HIGH-ORDER LUMPED-PARAMETER MODEL FOR FOUNDATION BASED ON CONTINUED FRACTION

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ABSTRACT: A new high-order lumped-parameter model (LPM) for foundation vibrations is proposed in this paper. Instead of partial fraction expansion used widely, the continued-fraction expansion is applied as a mathematical tool to construct the new LPM, so that its parameters can be obtained by only real-number operations and its configuration is independent of problem analyzed. The new LPM has a condensed physical configuration with few parameters as possible and is easily extended to higher order. Moreover, due to the LPM with mass on each internal degree of freedom, the resulting structure-foundation-soil system can be solved by an explicit time-integration method. The effectiveness of new LPM is verified by analyzing the benchmark problem of semi-infinite rod on elastic foundation.

KEYWORDS: foundation vibration, lumped-parameter model, continued fraction, explicit time integration

1. INTRODUCTION

In time-domain foundation-vibration or general soil-structure-interaction analysis, the *lumped-parameter model* (LPM) [1] is an effective method to represent the foundation-soil system in time domain, which consists of one or several springs, dashpots, and masses with frequency-independent real but not necessarily positive parameters, and contains the additional internal degrees of freedom introduced. The existing LPMs can be divided into two types: the simple or semi-empirical LPMs [2-6] and the so-called high-order LPMs [1, 7-10]. This paper concentrates on the latter.

The high-order LPMs are constructed by a systematic procedure proposed by Wolf [7]. The procedure can be summarized as two successive steps: the foundation frequency response (dynamic stiffness or flexibility) is first approximated by a rational function in complex frequency with real parameters, and the rational function is then realized as various types of LPMs. Here, the rational function is called as *continuous-time rational approximation* (CRA), and the resulting LPMs are called as *high-order* due to the asymptotic exactness of CRA with its order increasing. In this systematic procedure, the CRA determined the stability and accuracy of the resulting LPMs, which has been studied in great detail in [11]. The differences of various LPMs obtained from same CRA mainly lie in the aspects of physical configuration, time-integration method, earthquake input and so on. This paper aims at the second step to develop a new high-order LPM.

The partial-fraction expansion of rational function is an effective and widely used mathematical tool to construct various high-order LPMs. Wolf [1, 7, 8] proposed a set of parallel-form *consistent* LPMs based on the partial-fraction expansion of CRA of dynamic stiffness. Wu and Lee [9] proposed a series-form *systematic* LPM based on the partial-fraction expansion of CRA of dynamic flexibility. The partial-fraction expansion requires the complex analysis in determining the poles and residues. Moreover, the number of its first- and second-order terms is dependent of problem analyzed, so that the configuration of resulting LPM is also problem-dependent.

Alternatively, the *continued-fraction expansion* [12] of rational function can be used to construct high-order LPMs instead of a partial-fraction expansion. Wu and Lee [10] proposed a set of *nested* LPMs based on the so-called nested division (that is actually a type of continued-fraction expansion) of CRA of dynamic stiffness. Based on continued-fraction expansion, the construction of LPM involves only operations of real numbers and its configuration is independent of problem analyzed.

Moreover, an important problem of designing a LPM is whether it should contain masses. Wu and Lee [9, 10] developed a set of LPMs without any mass. Wolf [1, 7, 8] developed several different discrete-element models, some of which do not contain any mass and the others contain mass on each internal degree of freedom. The advantage of a LPM without any mass is that no extra work is needed for determining the input motion or the driving loads in their applications to seismic analysis. In contrast, the advantage of a LPM with mass on each internal degree of freedom lies in that the dynamic equation of the resulting structure-foundation-soil system can be solved by an explicit time-integration method.

In this paper, a mainly series-form high-order LPM with mass on each internal degree of freedom is proposed based on the continued-fraction expansion of decomposed CRA of foundation dynamic stiffness.

