_A and $\sigma_{jk} \to 0$, and therefore $\mathbf{A} \to \bar{\mathbf{A}}$ probabilistically.

2.2. Random Matrix Models for Dynamic Systems and Stationary Random Vibration Analysis

The random uncertainties of mass, damping and stiffness matrices in finite element models are modeled by random matrices. Under stationary random excitations structural equation of motion can be written as

$$\mathbf{M}\ddot{\mathbf{Y}}(t) + \mathbf{C}\dot{\mathbf{Y}}(t) + \mathbf{K}\mathbf{Y}(t) = \mathbf{p}w(t) \tag{2.12}$$

in which $\mathbf{M}$、$\mathbf{C}$ and $\mathbf{K}$ are the $n \times n$ symmetric positive-definite random matrices; $\mathbf{p}$ is the force index vector; $w(t)$ is a stationary random process and its PSD matrix $S_{ww}(\omega)$ is known. Here random matrices $\mathbf{M}$、$\mathbf{C}$ and $\mathbf{K}$ are assumed to be independent on random process $w(t)$. Corresponding to Eq. (2.12) the motion equation of the mean-valued finite element model is

$$\bar{\mathbf{M}}\ddot{\bar{\mathbf{Y}}}(t) + \bar{\mathbf{C}}\dot{\bar{\mathbf{Y}}}(t) + \bar{\mathbf{K}}\bar{\mathbf{Y}}(t) = \mathbf{p}w(t) \tag{2.13}$$

For any random sample matrix in accordance with the probability distribution defined in Section 2.1, i.e. the mass, damping and stiffness matrices, which satisfy $\mathbf{M} \to \mathbf{m}$, $\mathbf{C} \to \mathbf{c}$ and $\mathbf{K} \to \mathbf{k}$, Eq. (2.12) is transformed into a determinate linear dynamic equation system for the mean-valued matrices

$$\mathbf{m}\ddot{\mathbf{y}}(t) + \mathbf{c}\dot{\mathbf{y}}(t) + \mathbf{k}\mathbf{y}(t) = \mathbf{p}w(t) \tag{2.14}$$

In order to compute the PSD functions of various linear responses due to the random excitation vector on its right-hand side, the sinusoidal pseudo excitation $\tilde{w}(t) = \sqrt{S_{ww}(\omega)}e^{i\omega t}$ should be substituted into Eq. (2.14) to replace $w(t)$, that leads to the following harmonic equation

$$\mathbf{m}\ddot{\tilde{\mathbf{y}}}(t) + \mathbf{c}\dot{\tilde{\mathbf{y}}}(t) + \mathbf{k}\tilde{\mathbf{y}}(t) = \mathbf{p}\sqrt{S_{ww}(\omega)}e^{i\omega t} \tag{2.15}$$

That means the stationary random vibration analysis is transformed into determinate harmonic vibration analysis. The pseudo harmonic responses of the system obtained can be expressed in the form

$$\tilde{\mathbf{y}} = \hat{\mathbf{y}}e^{i\omega t} \tag{2.16}$$

The corresponding PSD matrix $\mathbf{S}_{yy}$ can then be computed by using the pseudo harmonic responses(Lin, Zhao, Zhang, 2001, 2004)

$$\mathbf{S}_{yy}\left(\omega; \mathbf{m}, \mathbf{c}, \mathbf{k}\right) = \tilde{\mathbf{y}}^*\tilde{\mathbf{y}}^T \tag{2.17}$$

The corresponding variance is

$$\sigma_y^2(\mathbf{m},\mathbf{c},\mathbf{k}) = 2\int_0^\infty S_{yy}(\omega;\mathbf{m},\mathbf{c},\mathbf{k})\mathrm{d}\omega \tag{2.18}$$

