AN EXPLICIT METHOD FOR NUMERICAL SIMULATION OF WAVE EQUATIONS

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ABSTRACT:
A method to develop a hierarchy of explicit recursion formulas for numerical simulation in irregular grids for scalar wave equations is presented and then is used to construct the formulas for the one-dimensional case in this paper. Numerical simulation of the one-dimensional scalar wave equation in a regular grid is discussed for understanding its accuracy and stability, and an approach is then proposed to construct the stable formulas which are of 2M-order of accuracy both in time and space with M being a positive integer and the recursion formulas of the second order (M=1) and the fourth order (M=2) are given as an example. Theoretical results of the method are demonstrated by a series of numerical tests.

KEYWORDS:
wave equation, numerical simulation, explicit recursion formula, finite element method (FEM)

1. RECURSION FORMULAS AND SOLUTIONS OF THE INITIAL-VALUE PROBLEM

Our starting point is a concept that wave speed is finite. According to this concept we will first clarify the relationship between the exact solution of an interior point of a finite homogeneous area within a short time window and solution of the initial-value problem for an infinite homogeneous space in the section, and the recursion formulas for an irregular grid are then derived by interpolation approximation. Assuming the wave speed 1 is a constant in a finite spatial domain, the field of displacement and that of velocity satisfying a scalar wave equation are denoted by $u(x,t)$ and $v(x,t)$ respectively, where x denotes a coordinate vector, and t means time. If the displacement distribution function $u(x,0)$ and the velocity distribution function $v(x,0)$ are known, we investigate $u(0,\Delta t)$ and $v(0,\Delta t)$ at a point P_0 which is assumed to be located at the coordinate origin without losing generality. We assume that the shortest distance from P_0 to the boundary of the area is l, the neighborhood of P_0 is defined as $|x| \le \Delta x$, $0 < \Delta x \le l$. Let $\Delta t \le \Delta x / c$, the following judgments can be drawn from the concept of finiteness of wave speed: $u(0,\Delta t)$ and $v(0,\Delta t)$ are determined completely by $u(x,0)$ and $v(x,0)$ on the interval $|\boldsymbol{x}| \le \Delta x$, and have nothing to do with the motion of all the other points outside the neighborhood of P_0 as $t=0$ for their effects have not reached the point P_0 at $t=\Delta t$. Therefore, we can extend the neighborhood of P_0 with a constant speed to an infinite homogeneous space as far as computing the motion of P_0 at an adjacent next time is concerned. So, it implies the motion of P_0 can be computed using the solution of Cauchy problem and the form of the computing formulas are as follows:

$$
\begin{aligned}
u(0,\Delta t) &= J_u\left(u(\boldsymbol{x},0), v(\boldsymbol{x},0)\right) \\
v(0,\Delta t) &= J_v\left(u(\boldsymbol{x},0), v(\boldsymbol{x},0)\right)
\end{aligned}
\tag{1.1}
$$

Where the function $u(\boldsymbol{x},0)$ and $v(\boldsymbol{x},0)$ are defined on the interval $|\boldsymbol{x}| \le c\Delta t$, and the expressions of functional $J_u\left(u(\boldsymbol{x},0), v(\boldsymbol{x},0)\right)$ can be derived from solutions of the Cauchy problem. The solutions are well known for the wave equations in 1-D, 2-D and 3-D cases, which are called as *D′Alembert*, *Poisson* and *Kirchhoff* formula respectively [1]; and $J_v(.,.)$ can be derived from $J_u(.,.)$ by differentiation with respect to time.

As far as the numerical simulation of wave equation in time domain is concerned, a continuous space is required to be discretized via a grid and to construct the recursion formulas of the nodal points. If data of the motion are known at the point P_0 and the adjacent nodal points when $t=0$, the distribution function $u(\boldsymbol{x},0)$ and $v(\boldsymbol{x},0)$ in the neighborhood of $P_0(|\boldsymbol{x}|\le c\Delta t)$ can be approximated in terms of these discrete data via interpolation. Substituting these approximate distribution functions into Eqn. 1.1, a basic form of the recursion formulas can be obtained.

