ALGORITHM FOR IMPROVING THE ACCURACY OF NEWMARK INTEGRATION METHOD IN THE STUDY OF ELASTOPLASTIC OSCILLATIONS

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ABSTRACT :

Damped forced vibrations of a single-degree-of-freedom elastoplastic system, are studied, by using Newmark-β integration method. A correction algorithm is proposed, in order to improve the accuracy of the calculations. The method is applied with and without correction and the results are compared for the case of the seismic excitation.

KEYWORDS: Newmark method, single-degree-of-freedom system, elastoplastic, seismic response

1. INTRODUCTION

A mechanical single-degree-of-freedom (SDOF) system is considered, as shown in Fig. 1. The system is composed of the following bodies:

1. The rigid solid of mass *m*, which can translate through a guide;
2. The linear viscous damper with the damping constant *c*;
3. The elastoplastic body (Prandtl body) defined by the stiffness *k* and the yield force Q_p

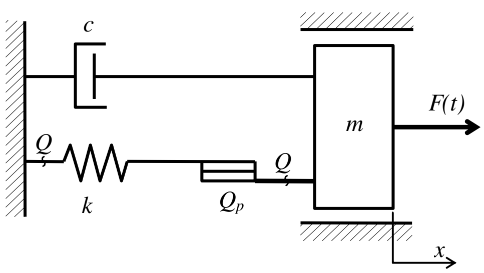

Figure 1. Single-degree-of-freedom system

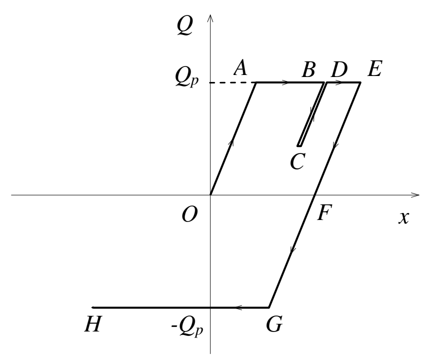

Figure 2. Elastoplastic force-displacement relationship

The configuration of the system is defined by the displacement *x* of the body *m* with respect to the initial (equilibrium) position.

Body *m* is also acted on by a known, time-variable force, $F(t)$.

The force in the elastoplastic element is denoted by Q.

The force-deformation characteristic of the Prandtl body is shown in Fig. 2, for an evolution consisting of the following steps:

1. Loading up to yield level *OA*;

2. Plastic deformation *AB*;
3. Partial unloading *BC*;
4. Re-loading up to the yield level *CD*;
5. Plastic deformation *DE*;
6. Unloading *EF* and loading in the opposite sense up to the yield level *FG*;
7. Plastic deformation *GH*.

The differential equation of motion of the system is (Ifrim, 1984)

$$m\ddot{x} + c\dot{x} + Q = F(t)\,, \tag{1.1}$$

where

$$Q(t+dt) = \begin{cases} Q(t) + k\,\mathrm{d}\,x, & \text{if } |Q(t)| < Q_p \ \text{ or } \begin{cases} |Q(t)| = Q_p \\ Q(t)\,\mathrm{d}\,x < 0 \end{cases} \\ Q(t), & \text{if } \begin{cases} |Q(t)| = Q_p \\ Q(t)\ \mathrm{d}\,x \geq 0. \end{cases} \end{cases} \tag{1.2}$$

The first form of $Q(t+\mathrm{d}t)$ corresponds to the elastic evolution (loading and unloading, respectively), while the second one corresponds to the plastic evolution (on the superior plateau $Q = Q_p$ or, on the inferior plateau $Q = -Q_p$).

2. NUMERICAL INTEGRATION

In order to integrate Eqns. 1.1 and 1.2, Newmark-β numerical method was used, with β = 1/4.