Next, the random responses of the dynamic system with random matrices require the calculation of the following multiple integration (Soize, 2005)

$$E\left[\mathbf{S}_{\mathrm{yy}}(\omega)\right] = \iiint \mathbf{S}_{\mathrm{yy}}(\omega;\mathbf{m},\mathbf{c},\mathbf{k}) \times p_{\mathbf{m}}(\mathbf{m}) \times p_{\mathbf{c}}(\mathbf{c}) \times p_{\mathbf{k}}(\mathbf{k})\tilde{\mathrm{d}}m\tilde{\mathrm{d}}c\tilde{\mathrm{d}}k \tag{2.19}$$

$$E\left[\boldsymbol{\sigma}_{\mathrm{y}}^2\right] = \iiint \boldsymbol{\sigma}_{\mathrm{y}}^2(\mathbf{m},\mathbf{c},\mathbf{k}) \times p_{\mathbf{m}}(\mathbf{m}) \times p_{\mathbf{c}}(\mathbf{c}) \times p_{\mathbf{k}}(\mathbf{k})\tilde{\mathrm{d}}m\tilde{\mathrm{d}}c\tilde{\mathrm{d}}k \tag{2.20}$$

In general, the above integration can be carried out by using Monte Carlo method (MacKeown, 1997). It will still be efficient enough as long as the random matrices are not very big and pseudo excitation method is used under the current high performance PC conditions.

In order to carry out the Monte Carlo simulation for random matrix $\mathbf{A}$, assume λ_A of Eq. (2.5) is a positive integer and introduce another positive integer m_A such that

$$m_A = n - 1 + 2\lambda_A \tag{2.21}$$

The Cholesky factorization of mean matrix $\bar{\mathbf{A}}$ yields

$$\bar{\mathbf{A}} = \mathbf{L}_{\mathrm{A}}^{\mathrm{T}}\mathbf{L}_{\mathrm{A}} \tag{2.22}$$

in which $\mathbf{L}_{\mathrm{A}}$ is an upper triangular matrix. Consequently, random matrix $\mathbf{A}$ can be written as (Soize, 2000)

$$\mathbf{A} = \frac{1}{m_{\mathrm{A}}}\sum_{j=1}^{m_{\mathrm{A}}}\left(\mathbf{L}_{\mathrm{A}}^{\mathrm{T}}\mathbf{u}_j\right)\left(\mathbf{L}_{\mathrm{A}}^{\mathrm{T}}\mathbf{u}_j\right)^{\mathrm{T}} \tag{2.23}$$

in which $\mathbf{u}_j$（$j = 1,\cdots,m_A$） is an independent random vector, of which any element is a normalized Gaussian random variable (with zero mean value and unit variance). By using Eq. (2.23), the corresponding random matrix samples for $\mathbf{M}$、$\mathbf{C}$ and $\mathbf{K}$ are generated.

3. NUMERICAL EXAMPLE

The 25-bar spatial truss of Fig.1 subjected to seismic loading is investigated. The parameters of the bars are: Young's Module 2.058×10^{11}Pa, sectional area 4.0×10^{-4}m^2, and a lumped mass of 20Kg is attached to each non-support node. The structure is subjected to a horizontal y-direction seismic acceleration which is regarded as a stationary Gaussian random process. Consider the two loading cases: (1) The seismic acceleration has a white-noise spectrum with intensity 100 m^2/s^3; (2) The seismic acceleration has a Kanai-Tajimi filtered white-noise spectrum with the form

$$S_{ww}(\omega) = \frac{1+4(\zeta_g\omega/\omega_g)^2}{(1-\omega^2/\omega_g^2)^2+4(\zeta_g\omega/\omega_g)^2}S_0 \tag{3.1}$$