2. THE BASIC RECURSION FORMULA FOR THE 1-D CASE

For the numerical simulation of the 1-D wave equation, the continuous x axis is discretized by a sequence of spatial discrete points P_j with coordinate $x=x_j, j=0,\pm1,\pm2,\cdots$. Point P_0 and the adjacent nodal points P_j, $j=\pm1,\cdots,\pm M$ with M being a positive integer consist of a local system of nodal points and Fig.1shows the case of $M=2$.

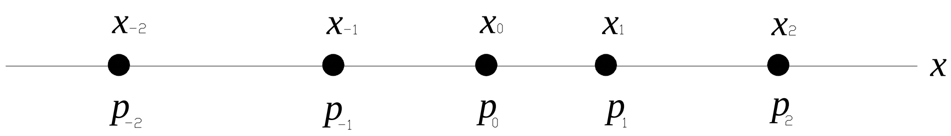


Figure 1 Schematic for a local system of nodal points in a 1-D irregular grid

Introducing the discrete time $t_p = p\Delta t$ with p being an integer number, we define that

$$u_j^p = u(x_j, t_p), \quad v_j^p = \Delta t v(x_j, t_p) \tag{2.1}$$

Applying the aforesaid method to the 1-D wave equation, the basic form of recursion formulas of a nodal point for a 1-D irregular grid is obtained [2]

$$\begin{aligned} u_j^{p+1} &= u_j^p + v_j^p + \sum_{i=-M}^{M} \xi_i u_{j+i}^p + \sum_{i=-M}^{M} \eta_i v_{j+i}^p \\ v_j^{p+1} &= v_j^p + \sum_{i=-M}^{M} \zeta_i u_{j+i}^p + \sum_{i=-M}^{M} \gamma_i v_{j+i}^p \end{aligned} \tag{2.2}$$

If the grid is regular, then $x_j = j\Delta x$, so Eqn.2.2 can be reduced to

$$\begin{aligned} u_j^{p+1} &= u_j^p + v_j^p + \sum_{m=1}^{M}\left(\xi_m (u_{j-m}^p - 2u_j^p + u_{j+m}^p) + \eta_m (v_{j-m}^p - 2v_j^p + v_{j+m}^p)\right) \\ v_j^{p+1} &= v_j^p + \sum_{m=1}^{M}\left(\zeta_m (u_{j-m}^p - 2u_j^p + u_{j+m}^p) + \gamma_m (v_{j-m}^p - 2v_j^p + v_{j+m}^p)\right) \end{aligned} \tag{2.3}$$

After analyzing accuracy and *Von-Neumann* stability [3] of Eqn.2.3, we found that the formulas are of $2M+1$-order of accuracy but unstable when coefficients ξ_m, η_m, γ_m and ζ_m, respectively, satisfy the following four systems of equation

$$\sum_{m=1}^{M} d_{lm}\xi_m = \frac{1}{2}\Delta\tau^{2l}, \quad \sum_{m=1}^{M} d_{lm}\eta_m = \frac{\Delta\tau^{2l}}{2(2l+1)} \qquad l=1,\cdots,M \qquad (2.4)$$
$$\sum_{m=1}^{M} d_{lm}\zeta_m = l\Delta\tau^{2l}, \quad \sum_{m=1}^{M} d_{lm}\gamma_m = \frac{1}{2}\Delta\tau^{2l}$$

Where $d_{lm} = m^{2l}$, $\Delta\tau = c\Delta t/\Delta x$. And we also found that the two systems of equation about coefficients ξ_m and η_m guarantee the error terms of even and odd order in eu (truncation error of u) equal to zero up to the order of $2M+1$ respectively; so does the group of coefficients ζ_m or γ_m for ev (truncation error of v); each system of equation contains M equations numbered as $l=1,\cdots,M$, and each equation in the system only ensures the error term of the corresponding order equal to zero in the truncation error. For example, the l th equation in the system about η_m ensures the error term of the ($2l+1$)th order in eu equal to zero, while is not related to the other lower-order terms; particularly, the last equation with the sequence number $l=M$ only ensures the ($2M+1$)-order term in eu equal to zero, but has nothing to do with other error terms of order lower than $2M+1$. Therefore, Eqn.2.4 makes it possible to adjust the coefficients of the recursion formulas while satisfying requirement to the lower order of accuracy. Thus if the last equation in the system about η_m is eliminated, that is, if we abandon requirement of ($2M+1$)th order error term in eu equal to zero, any value can be assigned to one of $\eta_1,\cdots\eta_M$ .So, we propose an approach to construct stable recursion formulas of $2M$ -order in the next section.

