STABILITY AND IDENTIFICATION FOR CONTINUOUS-TIME RATIONAL APPROXIMATION OF FOUNDATION FREQUENCY RESPONSE

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ABSTRACT: Continuous-time rational approximation (CRA) of foundation frequency response is the first step of a systematic procedure for constructing various high-order lumped-parameter models (LPMs) in foundation vibration analysis. The stability and accuracy of CRA determine those of its LPMs as realizations. In this paper, the stability and identification of CRA are studied. The necessary and sufficient stability conditions are presented based on the linear-system stability theory and the input-output case of LPMs. A parameter identification method is further proposed by directly solving a nonlinear least-squares fitting problem using the hybrid genetic-simplex optimization algorithm, where the proposed stability conditions are considered by the penalty function method. A stable and accurate CRA is obtained by this method and is then realized as Wu-Lee's and Wolf's LPMs. The proposed stability theory and identification method is verified by analyzing several typical foundation vibration problems and comparing with Wu-Lee' and Wolf's results.
KEYWORDS: foundation vibration, lumped-parameter model, continuous-time rational approximation, stability, identification

1. INTRODUCTION

The soil-structure-interaction analysis considering the dynamic behavior of foundation on infinite soil requires extending the study object from only the structure to a structure-foundation-soil system [1]. The time-domain method is preferred to numerically solve such system because any possible nonlinearity in superstructure can be easily considered. At present, the time-domain representations for foundation-soil system include two types of methods: *lumped-parameter models* (LPMs) and *time-domain recursive evaluations* (TDREs).

A systematic procedure of constructing various LPMs was proposed by Wolf [2]. The procedure can be summarized as two successive steps: the foundation frequency response (dynamic stiffness or flexibility) is first approximated by a rational function in complex frequency with real parameters, and the rational function is then realized as various types of LPMs with real but not necessarily positive parameters. Here, the rational function is called as *continuous-time rational approximation* (CRA), and the resulting LPMs are called as *high-order* due to the asymptotic exactness of CRA with its order increasing. Several types of high-order LPMs [1-6] have been proposed. This systematic procedure is not only for LPMs but also for TDREs, where a rational function in z variable of z-transform, that is called as *discrete-time rational approximation* (DRA), is used instead of a CRA. Several types of TDREs [1, 7-10] have been also proposed. The systematic procedure can be interpreted by linear system theory [11]. The first step of the procedure is that the foundation-soil system is approximated as a differential linear time-invariant continuous-time (LTIC) (or difference linear time-invariant discrete-time (LTID)) system with a CRA (or DRA) as frequency response. The second step of the procedure is that the resulting linear system is realized as various types of LPMs (or TDREs) in many different ways. Although these LPMs (or TDREs) from the same CRA (or DRA) may be different in many aspects, their accuracy and stability are identical and determined by the CRA (or DRA).

So far all works concentrate mainly on the second step of the systematic procedure to construct various types of LPMs (or TDREs). However, the stability and identification of CRA (or DRA) as a key start point are not studied specially. This is one of the reasons that the development and application of LPMs (or TDREs) based on the systematic procedure are limited. At the aspect of stability, no systematic stability theory has been seen by authors so far. At the aspect of parameter identification, the linearized method is used widely [1-5, 7]. Recently, the iteratively linearized method is used by Şafak [8]. The linearized method is an approximate to the original nonlinear parameter identification problem. Moreover, the instable LPMs (or TDREs) may be obtained due to without any stability constraint considered in the identification procedure, such as several LPMs in [4] and TDREs in [8] (which may accurately fit foundation frequency response in frequency domain but cause the time-domain results instable rapidly and severely). Recently, Du et al. [9] and Zhao and Du [10] apply the genetic-simplex algorithm to the original nonlinear identification problem for DRA, where the stability condition can be considered as penalty function.

This paper and an accompanying paper [12] concentrate on the first step of the above systematic procedure. The stability and identification of CRA realized as LPMs are studied in this paper, and those of DRA realized as TDREs do in the accompanying paper [12].