This implies the assumption that, on the time interval Δt, the response acceleration $\ddot{x}$ has a constant value, equal to the arithmetic mean of the values at the extremities of the interval (Fig. 3) (Ifrim, 1984):

$$\ddot{x}(t) = \frac{\ddot{x}_i + \ddot{x}_{i+1}}{2} \quad \text{if } \ t \in (t_i, t_{i+1})\,. \tag{2.1}$$

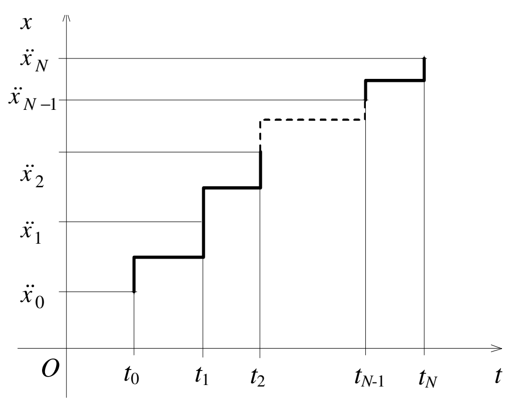

Figure 3. Approximation of the acceleration response of the system

By denoting the length of the time interval

$$\Delta t_{i+1} = t_{i+1} - t_i \,, \tag{2.2}$$

it can be shown (Ifrim, 1984) that the variation of the displacement x on the considered interval can be calculated as

$$\Delta x_{i+1} = \frac{\ddot{x}_i + \frac{F_{i+1}}{m} + \left(\frac{4}{\Delta t_{i+1}} + \frac{c}{m}\right)\dot{x}_i - \frac{Q_i}{m}}{\frac{4}{\Delta t_{i+1}^2} + \frac{2}{\Delta t_{i+1}}\frac{c}{m} + r_{i+1}\frac{k}{m}} \,, \tag{2.3}$$

where

$$r_{i+1} = \begin{cases} 1, & \text{if} \quad |Q_i| < Q_p \quad \text{or} \quad \begin{cases} |Q_i| \geq Q_p \\ Q_i \Delta x_{i+1} < 0 \end{cases} \\ 0, & \text{if} \quad \begin{cases} |Q_i| \geq Q_p \\ Q_i \Delta x_{i+1} \geq 0. \end{cases} \end{cases} \tag{2.4}$$

3. SOURCES OF ERRORS

The chosen integration method causes the results to be affected by errors.

The first source of errors is due to the basic assumption of the method, i.e. that the acceleration $\ddot{x}$ is constant over the time intervals (t_i, t_{i+1}). The errors produced by it can be reduced by increasing the discretization of the total time interval (t_0, t_N) on which the external excitation force is acting. This method, called in the following "nonselective subdivision", has the inconvenient to be time-consuming.

Moreover, if the excitation force does not have an analytical expression (usually it is defined by discrete values F_i), when the intervals (t_i , t_{i+1}) are divided in a number of subintervals, the intermediate values of the force must be estimated by interpolation.

The second source of errors is due to the shape of the force-deformation characteristic of the elastoplastic body (Fig. 2), which has angular points in A, D, G (yielding) and B, E (unloading). If such a point is situated in the interval (t_i , t_{i+1}), errors greater than those obtained on ordinary intervals (with perfectly elastic or perfectly plastic evolution) are to be expected.

In particular, if a yielding point is encountered, the value Q_{i+1} given by the algorithm will be superior in absolute value to the yielding force, which is physically impossible (Fig. 4). This type of error becomes significant if the level Q_p is small and/or the integration step is large.

The difference between $|Q_{i+1}|$ and Q_p is called "overshoot" in the scientific literature (Kanaan and Powell, 1973).

It should be noticed that Eqn. 2.4 has been deduced in the assumption that the elastic or plastic character of the evolution on the interval (t_i , t_{i+1}) is determined by the initial state A_i and, if A_i is in the plastic range, also by the sense of the evolution. Therefore, when a yield point is situated in the considered interval (Fig. 5), the evolution is approximated by a perfectly elastic one, $A_iA^P{}_{i+1}$, though it is actually elastic only in the first subinterval A_iP_{i+1} and plastic afterwards. This causes the occurrence of the overshoot.

A solution to diminish the overshoot consists again in increasing the discretization, with the above-mentioned disadvantage of time consumption. This disadvantage can be diminished, without totally eliminating it, by subdividing only the time intervals that contain yield points. This method will be called in the following “selective subdivision”.

In the next paragraph a new solution for the complete elimination of the overshoot, without a significant increase of the computation time is proposed.