2. CONTINUOUS-TIME RATIONAL APPROXIMATION OF DYNAMIC STIFFNESS

The dynamic stiffness of a massless rigid foundation is a complex-valued function dependent of the Fourier radian frequency ω. It is conventionally normalized with respect to its static stiffness S_0 and given as a function of the dimensionless frequency $\overline{\omega} = \omega d / c_S$ with a characteristic length d (of foundation) and the (shear) wave velocity c_S (of soil), as follows

$$S(\overline{\omega}) = S_0 \left[k(\overline{\omega}) + i\overline{\omega} c(\overline{\omega}) \right] \tag{2.1}$$

where $i = \sqrt{-1}$, $k(\overline{\omega})$ and $c(\overline{\omega})$ are the dimensionless spring and damping coefficients dependent of dimensionless frequency, respectively. The static stiffness leads to $k(0) = 1$. For many foundation-soil systems, the high-frequency limits $c_\infty = c(\infty)$ of their damping coefficients are known.

The CRA of foundation dynamic stiffness can be written as [11]

$$S(\overline{\omega}) \approx S_C(\overline{\omega}) = S_C(s) = S_0 \frac{1 + p_1 s + \ldots + p_{N+1} s^{N+1}}{1 + q_1 s + \ldots + q_N s^N} \tag{2.2}$$

where the order of the numerator polynomial $N+1$ is one more than that of the denominator N, s is the dimensionless complex frequency, $s = i\overline{\omega}$ here. If c_∞ is known, $p_{N+1} = c_\infty q_N$ in Eqn. 2.2 so that the CRA is exact (doubly asymptotic) at high- and low-frequency limits. The real parameters p_j and q_j are determined by fitting the rigorous dynamic stiffness values. An effective parameter identification method sees [11].

3. CONTINUED-FRACTION EXPANSION

It is inconvenient to perform the continued-fration expansion directly for Eqn. 2.2 due to the order of its numerator polynomial one more than that of its denominator. Therefore, Eqn. 2.2 is first decomposed into a sum of a linear polynomial and a product of complex frequency and a new rational function, in a way that the highest-order and constant terms in numerator are simultaneously removed and then the common factor s in the new numerator is extracted, as follows

$$\frac{S_C(s)}{S_0} = 1 + \frac{p_{N+1}}{q_N} s + \frac{P^{(1)}(s)}{Q^{(1)}(s)} s = 1 + \frac{p_{N+1}}{q_N} s + \frac{p_0^{(1)} + p_1^{(1)} s + \ldots + p_{N-1}^{(1)} s^{N-1}}{q_0^{(1)} + q_1^{(1)} s + \ldots + q_N^{(1)} s^N} s \tag{3.1}$$

where

$$q_j^{(1)} = q_j \quad \text{for} \quad j = 0,...,N \quad \text{and} \quad p_j^{(1)} = p_{j+1} - q_{j+1} - \frac{p_{N+1}}{q_N} q_j \quad \text{for} \quad j = 0,...,N-1 \tag{3.2}$$

In Eqn. 3.1, a superscript is added to the parameters of the new rational function to differ from the original one, and also denotes how many rounds of the division operation have been performed. Note that the constant and linear terms of the linear polynomial represent the dimensionless static stiffness and high-frequency limit of dimensionless dynamic stiffness, respectively, which ensures the doubly asymptotic feature of the resulting LPM. On the other hand, the product of complex frequency and new rational function accurately replicates the rigorous dimensionless dynamic stiffness values between the static and the high-frequency limits.

