A New Implementation of the Multi-Transmitting Formula for Numerical Simulation of Wave Motion

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ABSTRACT :

A new implementation scheme of the multi-transmitting formula(MTF) is provided as an absorbing boundary condition(ABC) for the numerical simulation of wave motion. The scheme not only improves accuracy of the boundary condition and reduces the computational effort, but also reveals clearly a relation between the order of accuracy of the scheme and the exact numerical solution in which effects of the ABC have been completely removed. Finally the new scheme is compared with the Givoli-Neta ABC based on Higdon ABC.

KEYWORDS: Numerical simulation, Wave motion, Absorbing boundary condition(ABC), Multi transmitting formula, Givoli-Neta ABC, Higdon ABC

1. Introduction

Problems of numerical simulation of waves in unbounded media are encountered in many fields of application, such as earthquake engineering, geophysics, electromagnetics, oceanography, aerodynamics, etc. Numerical methods for such problems often involve truncating a finite computational domain and applying absorbing boundary conditions(ABCs) on the truncation boundary. Many ABCs have been proposed since the 1970s, which can be classified into two broad classes: differential-equation-based and material-based; see surveys(Hagstrom 1999, 2003; Givoli 2004; Hagstrom and Lau 2007). Although many valuable results have been made in this field, there is still no consensus on the optimal ABC. Here we focus on MTF which was proposed in 1984 and commonly referred to as Liao's ABC by researchers in different fields (Liao et al. 1984; Liao 2002). This is because of its applicability to various fields and its easy implementation. In this paper we give a new implementation scheme of MTF, which not only improves accuracy of the boundary condition and reduces the computational effort, but also reveals clearly a relation between the order of accuracy of the scheme and the exact numerical solution in which effects of the ABC have been completely removed. The relation shows that improving accuracy of ABCs is still an important topic for the numerical simulation. Following is the outline of the rest of this paper. In Section 2 we give a brief description of MTF. In Section 3 we illustrate the new implementation scheme of MTF and discuss its implication. We demonstrate the performance of the new scheme via some numerical examples in Section 4 and compare it with the G-N ABC based on Higdon's ABC(Givoli and Neta 2003). We close with concluding remarks in Section 5.

2. Brief description of MTF

Without loss of generality, we consider a two-dimensional wave problem in unbounded media. A Cartesian coordinate system (x, y) is introduced such that the artificial boundary is parallel to the y-direction. The setup is shown in Fig. 1. It is assumed that the wave field u(t, x, y) in the neighboring area of a boundary point (0, y_k) consists of the one-way waves propagating from the interior of the computational domain into the exterior:

$$u(t,x,y)=\sum_l u_l(t,x,y)$$
$$u_l(t,x,y)=f_l(c_{xl}t-x,y) \tag{2.1}$$

It is known in (2.1) that $f_l(c_{xl}t - x, y)$ is a one-way wave satisfying the interior wave equation, and that the apparent speed c_{xl} is a real positive number which may be frequency-dependent. However, the exact form of $f_l(c_{xl}t - x, y)$ and the value of c_{xl} are unknown. An ABC, namely, the MTF has been derived based on a straightforward simulation of (2.1) via introducing a common artificial transmitting speed for all one-way waves in (2.1) and the concept of the error waves (Liao et al. 1984). The MTF for the boundary point $(0, y_k)$ can be written as

$$u((P+1)\Delta t,0,y_k)\approx\sum_{m=1}^{N}(-1)^{m+1}C_m^N u((P+1-m)\Delta t,-mc_a\Delta t,y_k) \tag{2.2}$$

$$C_m^N=\frac{N!}{(N-m)!m!} \tag{2.3}$$

where c_a is the artificial transmitting speed, $y_k = k\Delta y$, Δt and Δy are, respectively, the time-step size and grid spacing in the y-direction, N is the approximation order of MTF. It should be noticed that in derivation of (2.2), only Eq. (2.1) is used, thus MTF is not limited by a particular form of the interior wave equation.

