NUMERICAL SCHEME INFLUENCE ON PSEUDO DYNAMIC TESTS RESULTS

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ABSTRACT :

The pseudo-dynamic test method is a well established and widely spreading experimental technique for structural analysis but the convenience of the chosen time-stepping scheme is not often adequately stressed and reported in experimental results. In fact to obtain results reliable such as those granted by shaking table tests it is necessary to stand a noticeable computational effort to integrate step by step the dynamic equilibrium equation. In this survey a comparison between a number of numerical integration techniques and some numerical examples is presented. The integration methods have been chosen among the most established and newest ones with the aim to investigate their advantages and downsides particularly with regard to computational effort and error propagation for linear and non-linear structural systems. Furthermore some general information and definitions concerning numerical integration tecniques and their stability and accuracy properties are given; *Central Difference Method, Newmark Explicit Method,* the *α-Method,* as conceived by Hilber, Hughes and Taylor in 1977, and the *Time Discontinuous Galerkin Method,* applied by means of predictor-corrector tecnique by Bursi, Bonelli in 2001 and 2002, are discussed.

KEYWORDS: Pseudo-dynamic tests, errors, numerical algorithms, non-linear systems.

1. INTRODUCTION

A pseudo-dynamic test can be considered a simulation of a shaking table test by means of experimental equipment usually used to perform quasi-static tests. A greater computational effort though is demanded by pseudo-dynamic test because the equipment must be directed by an electronic controller to perform a closed loop test. In fact at each step the structural differential equation of motion has to be integrate to obtain the displacements to impose on the specimen in a quasi-static way. This computation employs specimen restoring forces measurements and thus it takes place during the test. The integration of the equation of motion and the minimization of the delay between actual and command displacements, are the main tasks performed by the controller.

Because of the integration of the dynamic equation of motion is the core of the pseudo-dynamic test method, it constitutes the current focus of the research. Two are the main research fields: the numerical analysis of stability and error propagation properties of the algorithms and the exploration of numerical schemes with higher performances. The ultimate target is performing pseudo-dynamic test on structural non-linear systems excited by random forces. In fact there are many circumstances influencing the results reliability, for example the choose of the temporal intervals Δt and ΔT which represent, respectively, the numerical integration step and the time span effectively running during a test between two ensuing loading steps. The choose of Δt is obviously linked to many experimental aspects as it has been discussed by Mahin and Shing (1985) while the numerical temporal step is strictly linked to the scheme properties. The level of viscous damping, the kind of external excitation and the structural system characteristics are others determinant elements in the choose of the proper numerical integration scheme.

The algorithms reported in this paper are the Central Difference Method, the Explicit Newark Algorithm, the α-Method (Hilber, Hughes, Taylor) and the Time Discontinuous Galerkin Method. Their description has the aim to provide a useful tool to approach the pseudo-dynamic test method and to show through some numerical comparisons the influence of the used algorithm on the results obtained.

Central Difference Method and Explicit Newmark Algorithm (1959) are the first numerical schemes which have been used to carry out a pseudo-dynamic test. Referring to Newmark algorithms, interesting analyses have been done by different authors. Xie and Steven (1994) have analyzed its stability, in the case of non-linear systems while Shing and Mahin (1987 b) have studied the possibility of modifying the scheme to make it more effective in connection with higher spurious modes. The so called α-Method (Hilber, Hughes and Taylor 1977) too has been subjected to several implementations and comparisons (e.g. Shing and Mannivan 1990, Shing, Vannan and Cater 1991, Bonelli and Bursi 1994, Bursi and Shing 1996, Bonelli and Bursi 2004). A weak formulation of the Galerkin method has been used and tested via numerical and symbolic applications not long ago (Bonelli, Bursi and Mancuso 2001 and 2002) and is here described since it uses a very different approach to the problem.