Now, the new rational function $P^{(1)}(s) / Q^{(1)}(s)$ can be expanded into continued frations. Performing the division for $Q^{(1)}(s)$ with respect to $P^{(1)}(s)$ in a way that the highest-order two terms of $Q^{(1)}(s)$ are simultaneously removed, a new generation of rational function $P^{(2)}(s) / Q^{(2)}(s)$ can be obtained. This new rational function holds exactly the same form as that before division, only with one term reduced in the highest order for both the numerator and denominator polynomials. Accordingly, the l-th time division from $P^{(l)}(s) / Q^{(l)}(s)$ to $P^{(l+1)}(s) / Q^{(l+1)}(s)$ can be generally written as

$$\frac{P^{(l)}(s)}{Q^{(l)}(s)} = \cfrac{1}{g_l + h_l s + \cfrac{p_0^{(l+1)} + p_1^{(l+1)} s + ... + p_{N-l-1}^{(l+1)} s^{N-l-1}}{q_0^{(l+1)} + q_1^{(l+1)} s + ... + q_{N-l}^{(l+1)} s^{N-l}}} = \cfrac{1}{g_l + h_l s + \cfrac{P^{(l+1)}(s)}{Q^{(l+1)}(s)}} \quad \text{for} \quad l = 1,...,N \tag{3.3}$$

where

$$h_l = \frac{q_{N-l+1}^{(l)}}{p_{N-l}^{(l)}} \quad \text{and} \quad g_l = \frac{q_{N-l}^{(l)}}{p_{N-l}^{(l)}} - \frac{p_{N-l-1}^{(l)} q_{N-l+1}^{(l)}}{p_{N-l}^{(l)2}} \tag{3.4}$$

$$q_{N-l}^{(l+1)} = p_{N-l}^{(l)}, \quad q_j^{(l+1)} = p_j^{(l)}, \quad p_j^{(l+1)} = q_j^{(l)} - \frac{q_{N-l+1}^{(l)}}{p_{N-l}^{(l)}} p_{j-1}^{(l)} - \left(\frac{q_{N-l}^{(l)}}{p_{N-l}^{(l)}} - \frac{p_{N-l-1}^{(l)} q_{N-l+1}^{(l)}}{p_{N-l}^{(l)2}} \right) p_j^{(l)} \quad \text{for } j = 0,...,N-l-1 \tag{3.5}$$

with the definition of $p_{-1}^{(l)} = 0$. Applying Eqn. 3.3~3.5 recursively, the continued-fraction expansion of $P^{(1)}(s) / Q^{(1)}(s)$ can be obtained as

$$\frac{P^{(1)}(s)}{Q^{(1)}(s)} = \cfrac{1}{g_1 + h_1 s + \cfrac{1}{g_2 + h_2 s + \cdots + \cfrac{1}{g_N + h_N s}}} \tag{3.6}$$

4. HIGH-ORDER LUMPED-PARAMETER MODEL

To realize Eqn. 3.1 with Eqn. 3.6, a new high-order LPM shown in Figure 1 is designed, which consists of an end-fixed parallel spring-dashpot model in parallel with an end-free in-series system including N in-series dashpot-mass models, and introduces N internal nodes each of which has an internal degree of freedom. Note that this LPM is mainly in-series and with mass on each internal node. Its time-domain motion equations are

$$-f + ku + (c + c_1)\dot{u} = c_1 \dot{u}_1 \quad \text{and} \quad m_l \ddot{u}_l + (c_l + c_{l+1})\dot{u}_l = c_l \dot{u}_{l-1} + c_{l+1} \dot{u}_{l+1} \quad \text{for} \quad l = 1,...,N \tag{4.1}$$

with $u_0=u$, $u_{n+1}=c_{n+1}=0$, where f is a force exerted on foundation, u is the corresponding foundation displacement, u_l are the auxiliary internal degrees of freedom introduced, the dot on variable denotes the derivative to time, k, c (c_l), and m_l are the undetermined spring, dashpot, and mass parameters, respectively.
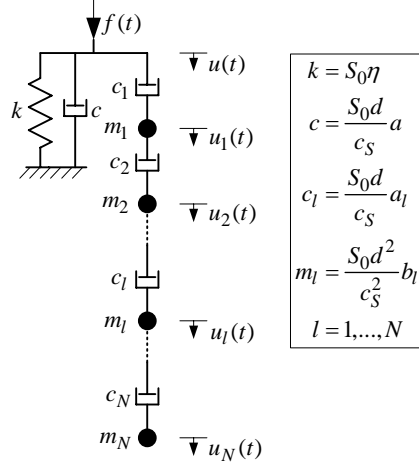