2. CONTINUOUS-TIME RATIONAL APPROXIMATION

From a viewpoint of system, the foundation-soil system form two interinvertible LTIC systems with single input, single output and zero-initial condition, although it is only a physically real system intuitively (see Figure 1). The foundation-soil system can not be conveniently solved in time domain, mainly because it is not a differential system although is a LTIC system. To achieve this, the known exact foundation frequency response (dynamic stiffness or flexibility) is approximated as a rational function in complex frequency in continuous-time case. This rational function is called as CRA here. The *dynamic-stiffness-form CRA* (SCRA) can be written as

$$S(\overline{\omega}) \approx S_C(\overline{\omega}) = S_C(\overline{s}) = S_0 \frac{1 + p_1\overline{s} + ... + p_{N+1}\overline{s}^{N+1}}{1 + q_1\overline{s} + ... + q_N\overline{s}^N} \tag{2.1}$$

where the order of the numerator polynomial N+1 is one more than that of the denominator N, S_0 is static stiffness, $\overline{s}$ is the dimensionless complex frequency, $\overline{s} = i\overline{\omega}$ here, $i = \sqrt{-1}$, $\overline{\omega} = \omega d / c_S$ is the dimensionless frequency with the characteristic length d (of foundation) and the (shear) wave velocity c_S (of soil), and p_j, q_j are the undetermined real parameters which will be identified in next section. If the high-frequency limit c_∞ of dimensionless damping coefficient of dynamic stiffness is known, $p_{N+1} = c_\infty q_N$ so that SCRA is exact (doubly asymptotic) at high- and low-frequency limits. Correspondingly, the *dynamic-flexibility-form CRA* (FCRA), reciprocal of SCRA, is

$$F(\overline{\omega}) \approx F_C(\overline{\omega}) = F_C(\overline{s}) = F_0 \frac{1 + q_1\overline{s} + ... + q_N\overline{s}^N}{1 + p_1\overline{s} + ... + p_{N+1}\overline{s}^{N+1}} \tag{2.2}$$

where $F_0(=1/S_0)$ is static flexibility. Once CRA is obtained, it can be realized as various types of LPMs. Correspondingly, LPM also forms to two interinvertible systems in mathematics (see Figure 1).

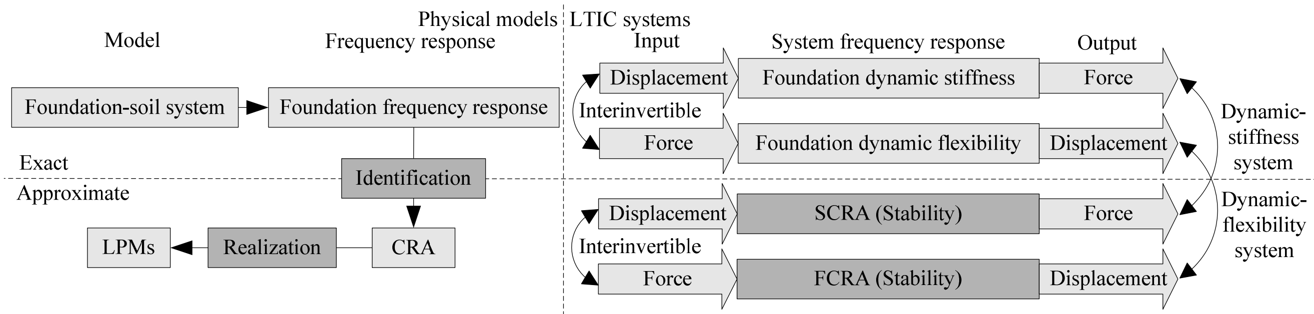


Figure 1 Foundation-soil system and its LPMs

3. STABILITY AND IDENTIFICATION

3.1. Stability

The stability of LPM is that of its dynamic-stiffness or -flexibility system, i.e. the stability of SCRA or FCRA. In soil-structure-interaction analysis, due to LPM with force as input and displacement as output, the stability of FCRA can guarantee the structure-foundation-soil system dynamically stable. Assume that the poles of SCRA are s_j^S for $j=1,...,N$, and the poles of FCRA are s_j^F for $j=1,...,N+1$, where the poles are zeros of their denominator polynomial. According to the linear-system stability theory [11], the stability conditions of a LPM can be stated as follows:

(1) When applied to compute force from displacement, a LPM is dynamically stable if and only if the real parts of all poles of its SCRA are negative, i.e. $\mathrm{Re}\left(s_j^S\right)<0$ for $j=1,...,N$.