in which S_0=142.75m^2/s^3，ω_g=19.07s^{-1}，ζ_g=0.544。

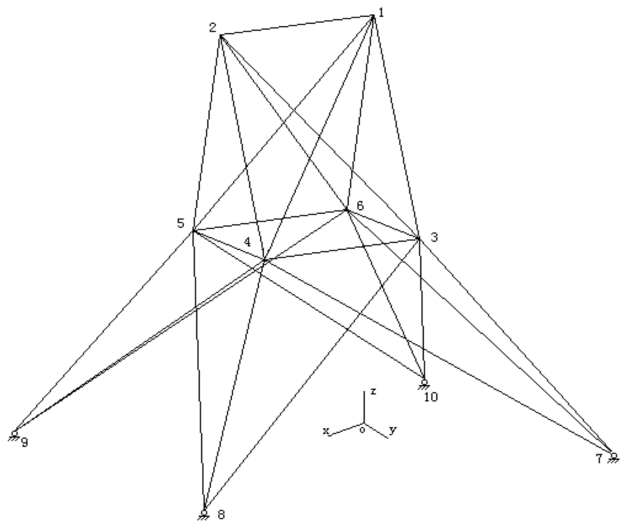

Figure 1 25-bar spatial truss

By comparing the PSD and variance responses of the mean valued model(MVM) and of the non-parametric model(NPM) under the above mentioned excitations, the frequency domain is restricted within $\omega \in [0, 800]$ s^{-1}, and the frequency interval is taken as: for mean valued models, the motion equation (2.13) can be solved directly by means of PEM; for non-parametric probabilistic models, the motion equation (2.12) adopts two groups of parameters: i.e. with $\delta_M = \delta_C = \delta_K = \delta = 0.02$ and with $\delta_M = \delta_C = \delta_K = \delta = 0.2$. The structural responses were computed by PEM combined with Monte Carlo method. The damping ratios for all participant modes take 0.02.

To carry out Monte-Carlo simulation, 50、150、250、500、750 and 1000 samples are used. The results with 250 samples are found to be precise enough for practical use. In order to achieve more accurate computations, Monte-Carlo method with 500 samples were also performed, which uses 2230s (for standard deviation 0.02) or 930s (for standard deviation 0.2)。The above computations were all executed on a personal computer with main frequency 3.0 GHZ.

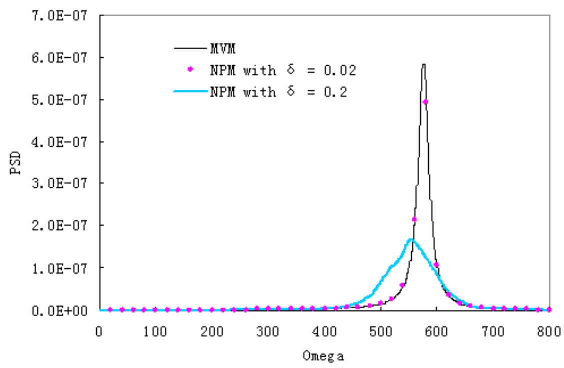

Figure 2 The PSD response of the white noise

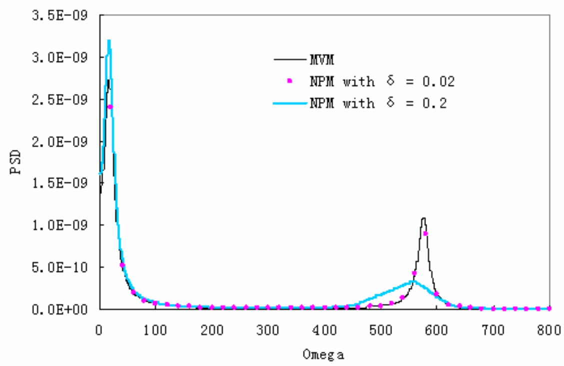

Figure 3 The PSD response of the filtered noise

The displacement PSD functions of node 1 in y-direction are shown in Figs 2 and 3, for loading cases 1 and 2, respectively. In fact, the mean-valued model can be regarded as a non-parametric probabilistic model with standard deviation δ=0. When the deviation is small, say δ=0.02, the responses computed based on the two models are very close. For bigger deviations, say δ=0.2, however, the differences in the response peak values or in the response PSD shapes due to different models are very big. Numerical results show that (1) for the first loading case, the response PSD peak value is $1.675\times10^{-7}m^2s$ at frequency $554s^{-1}$when the non-parametric probabilistic model with δ=0.2 is used, while the response PSD value is 5.845×10^{-7} m^2s at frequency 575.8 s^{-1}