Figure 1 The new high-order lumped-parameter model

Performing Fourier transform to Eqn. 4.1 and defining the dimensionless spring parameter η, the dimensionless dashpot parameters a and a_l, and the dimensionless mass parameters b_l, all of which satisfy

$$k = S_0\eta \ , \quad c = \frac{S_0 d}{c_S}a \ , \quad c_l = \frac{S_0 d}{c_S}a_l \quad \text{and} \quad m_l = \frac{S_0 d^2}{c_S^2}b_l \quad \text{for} \quad l = 1,...,N \tag{4.2}$$

the normalized dynamic-stiffness equation of the new LPM is obtained as

$$\frac{S_L(s)}{S_0} = \eta + (a + a_1)s + B_1 s \quad \text{and} \quad B_l = \frac{-a_l^2}{a_l + a_{l+1} + b_l s + B_{l+1}} \quad \text{for} \quad l = 1,...,N \tag{4.3}$$

with $B_{N+1} = a_{N+1} = 0$, where $B_l = -a_l u_l(\omega)/u_{l-1}(\omega)$, B_1 can be further written as the following continued-fraction expansion

$$B_1 = \cfrac{-a_1^2}{a_1 + a_2 + b_1 s + \cfrac{-a_2^2}{a_2 + a_3 + b_2 s + \cdots + \cfrac{-a_N^2}{a_N + a_{N+1} + b_N s}}} \tag{4.4}$$

Now, the parameters of LPM can be obtained by comparing Eqn. 4.3 and Eqn. 4.4 with Eqn. 3.1 and Eqn. 3.6. Comparing Eqn. 4.3 with Eqn. 3.1, we have

$$\eta = 1 \quad \text{and} \quad a + a_1 = \frac{p_{N+1}}{q_N} \tag{4.5}$$

Comparing Eqn. 4.4 with Eqn. 3.6, the following relation of parameters can be obtained as

$$\frac{1}{a_1}\left(1 + \frac{a_2}{a_1}\right) = -g_1 \ , \quad \frac{b_1}{a_1^2} = -h_1 \ , \quad \left(1 + \frac{a_l}{a_{l+1}}\right)\left(1 + \frac{a_{l+2}}{a_{l+1}}\right) = -g_l g_{l+1} \quad \text{and} \quad \frac{b_l b_{l+1}}{a_{l+1}^2} = -h_l h_{l+1} \quad \text{for} \, l = 1,...,N-1 \tag{4.6}$$

with $a_{N+1} = 0$. A two-step recursive algorithm shown in Box 1 can be obtained from Eqn. 4.6 to compute the dimensionless parameters a_l and b_l in Eqn. 4.4 from the dimensionless parameters g_l and h_l in Eqn. 3.6. After a_l and b_l are determined, the dimensionless parameters η and a can be obtained from Eqn. 4.5. Finally, the physical parameters k, c, c_l, and m_l of LPM can be obtained from the dimensionless parameters η, a, a_l and b_l, respectively, via Eqn. 4.2.