(2) When applied to compute displacement from force, a LPM is dynamically stable if and only if the real parts of all poles of its FCRA are negative, i.e. $\mathrm{Re}\left(s_j^F\right)<0$ for $j=1,...,N+1$.

Moreover, according to Routh-Hurwitz theorem, a simple instability criterion can be stated as follows:

(1) When applied to compute force from displacement, a LPM is instable if one or more parameters of denominator polynomial of its SCRA are negative.

(2) When applied to compute displacement from force, a LPM is instable if one or more parameters of denominator polynomial of its FCRA are negative.

3.2. Parameter Identification

The parameters in CRA can be obtained by fitting the exact foundation frequency response. The dynamic stiffness and SCRA are used here. To achieve a stable result, the parameter identification requires solving the following nonlinear least-squares fitting problem

$$\min_{p_j,q_j}\left\{\sum_{l=1}^{L}\left|\left[S_C(\overline{\omega}_l)-S(\overline{\omega}_l)\right]/S_0\right|^2+\sum_{j=1}^{N}P_j^S\left[\mathrm{Re}\left(s_j^S\right)\geq e\right]\right\}\quad\text{for dynamic-stiffness system}\qquad(3.1)$$

$$\min_{p_j,q_j}\left\{\sum_{l=1}^{L}\left|\left[S_C(\overline{\omega}_l)-S(\overline{\omega}_l)\right]/S_0\right|^2+\sum_{j=1}^{N+1}P_j^F\left[\mathrm{Re}\left(s_j^F\right)\geq e\right]\right\}\quad\text{for dynamic-flexibility system}\qquad(3.2)$$

where L denotes the number of data points chosen of interest, l denotes the l-th data point, $\overline{\omega}_l=l\Delta\overline{\omega}$, the constraint condition is combined into the object function by penalty function method, P_j^S and P_j^F are the penalty factors for different poles, $P_j^S=P_j^F=1000$ here, and the real number $e(\leq 0)$ is zero in theory but can be negative to enlarge the penalty range so as to avoid a marginally stable time-domain result including noise. Note that no frequency-dependent weight is required here.

The hybrid genetic-simplex optimization algorithm combining the genetic algorithm [13] and the Nelder-Mead simplex method [14] is used to solve Eqn. 3.1 or 3.2. The flow chart of the hybrid algorithm that is implemented into MATLAB sees Figure 2. The following three reasons make such hybrid algorithm become a proper choice for Eqn. 3.1 or 3.2. First, this algorithm is a direct search method without the derivative of object function required. Second, this algorithm can optimize a high-dimension problem (the dimension of problem is the number of parameters to optimize). Third, no “exact” initial values are required since the genetic algorithm is a global optimization method with random initial values.