3. AN APPROACH TO DEVELOP STABLE RECURSION FORMULAS OF $2M$ -ORDER FOR THE REGULAR GRID

The order of accuracy of Eqn.2.3 depends on values of the coefficients ξ_m, η_m, ζ_m and γ_m, whether Eqn.2.3 satisfies the $Von-Neumann$ stability condition depends on the values as well. Consequently, it is possible to develop stable recursion formulas of $2M$ -order of accuracy by selecting values of ξ_m, η_m, ζ_m and γ_m reasonably. There are several approaches to adjust the coefficients, and the one we propose is as following: values of ξ_m, ζ_m and γ_m ($m=1,\cdots,M$) are still solved by Eqn.2.4; new values η_m^* are assigned to η_m ($m=1,\cdots,M$). Substituting $\eta_m = \eta_m^*$ into the second system of equations in Eqn.2.4 and abandoning the last one lead to

$$\sum_{m=2}^{M} d_{lm}\eta_m^* = \frac{\Delta\tau^{2l}}{2(2l+1)} + \eta_1^*, \qquad l=1,\cdots,M-1 \qquad (3.1)$$

The new values of $\eta_2^*\cdots\eta_M^*$ can be solved from Eqn.3.1 as functions of η_1^*. Let $\eta_1^* = p_1\eta_1$, where η_1 is also solved by Eqn.2.4, and p_1 is an adjustable parameter. If the value of η_1^* selected satisfies the $Von-Neumann$ stability condition via adjusting p_1, Eqn.2.3 will be stable and of the $2M$ -order of accuracy as η_m in Eqn.2.3 is replaced by η_m^*.

Thus our problem now becomes to search values of p_1 which satisfy the $Von-Neumann$ stability condition for a given $\Delta\tau$ ($0<\Delta\tau\le 1$). A region composed of all such values of p_1 is called as the stability region of p_1, the one in the region closest to 1 is denoted as p_1^*, which is called the optimal value for the minimum loss of accuracy as $p_1 = p_1^*$.Substituting $\eta_1^* = p_1^*\eta_1$ into Eqn.3.1,we can obtain the optimal coefficients η_m^* which make Eqn.2.3 stable and of minimum accuracy loss with ξ_m, ζ_m and γ_m ($m=1,\cdots,M$) still solved by Eqn.2.4. As an

example, the coefficients and the stability region of p_1 for the stable recursion formulas of the second order ($M=1$) and the fourth order ($M=2$) are given as follows:
$M=1$:

$$\xi_1=\frac{\Delta\tau^2}{2},\quad \eta_1=\eta_1^*=p_1^*\frac{\Delta\tau^2}{6},\quad \zeta_1=\Delta\tau^2,\quad \gamma_1=\frac{\Delta\tau^2}{2} \tag{3.2}$$

where $p_1^*=1.5$ for all $\Delta\tau\in(0,1]$,and the stability region of p_1 shrinks to be a point $p_1^*=p_1=1.5$.
$M=2$:

$$\begin{cases}\eta_1=\eta_1^*=p_1^*\left(\frac{2\Delta\tau^2}{9}-\frac{\Delta\tau^4}{30}\right),\quad \eta_2=\eta_2^*=\frac{1}{4}\left(\frac{\Delta\tau^2}{6}-p_1^*\left(\frac{2\Delta\tau^2}{9}-\frac{\Delta\tau^4}{30}\right)\right)\\ \xi_1=\gamma_1=-\frac{\Delta\tau^4}{6}+\frac{2\Delta\tau^2}{3},\quad \xi_2=\gamma_2=\frac{\Delta\tau^4}{24}-\frac{\Delta\tau^2}{24},\\ \zeta_1=-\frac{2\Delta\tau^4}{3}+\frac{4\Delta\tau^2}{3},\quad \zeta_2=\frac{\Delta\tau^4}{6}-\frac{\Delta\tau^2}{12}\end{cases}. \tag{3.3}$$