Box 1. Algorithm for computing a_l and b_l from g_l and h_l

- First compute a_{l+1}/a_l from l=N to 1:

$$\frac{a_{N+1}}{a_N} = 0 \quad \text{and} \quad \frac{a_{l+1}}{a_l} = -\left[1 + g_l g_{l+1}\left(1 + \frac{a_{l+2}}{a_{l+1}}\right)^{-1}\right]^{-1} \quad \text{for } l = N-1,\dots,1$$

- Then compute a_l and b_l from l=1 to N:

$$a_1 = -g_1^{-1}\left(1 + \frac{a_2}{a_1}\right), \quad b_1 = -h_1 a_1^2, \quad a_l = \frac{a_l}{a_{l-1}} a_{l-1} \quad \text{and} \quad b_l = -h_{l-1} h_l \frac{a_l^2}{b_{l-1}} \quad \text{for } l = 2,\dots,N$$

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5. NUMERICAL TESTS

The benchmark problem of a semi-infinite rod on elastic foundation used very often in the literature is analyzed in this section. Due to with dispersive waves exhibiting a cutoff frequency below which the radiation damping vanishes, this system is a very stringent test for a LPM. The other reason why this problem is proper is that as mentioned in [11] the semi-infinite rod problem may have the stable CRAs and corresponding LPMs for all N in nature. As shown in Figure 2, the elastic modulus, cross-sectional area, and mass density of rod are E, A, and ρ, respectively. Foundation spring stiffness per unit length is k_g. f and u denote force and displacement on rod end, respectively. If the characteristic length $d = \sqrt{EA/k_g}$, the wave velocity $c_S = \sqrt{E/\rho}$, and the static stiffness $S_0 = \sqrt{EAk_g}$ are further defined, the exact dynamic stiffness of system can be obtained as

$$S(\overline{\omega}) = \begin{cases} S_0\sqrt{1-\overline{\omega}^2} & \text{for } \overline{\omega} \leq 1 \\ S_0 i\overline{\omega}\sqrt{1-\dfrac{1}{\overline{\omega}^2}} & \text{for } \overline{\omega} \geq 1 \end{cases} \tag{5.1}$$

It is clear from Eqn. 5.1 that the cutoff frequency is $\overline{\omega} = 1$ and the high-frequency limit of dimensionless damping coefficient is $c_\infty = 1$.
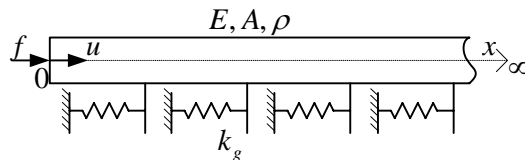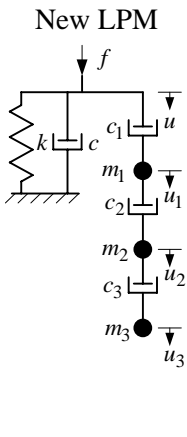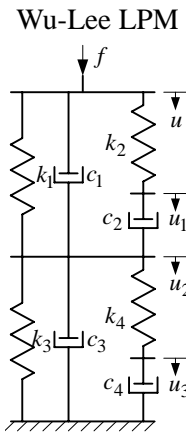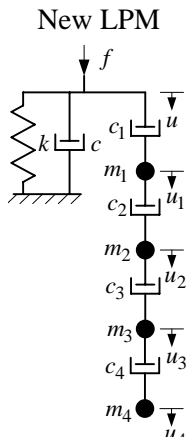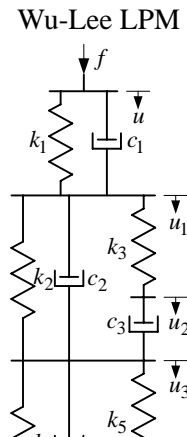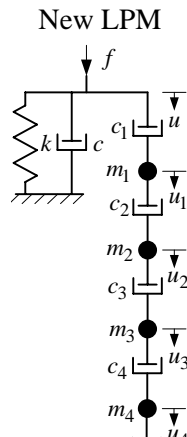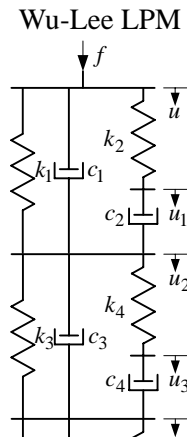