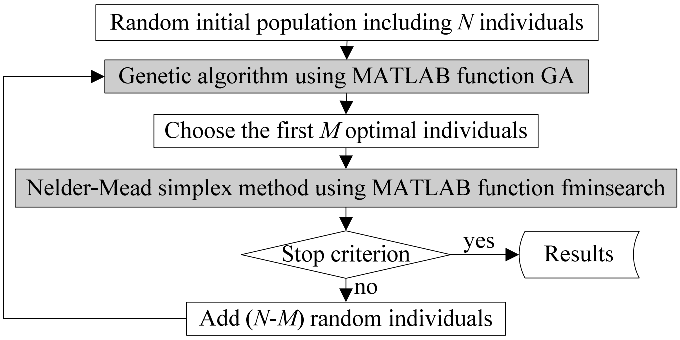


Figure 2 The flow chart of hybrid genetic-simplex algorithm

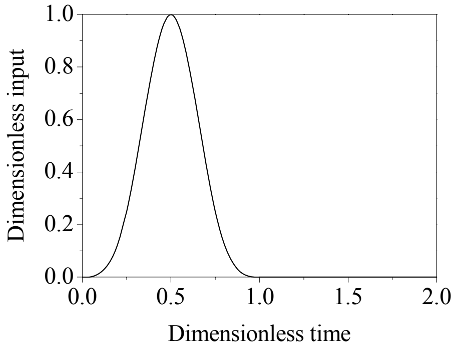
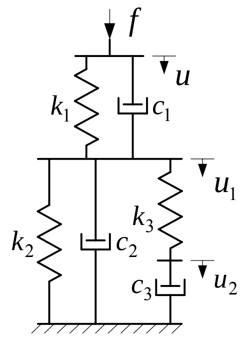
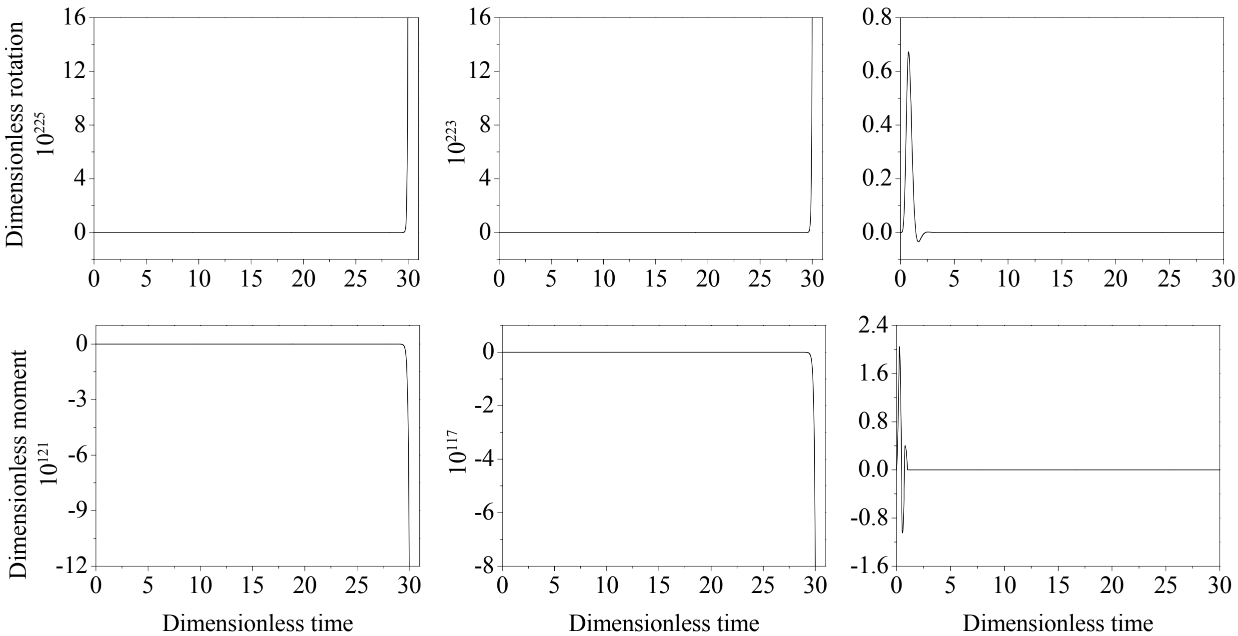


Figure 3 Dimensionless input impulse

Table 1 CRAs and corresponding Wu-Lee LPMs for rocking surface circular foundation (N=2)

	Wu-Lee	This paper (I)	This paper (II, III, IV)
p_1	**0.2756**	0.27772901	0.28231964
p_2	**0.0643**	0.05739021	0.04228887
q_1	**0.0455**	0.02621116	0.01297632
q_2	**−0.0161**	−0.01494088	0.00000002
s_j^S	9.4198	9.105156	−77.072673
	−6.5937	−7.350831	−645449.840300
s_j^F	17.5550	17.415225	−7142782.976057
	−1.9975 ± 2.8325 i	−2.186668 ± 2.875304 i	−3.337991 ± 3.536201 i
$\bar{k}_1$	**−10.28**	−9.78205698	2312731.29935156
$\bar{c}_1$	**0.5849**	0.56169571	0.32378576
$\bar{k}_2$	**0.9114**	0.90725332	1.00000043
$\bar{c}_2$	**0.5932**	0.61920192	3.25898995
$\bar{k}_3$	**−3.181**	−3.71381940	−230.39708171
$\bar{c}_3$	**−0.4067**	−0.41700709	−2.98964640
E	0.411270	0.396646	0.693334