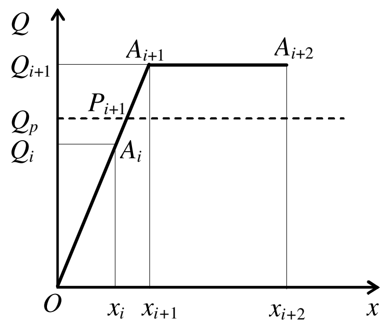

Figure 4. Evolution with overshoot

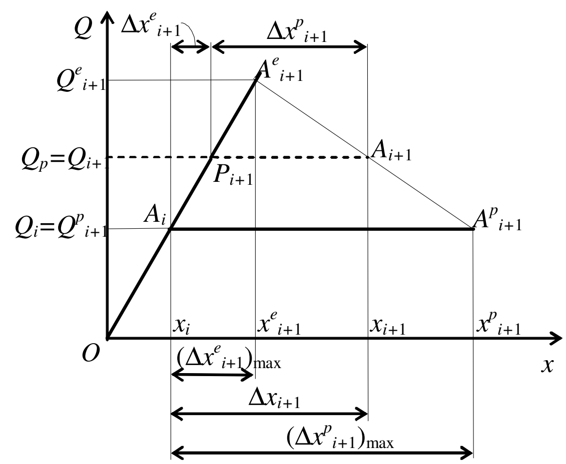

Figure 5. Elastoplastic evolution

4. OVERSHOOT ELIMINATION

The elastic approximation is more satisfactory and the overshoot is smaller when yielding occurs near the end of the interval (t_i , t_{i+1}). On the contrary, when yielding occurs near t_i, the elastoplastic evolution is better approximated by a perfectly plastic one, $A_iA^p_{i+1}$.

Therefore, the state A_{i+1} could be more accurately obtained by composing an elastic evolution, A_iP_{i+1}, with an elastic one, $P_{i+1}A_{i+1}$.

By considering that yielding occurs on the positive plateau Q=Q_p, with the notations in Fig. 5, the following relations can be written:

$$\Delta x^e_{i+1} = \frac{Q_p - Q_i}{k}\,, \tag{4.1}$$

$$Q_{i+1} = Q_p\,. \tag{4.2}$$

Because the determination of the time at which the point P_{i+1} is attained requires the solving of a nonlinear algebraical equation, Δx^p_{i+1} will be calculated approximately, by assuming that Δx^p_{i+1} is a fraction of $(\Delta x^p_{i+1})_{\max}$, equal to the ratio of the overshoot to the variation of Q in the elastic approximation, i.e.

$$\frac{\Delta x^p_{i+1}}{(\Delta x^p_{i+1})_{\max}} = \frac{Q^e_{i+1} - Q_p}{Q^e_{i+1} - Q_i}\,. \tag{4.3}$$

This relation verifies the likeness of the triangles $A^e_{i+1}P_{i+1}A_{i+1}$ and $A^e_{i+1}A_iA^p_{i+1}$, hence the collinearity of the points

A^e_{i+1}, A_{i+1} and A^p_{i+1}.

The following relation also results:

$$\frac{\Delta x^p_{i+1}}{(\Delta x^p_{i+1})_{\max}} = \frac{(\Delta x^e_{i+1})_{\max} - \Delta x^e_{i+1}}{(\Delta x^e_{i+1})_{\max}} . \tag{4.4}$$

Obviously,

$$\Delta x_{i+1} = \Delta x^e_{i+1} + \Delta x^p_{i+1} . \tag{4.5}$$

The displacements $(\Delta x^e_{i+1})_{\max}$ and $(\Delta x^p_{i+1})_{\max}$ are calculated from Eqn. 2.3. The results are introduced in Eqn. 4.4, together with Δx^e_{i+1} calculated from Eqn. 4.1. The expression of Δx^p_{i+1} is determined and then, from Eqn. 4.5, the following is obtained:

$$\Delta x_{i+1} = \frac{\ddot{x}_i + \dfrac{F_{i+1}}{m} + \left(\dfrac{4}{\Delta t_{i+1}} + \dfrac{c}{m}\right)\dot{x}_i - \dfrac{Q_p}{m}\,\mathrm{sgn}(\Delta x^e_{i+1})_{\max}}{\dfrac{4}{\Delta t^2_{i+1}} + \dfrac{2}{\Delta t_{i+1}}\dfrac{c}{m}} . \tag{4.6}$$

Signum function has been introduced in order to make Eqn. 4.6 valid for any time interval with yielding point, regardless of its position on the superior (Q=Q_p) or on the inferior (Q=-Q_p) plateau.