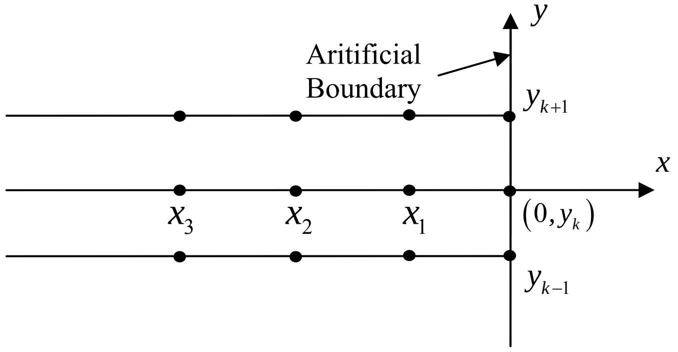

Figure 1 A neighboring area of a boundary point $(0, y_k)$

3. the new implementation of MTF

The original implementation is imposed (2.2) on the artificial boundary directly. The new scheme is a combination of Eq.(2.2) and the discrete formula of an interior field equation for the nodal points adjacent to the artificial boundary. As an example, we suppose the interior field equation is the two-dimensional SH wave equation

$$\left[\partial_t^2-c^2\left(\partial_x^2+\partial_y^2\right)\right]u=0 \tag{3.1}$$

where $\partial_a=\partial/\partial_a$, c is a given wave speed. Consider the explicit central difference scheme of (3.1)

$$u_{i,k}^{n+1}=\left(2-4\Delta\tau^2\right)u_{i,k}^n+\Delta\tau^2\left(u_{i+1,k}^n+u_{i-1,k}^n+u_{i,k+1}^n+u_{i,k-1}^n\right)-u_{i,k}^{n-1} \tag{3.2}$$

where $\Delta\tau = c\Delta t/\Delta x$, $u_{i,k}^{n} = u\left(n\Delta t, i\Delta y, k\Delta y\right)$. At the artificial boundary point (0, y_k), Eq. (3.2) can be rewritten as

$$u_{0,k}^{n+1} = \left(2-4\Delta\tau^2\right)u_{0,k}^{n} + \Delta\tau^2\left(u_{1,k}^{n} + u_{-1,k}^{n} + u_{0,k+1}^{n} + u_{0,k-1}^{n}\right) - u_{0,k}^{n-1} \tag{3.3}$$

MTF is used to update $u_{1,k}^{n}$ i.e.:

$$u_{1,k}^{n} = \sum_{m=1}^{N}(-1)^{m+1}C_m^N u_{1-m,k}^{n-m} \tag{3.4}$$

Where the binomial coefficient $C_m^N = \dfrac{N!}{(N-m)!m!}$. It should be noticed that in Eq.(3.4) the artificial speed $c_a = \Delta y/\Delta t$ is assumed for simplification . Substitute (3.4) into (3.3) leads to the new implementation scheme of MTF:

$$u_{0,k}^{n+1} = \left(2-4\Delta\tau^2\right)u_{0,k}^{n} + \Delta\tau^2\left(\sum_{m=1}^{N}(-1)^{m+1}C_m^N u_{1-m,k}^{n-m} + u_{-1,k}^{n} + u_{0,k+1}^{n} + u_{0,k-1}^{n}\right) - u_{0,k}^{n-1} \tag{3.5}$$

There are three reasons for us to provide such a scheme instead of the original one. The first reason is that the nodal points of the artificial boundary for the original implementation have been replaced by the interior nodal points adjacent to the boundary, thus the computational cost is reduced. This is not meaningless for three-dimensional problems of large scale. Second reason is it can improve accuracy of MTF, which will be shown in the following numerical tests, and further discussed in other reports. Third reason is that the new scheme reveals a relation between the order of accuracy of the scheme and the exact numerical solution in which effects of the ABC have been completely removed. We can see that as N increases, Eq.(3.5) approaches to the discrete form of interior wave equation. However, effects of the ABC can be completely removed as and only as N approaches infinity. Thus improving accuracy of ABCs is still a meaningful topic for the numerical simulation.