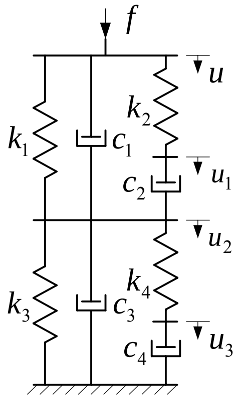


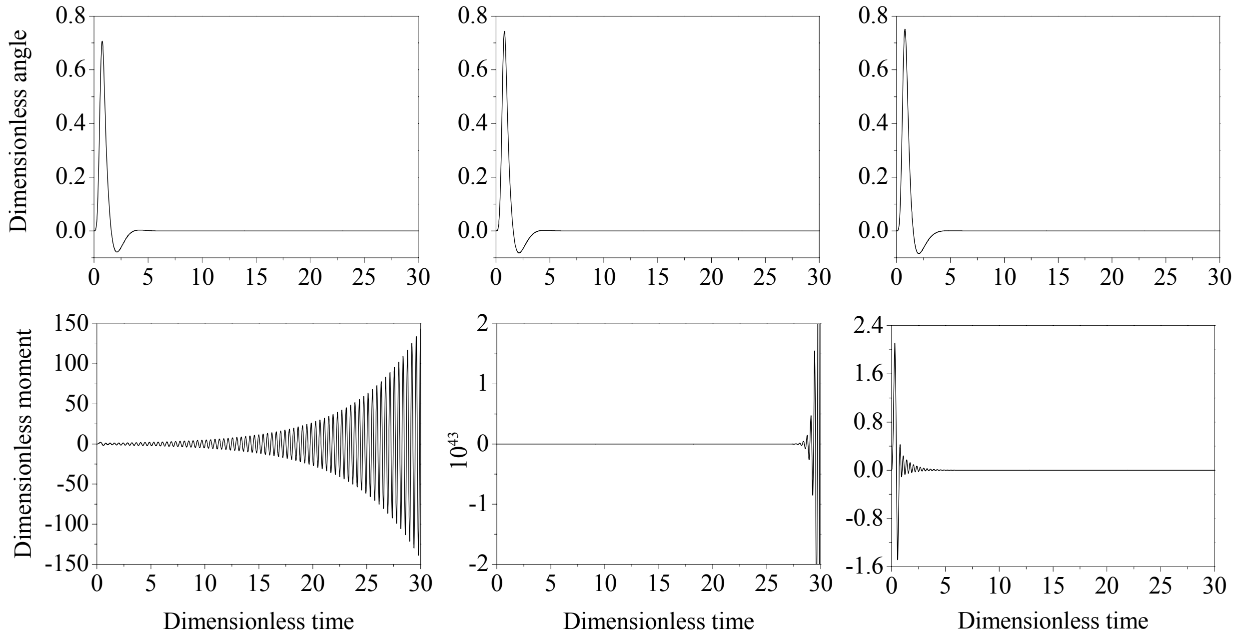
(a) Wu-Lee (b) This paper (I) (c) This paper (II, III, IV)

Figure 4 Time-domain results for rocking surface circular foundation (N=2)

Table 2 CRAs and corresponding Wu-Lee LPMs for rocking surface circular foundation (N=3)

	Wu-Lee	This paper	
		(Ⅰ, Ⅲ)	(Ⅱ, Ⅳ and $e=-1$)
p_1	**0.7019**	0.81039514	0.92971660
p_2	**0.2522**	0.29339136	0.33169604
p_3	**0.0125**	0.01763940	0.02905036
q_1	**0.6910**	0.83315133	1.00498595
q_2	**−0.0151**	−0.01426118	0.00753891
q_3	**0.0029**	0.00250200	0.00251293
s_j^S	−1.3934 3.3001 ± 15.3810 i	−1.171920 3.435913 ± 18.145015 i	−1.000040 −1.000002 ± 19.922989 i
s_j^F	−5.8149 ± 14.897 i −1.5025 ± 1.5233 i	−10.431980 ± 14.868793 i −1.536664 ± 1.323649 i	−17.901108 ± 0.171576 i −1.724339 ± 1.114734 i
$\bar{k}_1$	**−6.112**	−3.05032144	−1.71156278
$\bar{c}_1$	**−55.05**	−1.42318539	−0.46759778
$\bar{k}_2$	**127250**	192.42011405	25.91089713
$\bar{c}_2$	**54.77**	1.25009426	0.29593962
$\bar{k}_3$	**0.8594**	0.75310602	0.63120898
$\bar{c}_3$	**0.2930**	0.24402418	0.18070453
$\bar{k}_4$	**−0.4243**	−0.32838326	−0.22612104
$\bar{c}_4$	**−0.2794**	−0.24637972	−0.18734702
E	0.024246	0.018743	0.075270