Figure 2 Semi-infinite rod on elastic foundation

The CRA of foundation frequency response (dynamic stiffness or flexibility) as the first step of a systematic procedure of constructing LPMs mentioned in Introduction has been studied in great detail in [11]. Correspondingly, this paper concentrates on the second step of this procedure to propose new LPM. Therefore, the CRAs obtained in [11] are here used and listed in Table 1 for the cases of N=3, 4 and 5, respectively. These CRAs are all stable because all real parts of their poles are negative. They are also accurate for fitting exact foundation frequency responses, and the accuracy is considerably improved as N increasing. The corresponding

new LPMs are show in Table 1 for the cases of $N=3$, 4 and 5, respectively. For comparison, the Wu-Lee LPMs [9] realized from these same CRAs are also given in Table 1, which have been actually presented in [11].

Table 1 CRAs and corresponding LPMs

$N=3$	$N=4$	$N=5$
$p_1 = 1.73561311$	$p_1 = 2.05591649$	$p_1 = 2.21546530$
$p_2 = 2.57111362$	$p_2 = 3.19687897$	$p_2 = 4.12007810$
$p_3 = 1.80599507$	$p_3 = 3.50217395$	$p_3 = 4.66897021$
$q_1 = 1.82740854$	$p_4 = 2.16775170$	$p_4 = 4.54545837$
$q_2 = 1.81872359$	$q_1 = 2.01459812$	$p_5 = 2.43885475$
$q_3 = 1.38630902$	$q_2 = 2.83729241$	$q_1 = 2.24547891$
	$q_3 = 2.16158416$	$q_2 = 3.51659995$
	$q_4 = 1.38134744$	$q_3 = 3.82009771$
		$q_4 = 2.44255841$
		$q_5 = 1.41979314$

New LPM	Wu-Lee LPM	New LPM	Wu-Lee LPM	New LPM	Wu-Lee LPM
$\eta = 1$	$\bar{k}_1 = 3.18811174$	$\eta = 1$	$\bar{k}_1 = 2.64582619$	$\eta = 1$	$\bar{k}_1 = 6.82840008$
$a = -0.091795421$	$\bar{c}_1 = 2.78858518$	$a = 0.04131837$	$\bar{c}_1 = 3.22535900$	$a = -0.030013607$	$\bar{c}_1 = 6.45744183$
$a_1 = 1.09179542$	$\bar{k}_2 = -5.26271484$	$a_1 = 0.95868163$	$\bar{k}_2 = 4.62580931$	$a_1 = 1.03001361$	$\bar{k}_2 = -12.78027615$
$b_1 = 1.18117220$	$\bar{c}_2 = -6.11903670$	$b_1 = 0.92319240$	$\bar{c}_2 = 4.17164291$	$b_1 = 1.05816771$	$\bar{c}_2 = -13.53015579$
$a_2 = -0.44980557$	$\bar{k}_3 = 1.45701505$	$a_2 = -0.51880924$	$\bar{k}_3 = -8.13011005$	$a_2 = -0.48607457$	$\bar{k}_3 = 3.76775448$
$b_2 = -1.42561523$	$\bar{c}_3 = 1.55910113$	$b_2 = -0.87833541$	$\bar{c}_3 = -9.06894195$	$b_2 = -1.13664526$	$\bar{c}_3 = 3.21464622$
$a_3 = 2.88044126$	$\bar{k}_4 = -0.83082499$	$a_3 = 0.05997932$	$\bar{k}_4 = 2.46385625$	$a_3 = 1.19796464$	$\bar{k}_4 = -5.27258433$
$b_3 = 1.16458080$	$\bar{c}_4 = -1.05836569$	$b_3 = -0.01503575$	$\bar{c}_4 = 2.22102147$	$b_3 = 0.75483365$	$\bar{c}_4 = -6.45151021$
		$a_4 = -0.07293783$	$\bar{k}_5 = -2.70311907$	$a_4 = 0.047065694$	$\bar{k}_5 = 1.70026754$
		$b_4 = 0.24652541$	$\bar{c}_5 = -3.37780335$	$b_4 = 0.15309683$	$\bar{c}_5 = 1.87243585$
				$a_5 = -0.54995294$	$\bar{k}_6 = -0.69843329$
				$b_5 = -0.15857986$	$\bar{c}_6 = -0.86152478$