2. EQUATION OF MOTION OF A M.D.O.F. SYSTEM AND SOME DEFINITIONS FOR ITS NUMERICAL INTEGRATION

The dynamic equilibrium equation of a multi degrees of freedom structure can be expressed by Eqn. **Errore. L'origine riferimento non è stata trovata.** where $\mathbf{d}$, $\mathbf{v}$, $\mathbf{a}$ are respectively the displacements, velocities and accelerations vectors of the system, **M** e **C** are the mass and damping matrices, **R(d)** is the restoring forces vector and **F** the external forces vector.

$$\mathbf{M}\,\mathbf{a} + \mathbf{C}\,\mathbf{v} + \mathbf{R}(\mathbf{d}) = \mathbf{F} \tag{2.1}$$

Eqn. **Errore. L'origine riferimento non è stata trovata.** together with the corresponding initial data $\mathbf{d}(t_0) = \mathbf{d}_0$, $\mathbf{v}(t_0) = \mathbf{v}_0$ constitutes a Cauchy problem whose closed form solution is known only for special cases. To approach a numerical solution of this problem the first step is the discretization of the temporal domain by choosing a number N of time intervals lasting Δt and this way Eqn. (1) assumes at the time instant $t_i = i\,\Delta t$ the following form:

$$\mathbf{M}\,\mathbf{a}_i + \mathbf{C}\,\mathbf{v}_i + \mathbf{R}(\mathbf{d})_i = \mathbf{F}_i \tag{2.2}$$

In order to evaluate the numerical integration methods, used to simulate and carry out pseudo-dynamic tests, and to check their aptitude to simulate the dynamic response of a structural system, some definitions concerning the numerical properties and error propagation of the schemes can be useful (Quarteroni et al. 2002). The accuracy achieved by a numerical scheme is strictly linked to the *local truncation error* $\tau_i(\Delta t)$. This is the error existing at each step between the exact solution and the approximated one under the hypothesis that at the previous step numerical and exact solutions were coincident. The maximum value of the local truncation error is the *global truncation error* $\tau(\Delta t)$ and an integration algorithm is of k-th order consistency if this error is an infinitesimal of k-th order respect to Δt. To achieve convergence though, a global truncation error infinitesimal respect to Δt is not enough, in fact, the stability of the numerical scheme (the ability of working despite perturbations and errors accumulation) is a necessary condition. This definition of stability is related to the behavior of the algorithm when Δt is infinitesimal but stability concerns the asymptotic behavior of the scheme too. In this sense an algorithm is called stable if the numerical solution is bounded when time goes to infinity. Numerical schemes can be stable depending or not on the integration interval, their stability will be thus conditioned or unconditioned. Usually explicit schemes are conditionally stable and the implicit ones unconditionally stable, but this is only a general rule. For a structural system dynamically forced (which is a set of damped oscillators) a numerical scheme is stable if its free vibrations don't grow without boundaries regardless of the initial conditions. To analyze the stability of a numerical scheme, the free vibrations can be studied considering that the response can be expressed in the

following recursive form

$$\boldsymbol{x}_{i+1} = \boldsymbol{A}\,\boldsymbol{x}_i \tag{2.3}$$

where $\boldsymbol{A}$ is the amplification matrix of the numerical scheme, that is independent on i, but is strictly dependent on the adopted scheme, and $\boldsymbol{x}$ is the vector containing some or all state variables of the system such as displacements, velocities and accelerations. Some conditions on the eingenvalues of the matrix A give proof of stability of the algorithm.

Since the amplification matrix $\boldsymbol{A}$ depends on the dynamical characteristics of the system such as mass, stiffness and damping and on those attaining the scheme, these conditions convert on inequalities correlating the own pulsations ω_k of the system and the temporal step Δt.

3. CENTRAL DIFFERENCE METHOD

For the case of single degree of freedom systems, this numerical scheme assumes that velocity and acceleration at the i-th step can be described by the following equations (Mahin and Shing 1985)

$$v_i = \left(d_{i+1} - d_{i-1}\right)/2\Delta t; \qquad a_i = \left(d_{i+1} + d_{i-1} - 2d_i\right)/\Delta t^2; \tag{3.1}$$

Equilibrium Eqn.(2.1) specialized in the case of a s.d.o.f. system, among with Eqs. (3.1) can be rearranged to express the displacement in the form

$$d_{i+1} = \left[\Delta t^2\left(F_i - R_i\right) + \left(0.5\Delta t\,C - m\right)d_{i-1} + 2m\,d_i\right]\cdot\left(0.5\Delta t\,C + m\right)^{-1} \tag{3.2}$$