Table 1 Part of the numerical results of p_1^* versus $\Delta\tau$

$\Delta\tau$	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
p_1^*	1.220	1.221	1.223	1.227	1.232	1.239	1.250	1.265	1.288	45/34

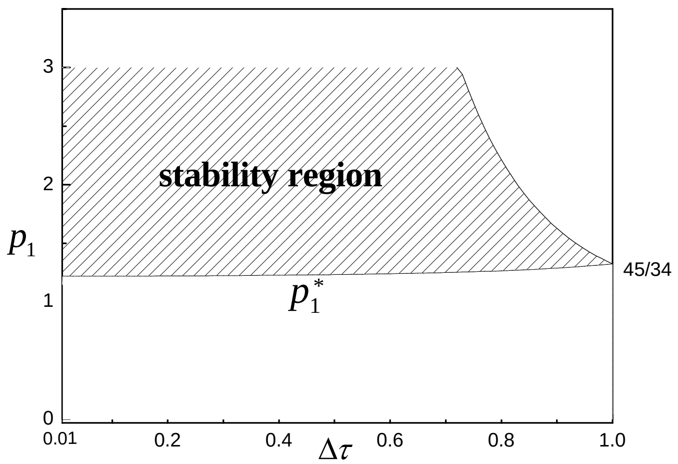


Figure 2 The stability region of p_1 and p_1^* versus $\Delta\tau$

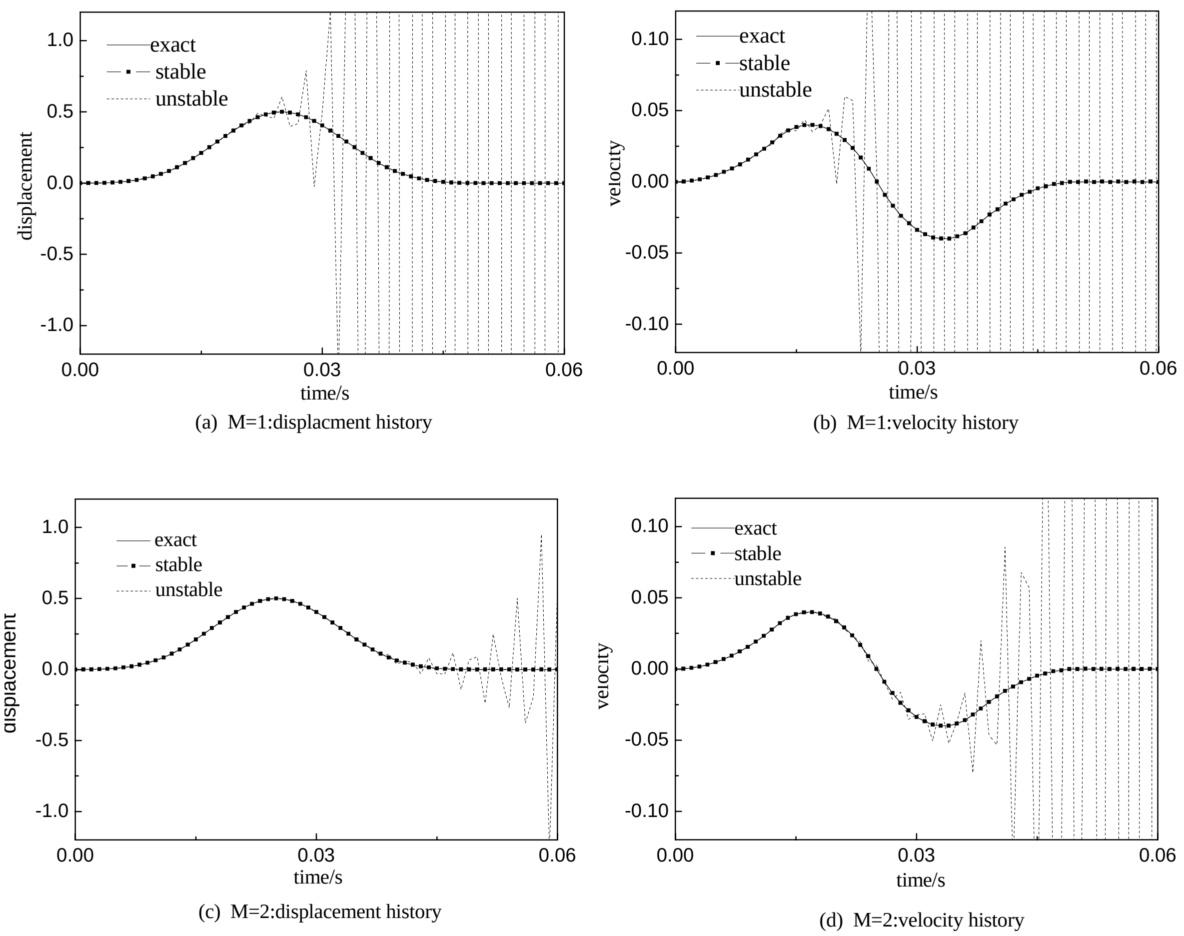
Figure 2 shows the stability region of p_1 (the shadow part) searched numerically within the range $p_1\in[0,3]$.