4. Numerical test

Here we present two numerical tests. In the first we compare the new scheme with the original, i.e., impose Eq.(3.4) directly upon the boundary. In the second the new scheme is compared with G-N ABC, which is the first alternative of a series high order ABCs recently developed based on the auxiliary variables. A semi –infinite two-dimensional wave-guide as a benchmark problem used by Givoli and Neta (2003) is illustrated in Fig. 2

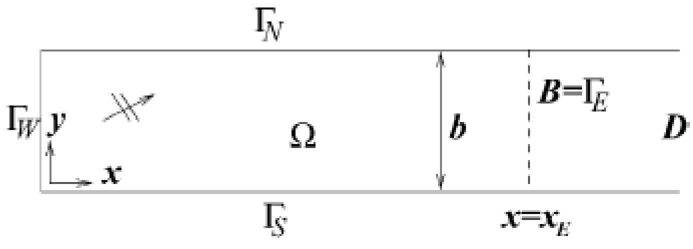

Figure 2 a semi-infinite wave-guide problem

The width of wave-guide is denoted by b. In the wave-guide we consider the linear homogeneous SH-wave equation i.e. Eq.(3.1). We set b=5m and c=1m/s. On the north and south boundaries Γ_S and Γ_N we specify the Dirichlet condition.

$$u = 0 \quad \text{on} \quad \Gamma_S \text{ and } \Gamma_N \tag{4.1}$$

On the west boundary Γ_W we prescribe u using a Dirichlet condition, i.e.,

$$u(0, y, t) = u_w(y, t) \quad \text{on} \quad \Gamma_W \tag{4.2}$$

$$u_w(y,t) = \begin{cases} \cos\left[\dfrac{\pi}{2r_s}(y - y_s)\right] & \text{if } |y - y_s| \le r_s \text{ and } 0 < t \le t_s \\ 0 & \text{otherwise} \end{cases} \tag{4.3}$$

We set $y_s = 1m$, $r_s = 1m$ and $t_s = 0.5s$. The artificial boundary Γ_E is introduced at x_E=5m. Thus, the computational domain Ω is a 5m×5m square. In Ω a uniform grid with 21×21 points is used. We discretize the SH wave equation using Eq.(3.2). In application of numerical simulation, engineer often like to set $\Delta\tau$ as close as, but a little smaller than the CFL stability limit. Here the CFL stability limit is $1/\sqrt{2}$, Thus we let $\Delta\tau$ vary from 0.4 to 0.6, Thus the time-step size Δt varies from 0.01s to 0.15s. On Γ_E we impose the new scheme of MTF, Eq.(3.4) and G-N ABC with different orders J. Initially we set C_N =c= 1m/s for all the N's in G-N ABC(In Givoli and Neta's paper, detail implementation information of G-N ABC is presented).

A reference solution which is regarded as the "exact solution" u_{ex} is obtained by solving the problem in extended domain $0 \le x \le 25$m, $0 \le y \le 5$m, using a 101×21 grid with the same resolution. During the simulation time t=40s the wave generated on Γ_W does not reach the remote (east) boundary of this large domain, and thus the issue of spurious reflection is avoided altogether, regardless of the boundary condition used on the remote boundary. Hence this will serve as a "reference solution" which is exact as far as the boundary condition treatment is concerned. Error measures of the numerical solutions are defined as the Eulerian norm of the point wise error at a nodal point (x_E, y_0), i.e.

$$E(x_E, y_0) = \sqrt{\frac{1}{N}\sum_{n=1}^{N}\left(u(x_E, y_0, n) - u_{ex}(x_E, y_0, n)\right)^2} \tag{4.4}$$

The error will be shown at the point $(x_E, y_0) = (5, 2.75)$ which is located on the artificial boundary, slightly above the center of the waveguide. Fig.3 shows the errors generated by the new scheme Eq.(3.5) with N=2, 3, 4, compared with the errors generated by the old scheme Eq.(3.4). In Fig.4 the errors generated by the new scheme with N=2, 3, 4 is plotted, compared with errors generated by the G-N ABC.