To investigate the performance of the new LPM applied to the time-domain analysis, the time-domain soil-structure-interaction analysis is performed by truncating the semi-infinite rod. The truncated infinite part is modeled by LPM and the residue finite part does by finite element method. The dimensionless physical models with load shown in Figure 3 of [11] exerted on the end of rod are used, where the dimensionless time is

$\bar{t} = tc_S/d$. To test stringently, only two finite elements with dimensionless element size $\Delta\bar{x} = 0.01$ are used in finite part. The implicit Newmark time-integration method [13] is used for structure-foundation-soil system including the new LPMs or Wu-Lee LPMs as does in [11]. Moreover, the explicit time-integration method [14] that can treat with non-diagonal damping matrix is used for the system including the new LPMs. The dimensionless time-step size is $\Delta\bar{t} = 0.005$. The extended solution is used as the reference one, which is obtained via taking the finite part large enough to prevent any reflection from truncated boundary to rod end before dimensionless time 50. Since the errors resulting from finite element discretization are also present in extended solution, the difference between LPM solution and extended solution is caused only by LPM. The time-domain results obtained by implicit Newmark method are show in Figure 3. It can be seen that in each case of N=3, 4 or 5 the result of new LPM is identical with that of Wu-Lee LPM, which indicates as a conclusion in [11] that the stability and accuracy of LPMs realized from same CRA are identical and determined by those of the CRA. Moreover, comparing with the extended solution, the accuracy of LPM solution is improved considerably as N increasing, and a very well fitting is obtained in the case of N=5. The time-domain results obtained by explicit method are shown in Figure 4. It can be seen that all explicit solutions are correspondingly identical with the implicit solutions, which indicates the advantage of the new LPM that can be solved by explicit time integration.
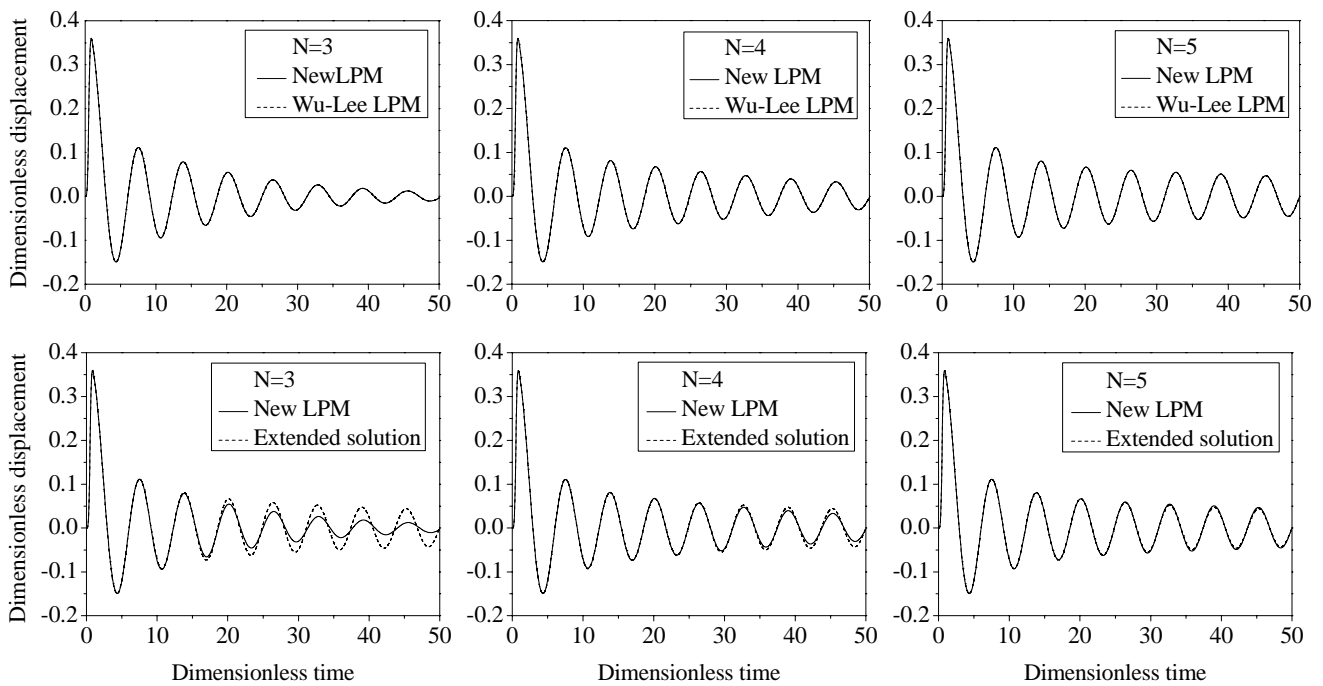