4. Numerical tests

Considering a Cauchy problem for the 1-D wave equation with wave speed $c=500m/s$, $v(x,0)=0$, $u(x,0)$ is a cubic B spline function (See Eqn. 135, P281 in [4]), whose non-zero part are located on the interval $x\in[0,25m]$. A series of numerical tests for different values of $\Delta\tau$ and Δx are carried out to verify the results presented in this paper, and the main results are as follows:

4.1. Stability Verification

The numerical results indicate that Eqn.2.3 with a value of p_1 in the stability region is stable, otherwise it's not. For example, Figure 3 shows the time histories of displacement (u_0^p) and velocity ($v_0^p/\Delta t$) at $x=0$ for the initial period $t\in[0,0.06s]$ with $\Delta\tau=1$, $\Delta x=0.5m$, $\Delta t=0.001s$. The value of p_1^* equals $1/4$ and $45/34$ for the stable recursion formulas of the second order and that of the fourth order, respectively, while $p_1=1$ for the unstable formula. Figure 3 indicates clearly that the instability phenomena occur quickly in the numerical results for the unstable scheme; meanwhile, the numerical results for the stable formula match the exact solution very well.



(a) M=1:displacment history (b) M=1:velocity history

(c) M=2:displacement history (d) M=2:velocity history

Figure 3 Comparison between the stable and the unstable recursion formulas

4.2. Accuracy Verification

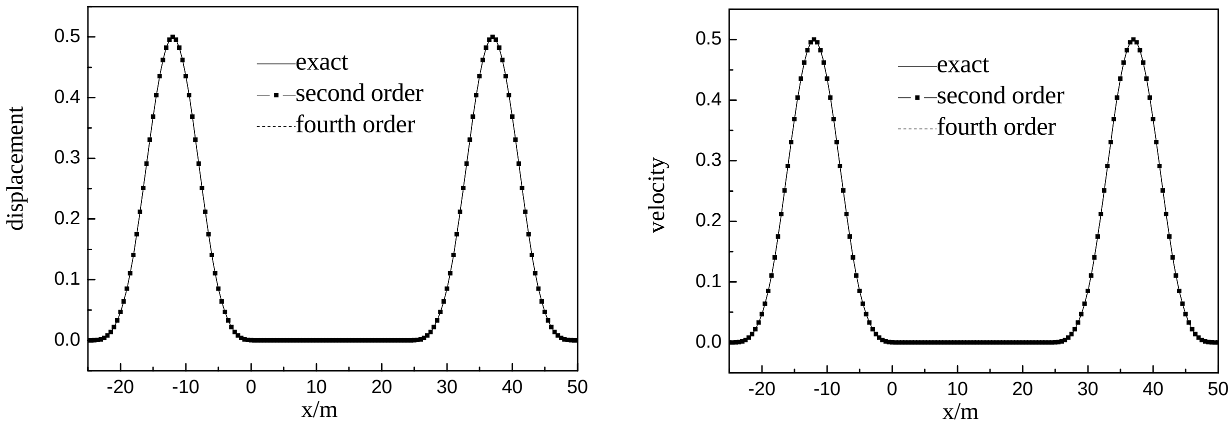
Let $u(x,0)$ be a cubic B spine function and $\Delta\tau=1$, $\Delta x=0.5m$, $\Delta t=0.001s$. Figure 4 shows the waveforms of displacement $\left(u_j^p\right)$ and velocity $\left(v_j^p/\Delta t\right)$ at $t=0.05s$ and at this time the initial disturbance has propagated $25m$ to the left and the right respectively. It can be seen from the figure that the numerical solutions of the stable recursion formulas of the second and fourth order are very close to the exact solution. In order to observe effects of the accuracy order on errors of the numerical solutions, the error function of displacement $\delta u(j,p,\Delta\tau)$ and that of velocity $\delta v(j,p,\Delta\tau)$ are defined by