For a linear s.d.o.f. system under an external force and with zero viscous damping, a direct step-by-step integration algorithm can be expressed in a recursive form as

$$\boldsymbol{x}_{n+1} = \boldsymbol{A}\,\boldsymbol{x}_n + \boldsymbol{l}\,f_n \tag{3.3}$$

where $\boldsymbol{l}$ is the load operator and $\boldsymbol{f_n}$ is the value of the external excitation at the n-th step but this formulation does not take into account the experimental and numerical errors which are always present in a numerical simulation or in an actual test. These errors are due for example to floating-point computation or they are inherent to the numerical scheme, but the main source is the restoring force feedback, further the latter is due both to measurement imprecision and to the difference between the actual displacements and the command ones which affects the restoring forces. By taking into account only the experimental errors Eqn. (3.3) assumes a modified form which depends not only on $\boldsymbol{A}$ and $\boldsymbol{l}$ but also on the "measured" restoring force, affected by the restoring force feedback error e_n^r introduced at the n-th step. For Central Difference Method a formulation of the cumulative error on the response system due to this feedback error has been given by Shing and Mahin (1987 a) and Shing and Mahin (1990). Furthermore Mahin and Shing (1984) showed how the natural frequency of the system is distorted by the error presence, in fact the apparent period of the oscillator undergoes a shrinkage which depends on the time step Δt. As it is shown in Fig. 1-a the numerical pulsation approaches the real one when the time step Δt goes to zero. In Fig. 1-b one can see the different response obtained for a single degree of freedom system, undamped and unforced, featured by an initial velocity $v_0 = 1$ for different integration time steps. The ratio between the maximum cumulative error $\left|\overline{e}_{n+1}\right|_{\max}$ and the standard deviation of the experimental random feedback errors at each instants S_e, estimated as a function of the number of steps n and the product $\omega\cdot\Delta t$ is shown in Fig. 1-c.

An example of numerical errors due to the scheme is given by a comparison with an alternative formulation of the Central Difference Method. In this case the summed variable $z_i = \left(d_i - d_{i-1}\right)/\Delta t$ instead of d_i (Eqn.(3.2)) is used

to avoid round-off errors caused i.e. by floating point computations or in the feedback process (Mahin and Shing 1985). A simple comparison of the difference between the two formulations is shown in Fig. 2-b where the displacement response of a s.d.o.f. system with natural angular frequency $\omega=2\pi$ (rad/s) and viscous damping $\zeta=0.01$ first and $\zeta=0.1$ after, is corrupted, step after step, by adding a random variable with zero mean ($\mu=0$) and standard deviation $\sigma = 0.001$ (Fig. 2-a). The system isn't externally excited and null initial conditions are imposed. The integration step used is $\Delta t=0.01$ (s). By using both formulation a response different from the attended null one is obtained because of the error influence but the second formulation achieves a smaller system response for its greater capacity of working in spite of perturbations. Observe that a better effect is obtained for a greater value of viscous damping.

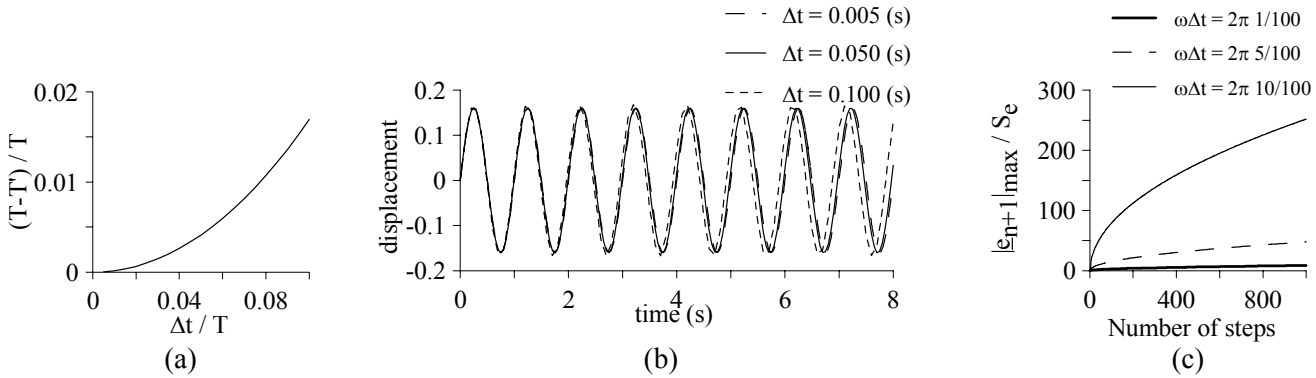

Figure 1. Period shrinkage by Central Difference Method (a);
Free vibration response of a s.d.o.f. system ($\omega = 2\pi$) for different time steps (b).
Normalized cumulative bounds for random errors ($\left|\overline{e}_{n+1}\right|_{\max} / S_e$) for different values of Δt (c).