5. Concluding remarks

We have illustrated a new implementation scheme of MTF and its advantages over the old one via a simple two-dimensional benchmark example. The new scheme is applicable to various cases just like the old one as long as the outgoing waves can be described by Eq.(2.1). The measures for stable implementation of the old (Liao, 2002) can be applied to the new scheme as well. As shown in this paper improving accuracy of MTF is still a meaningful topic. In addition, deeper studies on numerical stability of the new scheme are needed.

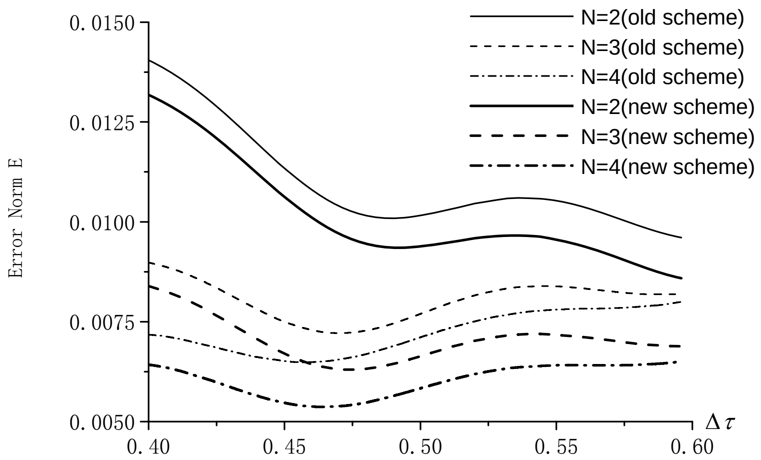

Figure 3 Errors calculated by the measure E defined by (4.4) at point (5, 2.75), the new scheme and the old scheme Eq.(3.4) of order N is used on Γ_E : N=2, N=3, N=4.

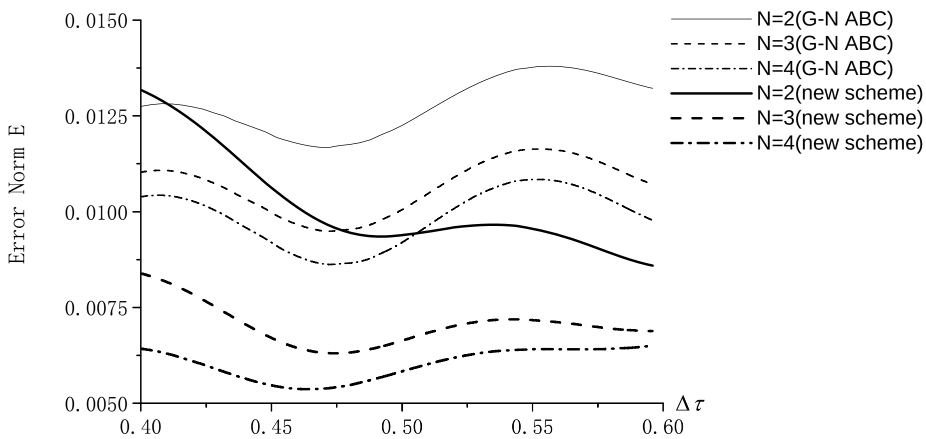

Figure 4 Errors calculated by the measure E defined by (4.4) at point (5, 2.75), the new scheme and G-N ABC of order N is used on Γ_E : N=2, N=3, N=4.

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