$$\begin{aligned}\delta u(j,p,\Delta\tau)&=u_j^p-u(j\Delta x,p\Delta t)\\ \delta v(j,p,\Delta\tau)&=v_j^p-\Delta t v(j\Delta x,p\Delta t)\end{aligned},\quad 0\le j\le J,\quad p\ge 0 \tag{4. 1}$$

The error norm of displacement $\|\delta u\|_2$ and that of velocity $\|\delta v\|_2$ are introduced to evaluate the numerical accuracy of the whole waveform at $t = p\Delta t$,

$$\|\delta u\|_2 = \sqrt{\sum_{j=0}^{J}(\delta u(j,p,\Delta\tau))^2}$$
$$\|\delta v\|_2 = \sqrt{\sum_{j=0}^{J}(\delta v(j,p,\Delta\tau))^2} \quad (4.2)$$

Where the value of J should cover all nodal points whose motion data was used by Eqn.2.3 till the given time t . The error norms are computed by Eqn.4.2 for each value of $\Delta\tau$ for $\Delta\tau = 0.01n,\ n = 1,\cdots,100$ .Figure 5 shows the change of $\|\delta u\|_2$ and $\|\delta v\|_2$ versus $\Delta\tau$ at $t = 0.05s$. It can be seen from it that the error norm of the fourth order formula is considerably smaller than that of the second order one for almost the entire interval $\Delta\tau \in (0,1]$.



(a) displacement wave form (b) velocity wave form

Figure 4 The waveform of displacement and velocity at $t = 0.05s$

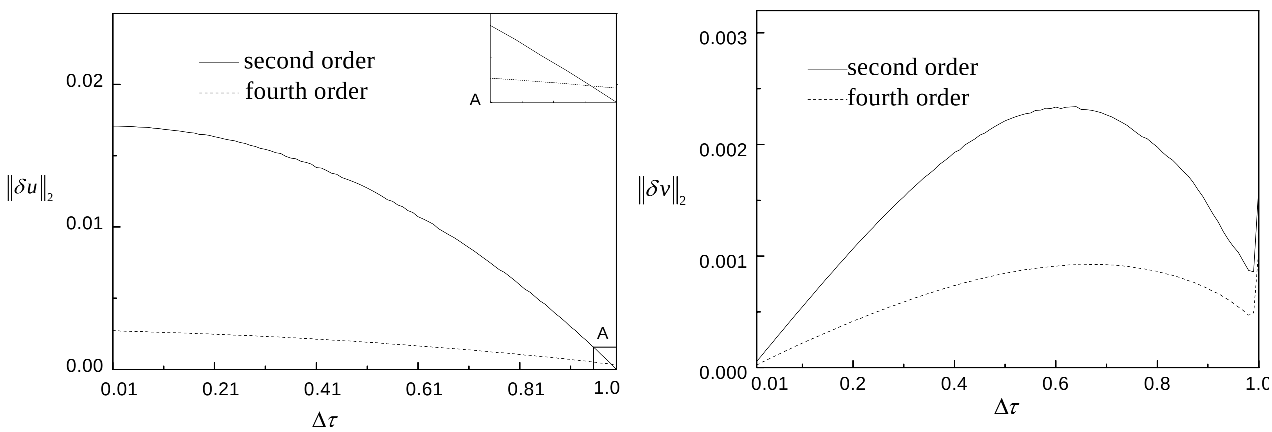


Figure 5 The error norm of displacement $\|\delta u\|_2$ and the error norm of velocity $\|\delta v\|_2$