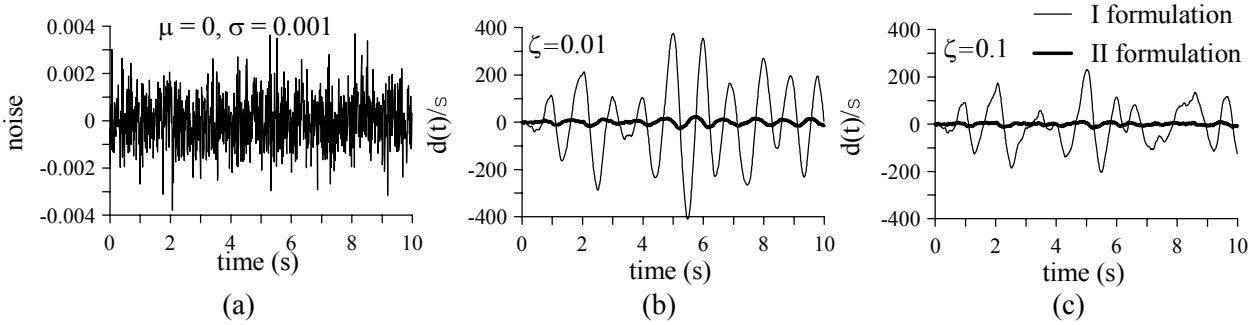

Figure 2. Noise with zero mean and standard deviation $\sigma = 0.001$ (a); Normalized displacements of a not excited s.d.o.f. system calculated by using the two different formulations of the Central Difference Method in presence of the noise and for two different value of viscous damping: $\zeta = 0.01$ (b) and $\zeta = 0.1$ (c).

4. NEWMARK ALGORITHMS AND α-METHOD

The α-Method family of algorithms (Hilber, Hughes and Taylor 1977) is based on the following set of equations:

$$\mathbf{M}\mathbf{a}_{i+1} + (1+\alpha)\mathbf{C}\mathbf{v}_{i+1} - \alpha\mathbf{C}\mathbf{v}_i + (1+\alpha)\mathbf{R}_{i+1} - \alpha\mathbf{R}_i = (1+\alpha)\mathbf{F}_{i+1} - \alpha\mathbf{F}_i \tag{4.1}$$

$$\mathbf{d}_{i+1} = \mathbf{d}_i + \Delta t\,\mathbf{v}_i + \Delta t^2\left[\left(0.5-\beta\right)\mathbf{a}_i + \beta\,\mathbf{a}_{i+1}\right] \tag{4.2}$$

$$\mathbf{v}_{i+1} = \mathbf{v}_i + \Delta t\left[\left(1-\gamma\right)\mathbf{a}_i + \gamma\,\mathbf{a}_{i+1}\right] \tag{4.3}$$

The algorithms of this family are a function of three parameters α, β, γ and they include the Newmark algorithms

when $\alpha = 0$. In its general form a Newmark algorithm is implicit because at the temporal step t_{i+1} the displacement depends on the acceleration at the same instant. The two parameters β and γ control the accuracy and stability properties of the scheme, i.e. by assuming $\beta = 0.25$ and $\gamma = 0.5$ we are using the trapezoidal rule or constant average acceleration method, whereas the choose $\beta = 0$ and $\gamma = 0.5$ provides an explicit formulation which is the one that has been first used to carry out pseudo-dynamic tests. This Explicit Newmark Method and the Central Difference Method possess the same mathematical properties that is they are both conditionally stable and their stability limit, in the case of a linear system, is $\omega_k \Delta t \leq 2$, where ω_k is the generic natural pulsation of the system. For example referring to errors influence, the results inserted in Fig. 1 are still valid for the Newmark algorithms family and if the simulation to which Fig. 2 refers is repeated, the same curve of the second formulation of the Central Different Method is obtained.