It can be noticed that, if yielding occurs at exactly the moment t_i, i.e. $P_{i+1} \equiv A_i$, Eqn. 4.6 determines a Δx_{i+1} equal to that which results in the assumption of plastic evolution, $(\Delta x^p_{i+1})_{\max}$. On the other hand, if yielding occurs at the moment t_{i+1}, i.e. $P_{i+1} \equiv A^e_{i+1}$, from Eqn. 4.6, a Δx_{i+1} results, equal to that which is obtained in the assumption of elastic evolution, $(\Delta x^e_{i+1})_{\max}$. If P_{i+1} is situated between A_i and A_{i+1}, the value of Δx_{i+1} will range between $(\Delta x^e_{i+1})_{\max}$ and $(\Delta x^p_{i+1})_{\max}$.

5. APPLICATION FOR THE CASE OF SEISMIC EXCITATION

The idealized model of a SDOF building structure is considered (Fig. 6), consisting of a particle of mass m, fixed at the upper end of a linearly elastic bar with negligible mass. The bar is connected to a rigid base (the ground-infrastructure ensemble) by a plastic hinge O, of yielding moment M_p (Ifrim, 1984). The plastic hinge is equivalent to a friction hinge. The end B is connected by a dissipative linear viscous element, of damping constant c, to a rigid wall, fixed on the base. Under the action of a seismic excitation, the rigid base moves with a known acceleration, $a(t)$.

By choosing as a generalized coordinate the displacement of the body m relative to the base, measured with respect to its initial position (of equilibrium), it can be shown that the motion equation is analogous to Eqn. 1.1:

$$m\ddot{x} + c\dot{x} + Q = -ma(t) . \tag{5.1}$$

The expression of Q is given by Eqn. 1.2. $F(t)$ in Eqn. 1.1 is replaced, in Eqn. 5.1, by -$ma(t)$.

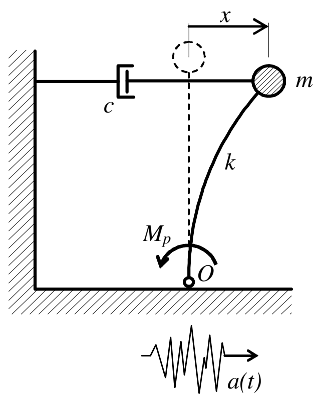

Figure 6. Simplified oscillating system

6. NUMERICAL EXAMPLE

Newmark-β method was applied for a system of the type described in the previous paragraph. Its mechanical characteristics are defined by the mass m, the damping factor ξ, the fundamental period of vibration T and the yield force Q_p:

$$\begin{cases} m = 1\,\text{kg} \\ \xi = 0.02 \\ T = 0.30\,\text{s} \\ Q_p = 0.5mg = 4.905\,\text{N}. \end{cases} \tag{6.1}$$

Thus:

$$\begin{cases} k = \dfrac{4\pi^2}{T^2} m = 438.65\ \text{N/m} \\ c = \dfrac{4\pi\xi}{T} m = 0.83776\ \text{Ns/m}. \end{cases} \tag{6.2}$$

The seismic accelerogram $a(t)$ is defined by discrete values (Fig. 7).

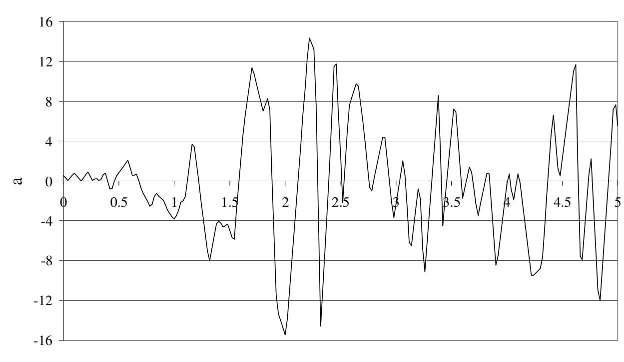

Figure 7. Accelerogram used in the study

The integration step, equal to the discretization interval of the function $a(t)$, is

$$\Delta t = 0.02\,\mathrm{s}\,, \tag{6.3}$$

and the integration is performed on the interval $[0, t_{max}]$, with

$$t_{max} = 2\,\mathrm{s}\,. \tag{6.4}$$

The following computation cases have been considered:

1. without elimination or reduction of the overshoot; this case corresponds to a subdivision to 1 of the time increment ($\Delta t/1$);
2. with reduction of the overshoot, by selectively subdividing to 10 the time increment ($\Delta t/10$);
3. with reduction of the overshoot, by selectively subdividing to 100 the time increment ($\Delta t/100$);
4. with reduction of all errors, by nonselectively subdividing to 10 the time increment ($\Delta t/10$);
5. with reduction of all errors, by nonselectively subdividing to 100 the time increment ($\Delta t/100$);
6. with elimination of the overshoot.

A computer program has been generated and run on an IBM-PC compatible computer.

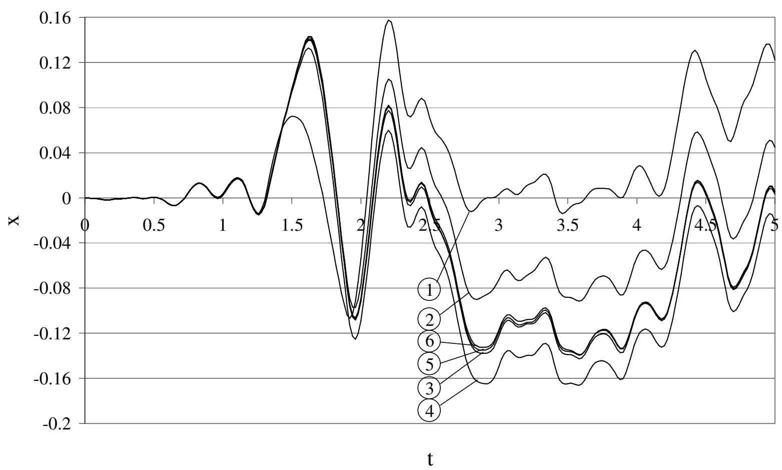

Figure 8. Displacement response of the SDOF system

The results are listed in Table 1 and displayed in Fig. 8. The accuracy is estimated by the maximum relative overshoot, obtained by dividing the maximum absolute value of the overshoot to Q_p. The efficiency is estimated by the computing time.

Table 1. Maximum relative overshoot and computation time, for the studied cases

Computation case	Computation method	Maximum relative overshoot [%]	Computation time [ms]
1	$\Delta t/1$	66.53	0.108
2	$\Delta t/10$, selectively	9.12	0.164
3	$\Delta t/100$, selectively	1.31	0.637
4	$\Delta t/10$, nonselectively	6.71	0.939
5	$\Delta t/100$, nonselectively	1.43	8.965
6	elimination of the overshoot	0	0.113

7. CONCLUSIONS

Case 5 is considered as a reference, because the nonselective subdivision of the time increment to 100 results in significant reduction of all errors, not only of those produced by the overshoot.

Case 1 is characterized by technically unacceptable errors, the values of the displacement x being severely altered. Therefore, the applying of a computation case which reduces or eliminates the overshoot is advisable.

By comparing cases 1, 2 and 3 with case 5, it can be noticed that a significant reduction of errors with the increase of the factor of selective division is accompanied by an increase of the computing time. The displacement values are still altered, but they tend to approach the reference values.

Cases 4 and 5 demonstrate that, if nonselective subdivision is performed, the overshoot remains near the values obtained by selective subdivision. The computation time, on the other hand, increases significantly. Case 5 is by far the least efficient (for the analyzed case, the computation lasts about 77 times more than in the case 1, which is the most efficient).

Case 6 completely eliminates overshoot and gives a very good approach of x-values, with respect to the reference case. The computation time is only slightly longer (with about 5%) than the time needed in case 1.

Curves 3, 5 and 6 are very close one to each other.

Therefore, the overshoot elimination method proposed in this paper satisfies successfully both the accuracy and the efficiency criteria.

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