(a) Wu-Lee (b) This paper (I, III) (c) This paper (II, IV and e=−1)

Figure 5 Time-domain results for rocking surface circular foundation (N=3)

4. NUMERICAL TESTS

The proposed stability theory and parameter identification method are verified via two examples used by Wu and Lee [4] and Wolf [1]. The comparison is performed with their results. The parameters from [4, 1] are boldface. The four constraint cases in our identification method are: (I) without any constraint; (II) constraint of

$\mathrm{Re}(s_j^S) < 0$; (III) constraint of $\mathrm{Re}(s_j^F) < 0$; (IV) simultaneous constraints. The object function without penalty in Eqn. 3.1 or 3.2 is defined as a global dynamic-stiffness error *E*. The dimensionless physical models with input impulse shown in Figure 3, and the implicit Newmark time-integration method [15] are used in time-domain analysis. For LPMs, the dimensionless spring parameter $\bar{k}_j = k_j / S_0$, dashpot parameter $\bar{c}_j = c_j c_S / (S_0 d)$ and mass parameter $\bar{m}_j = m_j c_S^2 / (S_0 d^2)$ are introduced, respectively.

4.1. Verifying Stability Theory

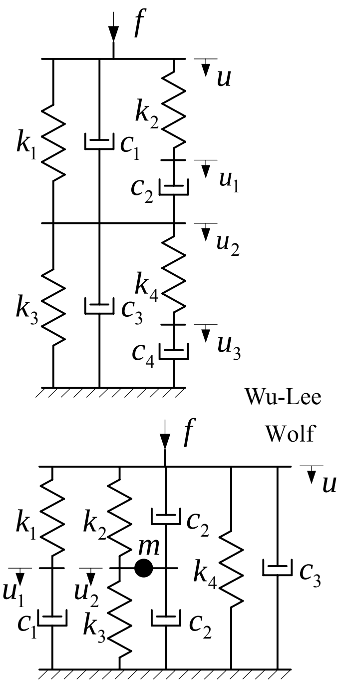
A rocking circular foundation on half-space elastic soil is analyzed. The radius of circular foundation (characteristic length) is *d*. The mass density, shear modules and Possion ratio of soil are ρ, *G*, and *v* (=0.5 here), respectively. The shear wave velocity of soil is $c_S = \sqrt{G/\rho}$. The static stiffness is $S_0 = 8Gd^3/[3(1-v)]$. The high-frequency limit of dimensionless damping coefficient is $c_\infty = 3\pi(1-v)/16$. The exact dynamic stiffness sees [16].

For the case of *N*=2, the paramters of CRAs and corresponding LPMs and the global dynamic-stiffness errors are listed in Table 1. The the time-domain results are shown in Figure 4. Wu-Lee parameters are instable for either dynamic-stiffness or dynamic-flexibility system. The stability of parameters of this paper (I) is same as that of Wu-Lee parameters due to without any stability constraint, but the global dynamic-stiffness error is less which indicats that the proposed method is more effective than the linearized identification method. The parameters of this paper (II, III, IV) are stable for both dynamic-stiffness and -flexibility systems but at the price of the light loss of accuracy.

For the case of *N*=3, the results see Table 2 and Figure 5. This time, Wu-Lee parameters are stable for dynamic-flexibility system but instable for dynamic-stiffness system. The stability of parameters of this paper (I, III) is same as that of Wu-Lee parameters, but the global dynamic-stiffness error is less. The parameters of this paper (II, IV) are stable for both dynamic-stiffness and -flexibility systems but at the price of the loss of accuracy, where $e = -1$ used to avoid the marginal stability.