The aim of α-Method family of algorithms is to overcome some limits of the Newmark algorithms regarding in specific the numerical dissipation, that is the ability of the numerical scheme to damp the so-called spurious higher modes of the amplification matrix, those produced by the discretization of the motion equations. Newmark algorithms possess this property which is partially controlled by the γ parameter. What the method wants to eliminate is the dissipation of the basic modes of the system when spurious higher modes are dissipated by means of the parameter γ. In this sense the introduction of the parameter α inverts the trend of the basic modes to be dissipated when the spurious higher modes are dissipated. The α parameter ($\alpha < 0$), introduces a negative damping aptitude which is similar to the viscous damping and, what's more important, is not effective on the higher modes. These properties led the authors of the method in object to arrange the γ-dissipation and a negative α-dissipation so that it's possible to get rid of drawbacks like the dissipation of the lowest modes and gain the benefits (the dissipation of spurious modes) of the γ-dissipation. The new algorithm while been function of the three afore mentioned parameters, is thus ruled by only one of them because of the expressions stating the unconditional stability ($\beta = 0.25\,(\gamma+0.5)^2$) and a positive global numerical dissipation ($\alpha + \gamma = 1/2$).

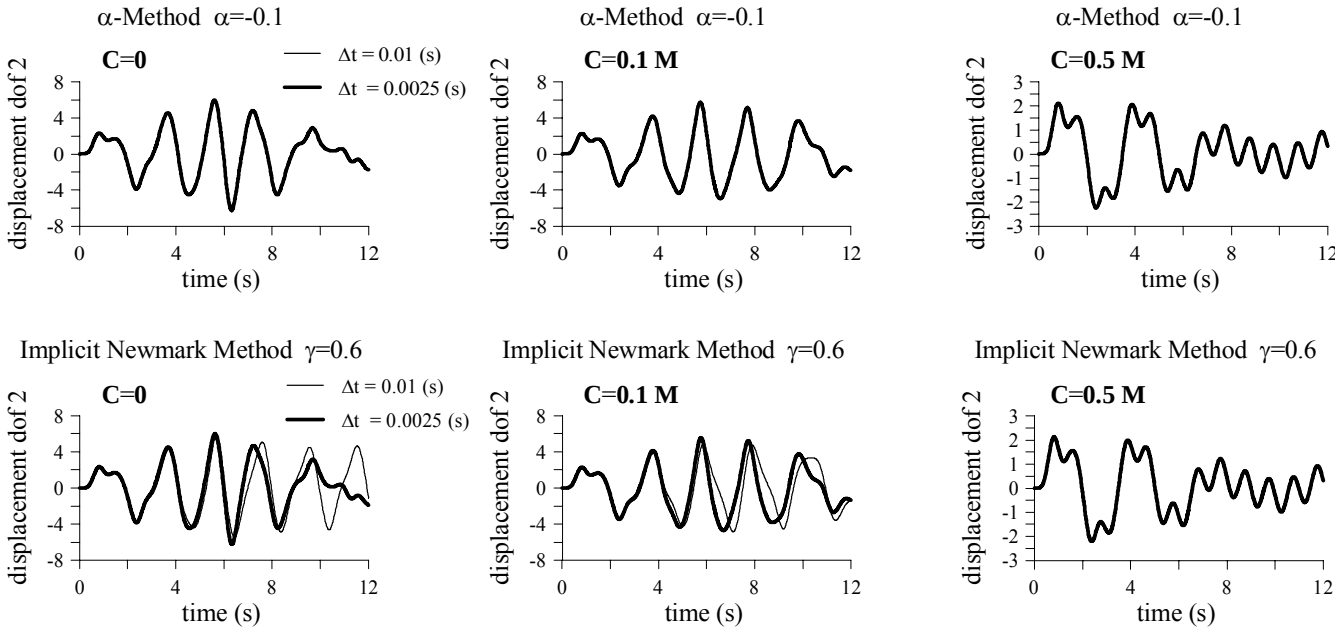

Figure 4. Displacements of a 2d.o.f. non-linear system calculated for two different values of Δt by means of Implicit Newmark Method and α-Method;

In the following numerical example the α–Method is tested in comparison with the Newmark scheme. The dynamical system studied is a 2d.o.f. system whose non-linear restoring forces are expressed by