Finally we design three schemes with $\Delta\tau = 0.5$ to reveal the relationship between the order of accuracy and the computational efficiency: (1) the second order stable formula with $\Delta x = 0.5m$ and $\Delta t = 0.0005s$; (2) the fourth order stable formula with $\Delta x = 1.0m$ and $\Delta t = 0.001s$; (3) the second order stable formula with $\Delta x = 1.0m$ and $\Delta t = 0.001s$ .Figure 6 shows the error $\delta u(0, p, 0.5)$ and $\delta v(0, p, 0.5)/\Delta t$,where the two groups of data have been normalized by the maximum of the exact solution of displacement and velocity, respectively. It shows that error of the scheme (2) shares the same order of magnitude as that of the scheme (1), and both of them are considerably smaller than error of the scheme (3). This indicates that the fourth order scheme has almost the same numerical accuracy as the second order stable one though the time step and space step of the former is one time larger than those of the latter. It means that the improvement of the order of accuracy can not only make the numerical simulation more accurate, but also improve computation efficiency for same precision.

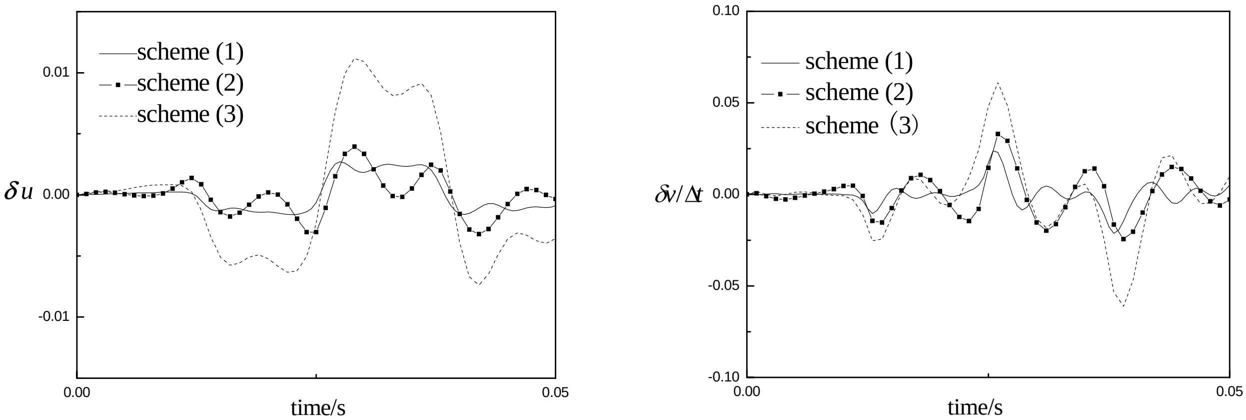


Figure 6 Comparison between the recursion formulas with different order of accuracy and different gird size $(\Delta\tau = 0.5)$

5. CONCLUSION

The paper presents a method to develop the recursion formula which is explicit, time-space decoupling, highly accurate, stable and single-step for numerical simulation of wave equation in irregular grids and in time domain, according to the concept of wave speed being finite; and demonstrates the feasibility for the method via 1-D model. Having noticed the well-known solutions for the 2-D and 3-D Cauchy problem, the method can be applied to constructing the recursion formula for multidimensional irregular grids. We have generalized this method to the 2-D model, and the generalization to the 3-D case is in progress. And the method is suitable for the numerical simulation in a space domain where wave speed varies smoothly with spatial coordinates as long as the wave speed in the recursion formula takes a value of that at a nodal point under consideration; furthermore, the method can be applied to developing the recursion formula of nodal points on an interface where an abrupt change in wave speed occurs if the solution of initial-value problems for the homogeneous infinite spatial domain is extended to that for an infinite spatial domain with the abrupt interface. Besides, the ideas and the approaches presented in the paper have certain reference to improving techniques for the numerical simulation of electromagnetic and elastic wave equations. The above generalization is possible as shown by some work we have done, the related research results will be presented in forthcoming papers.

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