Figure 3 Comparison of new LPMs with Wu-Lee LPMs using implicit Newmark time integration
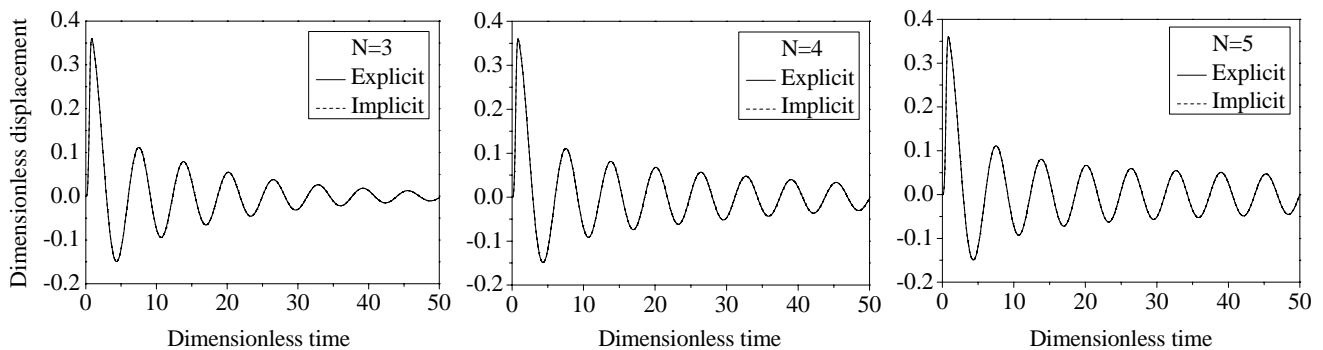

Figure 4 Results of new LPMs using explicit time integration

6. CONCLUSIONS

The systematic procedure based on *continuous-time rational approximation* (CRA) [11] is applied in this paper to develop new *lumped-parameter model* (LPM) for representing the dynamic foundation-soil system. Such LPM is here called *high-order* due to the asymptotic exactness of CRA as its order *N* increasing. A new high-order LPM is proposed in this paper based on the *continued-fraction expansion* of decomposed CRA of foundation dynamic stiffness instead of partial-fraction expansion widely used. This new LPM is mainly in-series and designed with mass on each internal node with an auxiliary internal degree of freedom. Comparing with the existing high-order LPMs, the advantages of the new LPM can be summarized as follows:

(1) Only real-number operations in construction result in the problem-independent configuration due to using the continued-fraction expansion.
(2) The condensed physical configuration with fewer parameters.
(3) The easy extension to higher order based on a parameter identification method for CRA proposed in [11].
(4) The explicit time integration can be applied due to with mass on each internal degree of freedom.

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