$R_1 = k_{11}\, d_1 + k_{12}\, d_2 + k_{13}\, d_1^3$ and $R_2 = k_{21}\, d_1 + k_{22}\, d_2 + k_{23}\, d_2^3$. The system has been excited by a sinusoidal force $F_1(t)=400\, m_1 \sin(2\pi\, t)$ applied at the first degree of freedom and the structural response has been obtained for two different values of the time step, Δt=0.01 s and Δt=0.0025 s. The parameters characterizing the system stiffness are k_{11}=100 N/m, k_{22}=20 N/m, k_{12}= k_{21}=-20 N/m, k_{13}=5 N/m^3, k_{23}=5 N/m^3 while the masses are m_1=m_2=10 kg. The simulation has been repeated for two different values of the viscous dissipation matrix: $\boldsymbol{C}$=0.1 $\boldsymbol{M}$ and $\boldsymbol{C}$=0.5 $\boldsymbol{M}$. In this second case the maximum restoring force value is about forty times the maximum viscous force value. The value of numerical dissipation chosen for the α-Method is α=-0.1 which implies, γ=0.6 for the implicit Newmark scheme.

In Fig. 4 the results obtained for different values of the matrix $\boldsymbol{C}$ are plotted. Each figure shows the displacements of the nodes 1 and 2 computed by using the implicit Newmark method and the α-Method. One can see how displacements calculated by using α-Method don't change for the different integration step values, both for the undamped system and for the damped one, meaning convergence has been achieved for the chosen Δt. This doesn't happen with the implicit Newmark method when the system is not damped by viscous forces. Further observe as the increasing of dissipation produces the agreement of the responses obtained with the two methods.

5. TIME DISCONTINUOUS GALERKIN METHOD

The application of the Galerkin technique to integrate differential equation as requested in pseudo-dynamic test arises from the will to extend this experimental technique to highly non-linear problems. Nevertheless the first formulation of the method is addressed to linear problems. Observe that the numerical properties such as unconditional stability or accuracy order, verified for a numerical scheme used to solve a linear problem, don't apply immediatly for non-linear systems for which stability and accuracy have to be proved for each case. The formulation of the Galerkin method used here is the one implemented in a predictor-corrector form by Bonelli and Bursi (2002) for non-linear cases. The procedure has been checked on classical stiff problems such as Duffing oscillator and stiff string pendulum.

The formulation is based on two independent fields, displacement $\boldsymbol{d}(t)$ and momentum $\boldsymbol{p}(t) = \boldsymbol{M}\cdot\boldsymbol{v}(t)$. The second order differential equation (2.1) and the correspondent initial conditions are rewritten in the following first order scheme where the superposed dots indicate time derivative

$$\begin{aligned} &\dot{\mathbf{p}}(t) + \mathbf{C}\mathbf{M}^{-1}\mathbf{p}(t) + \mathbf{R}(\mathbf{d}(t)) = \mathbf{F}(t); \qquad \mathbf{M}^{-1}\mathbf{p}(t) - \dot{\mathbf{d}}(t) = \mathbf{0}, \quad t \in I = (0, t_N); \\ &\mathbf{d}(0) = \bar{\mathbf{d}}_0; \qquad \mathbf{p}(0) = \bar{\mathbf{p}}_0 = \mathbf{M}\mathbf{v}_0; \end{aligned} \tag{5.1}$$

The numerical properties of the algorithm both in the linear version and in non-linear one are extensively discussed in Bonelli, Bursi and Mancuso (2001) and (2002) and the scheme in its non-linear version has been evaluated (Bonelli, Bursi and Mancuso 2002) on a Duffing system whose closed-form solution is known for initial conditions $\boldsymbol{d}(0) = \boldsymbol{d_0}$; $\boldsymbol{p}(0) = \boldsymbol{p_0}$ and no external excitation. The results appear to be satisfying in respect of accuracy and energy-decaying properties. Nevertheless non-linear problems, far from be solved, are underlined in the following numerical example showing the inability of the current schemes when non-linear systems are excited by random forces to reach a reliable solution.

The considered system is a s.d.o.f. oscillator whose restoring force is modeled by the Duffing expression $R(d) = S_1\, d(t)\left(1 + S_2\, d(t)^2\right)$. The system characteristics are m=1, S_1=10, S_2=2. The initial conditions of the system are null while excitation is given by a Gaussian white noise with zero mean (μ=0) and standard deviation σ = 79.2.