when the mean value model is used. The former is smaller than the latter by 71.3%. (2) for the second loading case, the second largest response PSD peak value is 2.972×10^{-10} m^2s at frequency 555.8 $s^{-1}$when the non-parametric probabilistic model with δ=0.2 is used, while the second largest response PSD value is 1.085×10^{-9} m^2s at frequency 575.6 s^{-1} when the mean value model is used. The former is smaller than the latter by 72.6%.

Table 1 The variance responses of different model (m^2)

Load cases	Mean valued model	Non-parametric model	
		δ = 0.02	δ = 0.2
White noise	4.199×10^{-5}	4.174×10^{-5}	3.600×10^{-5}
Filtered noise	2.418×10^{-7}	2.417×10^{-7}	2.662×10^{-7}

Table 1 gives the comparison of the variances of dynamic responses between the two models. Similar to the conclusions drawn above, with the smaller standard deviation δ=0.02, the variance responses for the non-parametric probabilistic model has a smaller difference, 0.595%, from those by using the mean-valued model subjected to white-noise excitations. And this difference reduces to a negligible level of only 0.041% when the truss is subjected to filtered white noise excitations. For larger standard deviations, say δ=0.2, the differences of the variance responses between the non-parametric probabilistic model and the mean valued model is 14.265% for white noise excitations, and 10.091% for filtered white noise excitations.

4. CONCLUSIONS

The non-parametric probabilistic model subjected to stationary random excitations is investigated. The numerical computations are based on the random mass, damping and stiffness matrices as well as the highly efficient pseudo excitation method for random vibration analyses. The proposed approach is quite efficient and may hopefully be used in the stochastic reliability analysis for some complicated engineering structures, such as bridges, aircrafts and others.

5. ACKNOWLEDGEMENTS

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REFERENCES

Capiez-Lernouta, E., Pellissettib, M. and Pradlwarterb, H. (2006). Data and model uncertainties in complex aerospace engineering systems. *Journal of Sound and Vibration* **295:** 923-938.

Ibrahim, R. (1987). Structural dynamics with parameter uncertainties, *Applied Mechanical Reviews* **40(3):** 309-328.

Soong, T. T. and Dargush, G. F. (2005). Passive Energy Dissipation Systems in Structural Engineering, Science Press, Bei Jing. (in Chinese)

Soize, C. (2005). A comprehensive overview of a non-parametric probabilistic approach of model uncertainties for predictive models in structural dynamics. *Journal of Sound and Vibration* **288:** 623-652.

Lin, J. H., Zhao, Y. and Zhang, Y. H. (2001). Accurate and highly efficient algorithms for structural stationary/non-stationary random responses. *Computer Meth in Appl Mech and Eng* **191 (1-2):** 103-111.

Soize, C. (2000). A nonparametric model of random uncertainties for reduced matrix models in structural dynamics. *Probabilistic Engineering Mechanics*, **15:** 277-294.

Soize, C. (2001). Transient responses of dynamical systems with random uncertainties. *Probabilistic Engineering Mechanic* **16:** 363-372.

Soize, C. (2005). Random matrix theory for modeling uncertainties in computational mechanics. Comput.

Methods Appl. Mech. Engrg. **194:** 1333-1366.
Lin, J. H. and Zhang Y. H. (2004) Pseudo excitation method of random vibration, Science Press, Bei Jing. (in Chinese)
MacKeown, P. K. (1997). Stochastic simulation in physics, Springer, New York.