Table 3 CRAs and corresponding Wu-Lee and Wolf LPMs for semi-infinite rod (*N*=3)

	Wu-Lee	This paper			Wolf *
p_1	**1.66**	1.73561311			1.84874893
p_2	**2.71**	2.57111362			2.39728651
p_3	**1.74**	1.80599507			1.81344415
q_1	**1.86**	1.82740854			1.84775798
q_2	**1.76**	1.81872359			1.81344415
q_3	**1.60**	1.38630902			1.05678564
s_j^S	−0.7066	−0.795079			**−0.930**
	−0.1967 ± 0.9197 i	−0.258419 ± 0.916774 i			**−0.393 ± 0.929 i**
s_j^F	−0.0380 ± 0.9880 i	−0.058047 ± 0.989903 i			−0.106062 ± 1.011391 i
	−0.5058 ± 0.6193 i	−0.593321 ± 0.617719 i			−0.751938 ± 0.591515 i
$\bar{k}_1$	**4.45**	3.18811174	$\bar{k}_1$	−0.71686216	**−0.605785**
$\bar{c}_1$	**4.44**	2.78858518	$\bar{k}_2$	0.04108326	**0.0559794**
$\bar{k}_2$	**−8.47**	−5.26271484	$\bar{k}_3$	−0.04684883	**−0.0639044**
$\bar{c}_2$	**−8.65**	−6.11903670	$\bar{k}_4$	0.66617291	**0.548601**
$\bar{k}_3$	**1.29**	1.45701505	$\bar{c}_1$	−0.90162326	**−0.651197**
$\bar{c}_3$	**1.29**	1.55910113	$\bar{c}_2$	−0.00164224	**−0.00306417**
$\bar{k}_4$	**−0.54**	−0.83082499	$\bar{c}_3$	1.00164224	**1.003064**
$\bar{c}_4$	**−1.27**	−1.05836569	$\bar{m}$	−0.00635496	**−0.00779299**



* In [1], the constant term in numerator polynomail of Eqn. 2.1 is 1.00032084 and $\bar{c}_2 = -0.00306417$.

For the cases of N=2 and 3, all conclusions obtained from the proposed stability theory accord with the results of numerical tests, which indicates the correctness of the theory. Moreover, for same problem, the stability may change for different N, and the accuracy is improved considerably with N increasing.

4.2. Verifying Accuracy of Identification

A semi-infinite rod on elastic foundation is analyzed. The elastic modulus, cross-sectional area, and mass density of rod are E, A, and ρ, respectively. The spring stiffness per unit length of foundation is k_g. The characteristic length is $d=\sqrt{EA/k_g}$. The wave velocity is $c_S=\sqrt{E/\rho}$. The static stiffness is $S_0=\sqrt{EAk_g}$. The high-frequency limit of dimensionless damping coefficient is $c_\infty=1$. The concrete description about this problem sees [1, 4, 6].

The CRAs and corresponding LPMs are listed in Table 3 and 4 respectively for N=3 and 4. All parameters by various identification methods are stable, which indicates that CRAs of semi-infinite rod problem may be stable in nature. Therefore, this problem is proper to verify the accuracy of identification. The global dynamic-stiffness errors are listed in Table 5. It is clear that the accuracy is improved with N increasing, and for same N the parameters in this paper are more accurate.

Table 4 CRAs and corresponding Wu-Lee LPMs for semi-infinite rod (N=4)

	Wu-Lee	This paper
p_1	**2.27**	2.05591649
p_2	**3.50**	3.19687897
p_3	**4.22**	3.50217395
p_4	**2.50**	2.16775170
q_1	**2.17**	2.01459812
q_2	**3.30**	2.83729241
q_3	**2.49**	2.16158416
q_4	**1.88**	1.38134744
s_j^S	−0.1148 ± 0.9601 i	−0.129403 ± 0.975620 i
	−0.5474 ± 0.5189 i	−0.653016 ± 0.566557 i
s_j^F	−0.6840	−0.820320
	−0.0234 ± 0.9935 i	−0.028225 ± 0.996635 i
	−0.2995 ± 0.8352 i	−0.346266 ± 0.876273 i
$\bar{k}_1$	**2.31**	2.64582619
$\bar{c}_1$	**3.38**	3.22535900
$\bar{k}_2$	**5.22**	4.62580931
$\bar{c}_2$	**4.09**	4.17164291
$\bar{k}_3$	**−8.21**	−8.13011005
$\bar{c}_3$	**−10.6**	−9.06894195
$\bar{k}_4$	**2.67**	2.46385625
$\bar{c}_4$	**2.17**	2.22102147
$\bar{k}_5$	**−2.76**	−2.70311907
$\bar{c}_5$	**−4.30**	−3.37780335