Displacements have been calculated by means of α-Method (α=-1/3) and Time Discontinuous Galerkin Method (a=b=1/6, number of iterations of the corrector k=2) and for two values of the viscous damping coefficient c=0.001S_1 and c=0.01S_1. The excitation time step is Δt_F=0.01 s and two different values of the integration step have been chosen, first Δt =Δt_F = 0.01 s and then Δt = 0.1·Δt_F = 0.001 s. For each method displacements obtained with

the two time steps are shown in Fig. 7. These evidence how both methods are reliable only for few seconds, but in the case of the Galerkin Method the response for the two chosen Δt remains coincident for a longer time. For both schemes differences between response obtained with different time step reduce when c increases. In spite of the convergence of the two methods for a sufficiently high value of c, the reliability of the response is still to be evaluated. In fact the balance of the energies of the system, the one given by the external forces, the one dissipated and that stored in terms of kinetic and potential energy, could give an indication but not a definitive response.

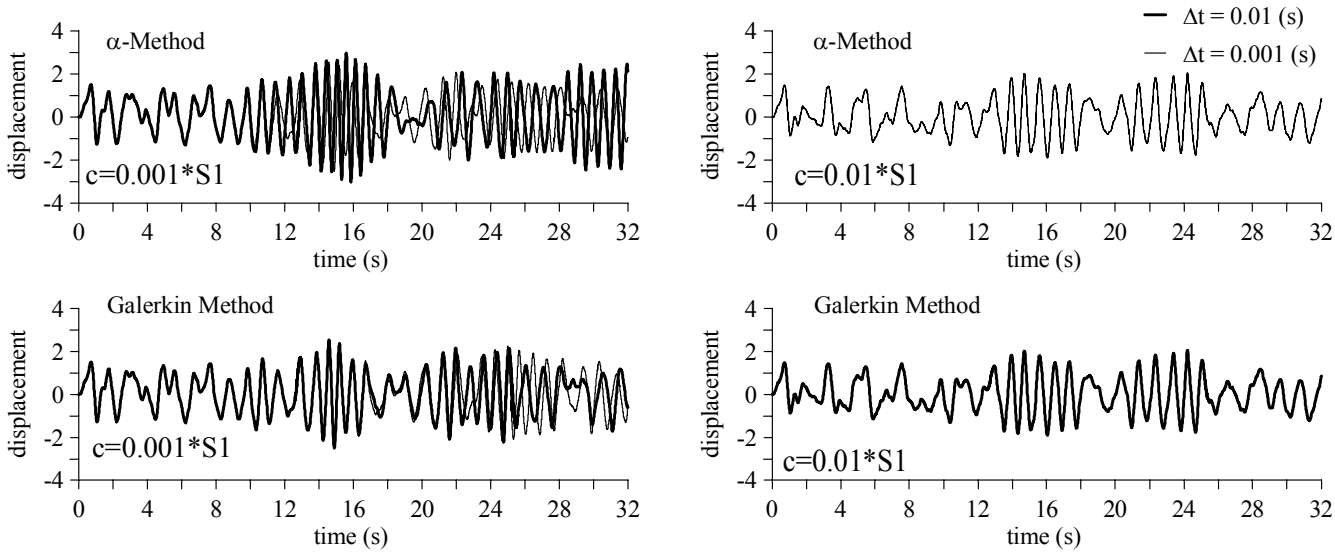

Figure 7. Duffing oscillator with viscous damping coefficient c=0.001*S_1 and c=0.01*S_1; displacements calculated with different values of integration step

6. CONCLUSIONS

In this paper Central Difference Method, Newmark Explicit Method, α-Method (Hilber, Hughes and Taylor 1977), Time Discontinuous Galerkin Method (Bursi, Bonelli 2001 and 2002) have been reported and some simulations of pseudodynamic tests have been carried out on s.d.o.f. and m.d.o.f. systems to assess the numerical scheme properties and their reliability when applied to linear and non-linear cases.

In spite of the differences among the analyzed schemes, from the above tests it mainly emerges that the increasing of the physical dissipation produces a reduction of the differences between those schemes. Alternatively, low levels of the dissipation coefficients pose the problem of the identification of a sufficiently reliable algorithm.

In the case of linear systems some rules which make it possible to evaluate the algorithm stability, are available. On the contrary, in the case of non linear systems, it is not possible to obtain rules valid in each case. Certainly some verification, for example by computing the energies involved in the test may be carried out at the end of a pseudo-dynamic test in order to verify its reliability.

What above makes some questions rise about pseudo-dynamic tests each time the algorithm used is not declared and a discussion about it is not offered before the results are proposed. Further details about the differences involving the numerical schemes analyzed can be found in (Amato and Cavaleri, 2007).

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