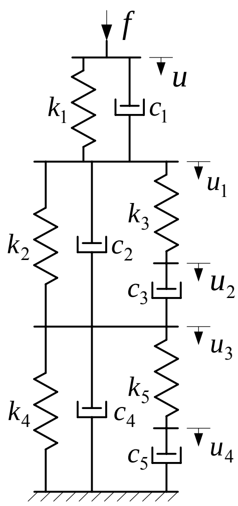


Table 5 Global dynamic-stiffness errors for semi-infinite rod (10^{-4})

N=3			N=4	
Wolf	Wu-Lee	This paper	Wu-Lee	This paper
9.956701	7.297607	5.894309	3.647233	2.964235

$0 \le \bar{\omega} \le 10, \Delta\bar{\omega} = 0.01$.

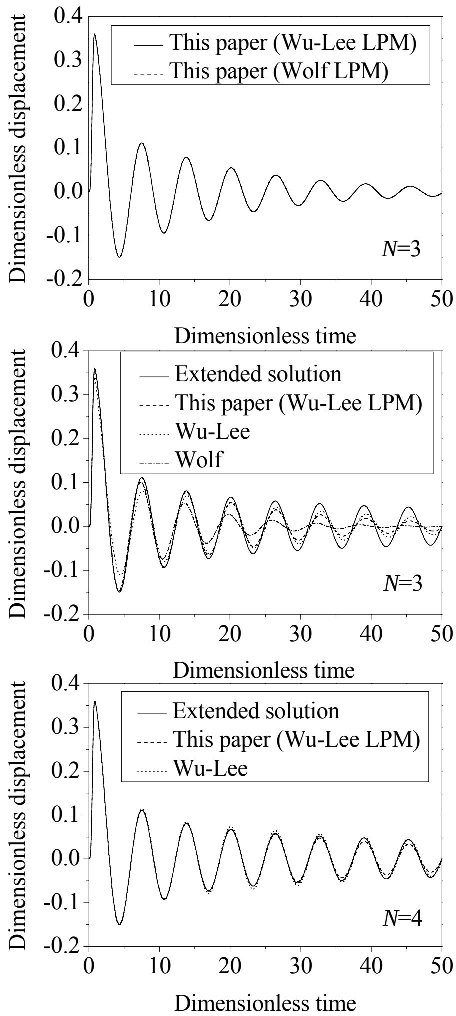


Figure 6 Time-domain soil-structure-interaction results for semi-infinite rod

The time-domain soil-structure-interaction analysis is performed by truncating the semi-infinite rod. The truncated infinite part is modeled by LPM and the residue finite part does by finite element method. To test stringently, only two finite elements with dimensionless element size $\Delta\bar{x} = 0.01$ are used in finite part. The extended solution is used as the reference one, which is obtained via taking the finite part large enough to prevent any reflection from truncated boundary to rod end before dimensionless time 50. The displacement results on rod end are shown in Figure 6. It is clear from the first figure of N=3 that for a same CRA obtained in this paper the result of Wu-Lee LPM is identical with that of Wolf LPM, which indicates that the accuracy and stability of CRA determine those of its resulting LPMs. In the second figure of N=3, all three results fit the extended solution with large errors. The accuracy is improved considerably as increasing N to 4.

5. CONCLUSIONS

The stability and identification of CRA of foundation frequency response realized as LPMs are studied in this paper. Some conclusions are summarized as follows:

(1) The accuracy and stability of CRA determine those of the resulting LPMs.

(2) A LPM corresponds to two interinvertible systems from a LTIC-system viewpoint: the dynamic-stiffness system with SCRA as frequency response and the dynamic-flexibility system with FCRA as frequency response. The stability of a LPM is determined by its SCRA or FCRA in terms of the case of its input and output. In soil-structure-interaction analysis, LPM is with force as input and displacement as output.

(3) To avoid any instable LPM, the proposed stability condition should be considered in parameter identification of CRA. A parameter identification method called as penalty genetic-simplex algorithm is proposed, which is more effective than the linearized identification method widely used.

(4) The accuracy of LPMs based on CRA is improved with the order N of CRA increasing. Moreover, if without any stability constraint, the stability of LPMs may vary with N for a foundation vibration problem. In particular, LPMs of the semi-infinite rod on elastic foundation may be stable for all N in nature.

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