A General Approach for Analysis of Actuator Delay Compensation Methods for Real-Time Testing

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ABSTRACT :

Real-time testing is a useful technique to evaluate the performance of structural systems subjected to seismic loading. Servo-hydraulic actuators can introduce an inevitable delay when applying command displacements to a structure during a real-time test due to their inherent dynamics. Various compensation methods have been developed and applied in real-time testing to minimize the effect of actuator delay. This paper proposes a general approach to analyze actuator delay compensation methods using an equivalent discrete transfer function. Discrete control theory is introduced and used to formulate the discrete transfer function in the frequency domain and the difference equation in the time domain for an actuator delay compensation method. The discrete transfer function is used to conduct frequency response analysis to provide insight into the expected performance of the compensation method by graphically illustrating the characteristics of the compensation method as a function of frequency of command displacement and the parameters for the compensator. The difference equation gives the form of the mathematical extrapolation in the time domain for the compensation method. Three popular compensation methods are selected to illustrate the proposed approach. Experiments with predefined sine sweep displacement are conducted to experimentally validate the analysis of the selected compensation methods using the proposed approach.

KEYWORDS: real-time testing, actuator delay, compensation, discrete transfer function, frequency response, difference equation, phase, magnitude

1. Introduction

Structural experiments are important for earthquake engineering research when the structural system or its components are difficult to model. Various methods have been developed for structural experiments, these include the displacement-based pseudodynamic and hybrid testing methods which provide viable techniques to evaluate the performance of structural system subject to dynamic loading (Mahin et al. 1985, Shing et al. 1996). Recently, several devices that are sensitive to loading-rate have been developed for seismic hazard mitigation, including viscous, viscoelastic and elastomeric dampers (Soong and Spencer 2002). An evaluation of the performance of structures with these rate-dependent devices requires the structural experiments be performed in real-time. Real-time pseudodynamic and hybrid testing therefore present themselves as economic and viable techniques to shake table testing and have been investigated by numerous researchers (Jung and Shing 2006, Chen et al. 2008).

In real-time testing, the equations of motion are numerically solved by an integrated algorithm. Servo hydraulic actuators are often utilized in real-time tests to apply the command displacement(s) to the experimental structure(s). Due to the inherent actuator servo-hydraulic dynamics, the actuator has an inevitable delay in response to the command displacement. This delay is usually referred to as actuator delay. The effect of actuator delay on real-time testing has attracted the attention of many researchers (Horiuchi et al. 1999, Wallace el. 2005, Chen and Ricles 2007). Their studies show that actuator delay is equivalent to a negative damping that can destabilize a real-time test if not compensated properly. Various compensation methods have been proposed to minimize the effect of actuator delay for real-time testing. Horiuchi *et al.* (1999, 2001) proposed two compensation schemes based on polynomial extrapolation and a linear acceleration assumption, respectively. Other compensation methods originating from control engineering practice have also been investigated such as derivative feedforward (Jung and Shing 2006). Chen (2007) proposed a simplified discrete transfer function model for the servo hydraulic actuator and applied the inverse of the model for actuator delay compensation for real-time testing.

The analysis of compensation methods is necessary in order to gain an understanding and insight into the behavior of the methods. This will enable an appropriate actuator delay compensation method to be selected and the values for the parameters of the compensation method to be assigned in order to achieve a successful real-time test. A general approach is proposed in this paper to analyze different actuator delay compensation methods. Discrete control theory is introduced to formulate the equivalent discrete transfer function for a selected compensation method. Frequency response analysis is conducted using the equivalent discrete transfer function to provide insight into the performance of the compensation method by graphically illustrating the characteristics of the compensation method as a function of frequency of the command displacement and the parameter values for the compensator. It is also shown that the difference equation corresponding to the equivalent discrete transfer function for the compensation method gives the expression for extrapolation in the time domain. Three popular actuator delay compensation methods are selected to illustrate the proposed approach to demonstrate the effectiveness of the proposed approach.

2. DISCRETE TRANSFER FUNCTION AND FREQUENCY RESPONSE

The discrete transfer function of a discrete system is defined as the ratio of the discrete z-transforms of the input and output (Ogata 1995). For a linear discrete-time system with input *F* and output *x*, the discrete transfer function *G*(*z*) can be written in the following general form:

$$G(z)=\frac{X(z)}{F(z)}=\frac{n_n\cdot z^n+\cdots+n_1\cdot z+n_0}{d_m\cdot z^m+\cdots+d_1\cdot z+d_0} \tag{2.1}$$

In Eqn. 2.1, $n_n,\cdots n_0$ and $d_m,\cdots d_0$ are coefficients of the linear difference equation; *z* is the complex variable in the discrete *z*-domain; and *X*(*z*) and *F*(*z*) are the discrete *z*-transforms of system input *x* and system output *F*, respectively; and *m* and *n* are integer numbers.

The frequency response of a dynamic system is usually represented by the magnitude and phase properties, where the magnitude is defined as the amplitude change of the output compared with the input and the phase is the phase shift between the two signals. For a discrete transfer function *G*(*z*) in Eqn. 2.1, the frequency response function can be obtained by substituting $z=e^{j\omega T}$, whereby the magnitude *M* and phase θ can be written as

$$M=M(\omega)=\left|G(e^{j\omega T})\right| \tag{2.2a}$$

$$\theta=\theta(\omega)=\angle G(e^{j\omega T}) \tag{2.2b}$$

In Eqns. 2.2a and 2.2b, *j* is the imaginary unit and defined as $\sqrt{-1}$; ω is the circular frequency of the input signal; and *T* is the sampling time of *G*(*z*). *M* and θ can be readily calculated using Matlab (2008).

In real-time testing when the actuator is subject to a sinusoidal command displacement $x^c(t)=A\sin(\omega t)$, where A and ω are the amplitude and circular frequency of the harmonic input, respectively, the predicted displacement output $x^p(t)$ from a compensator method can be written as $x^p(t)=A'\sin(\omega t+\phi')$, where A' and ϕ' are the amplitude and phase shift of $x^p(t)$. The actuator measured displacement $x^m(t)$ can be written as $x^m(t)=A''\sin(\omega t+\phi'')$, where A'' and ϕ'' are the amplitude and phase shift of $x^m(t)$. By definition, A'/A and ϕ' are frequency response properties of the compensation method, while A''/A' and

$\phi'' - \phi'$ are frequency response properties of the actuator dynamics. To achieve precise actuator control in real-time testing, the measured displacement should be the same as the command displacement, which leads to

$$\frac{A''}{A} = \frac{A'}{A} \cdot \frac{A''}{A'} = 1 \quad (2.3a)$$

$$\phi'' = \phi' + (\phi'' - \phi') = 0 \quad (2.3b)$$

3. EQUIVALENT DISCRETE TRANSFER FUNCTIONS FOR SELECTED COMPENSATION METHODS

Three different compensation methods are selected to illustrate the proposed approach, including the linear acceleration extrapolation method (Horiuchi et al. 2001), the derivative feedforward compensation method (Jung and Shing 2007) and the inverse compensation method (Chen 2007).

3.1 Linear Acceleration Extrapolation

Horiuchi *et al.* (2001) proposed an actuator delay compensation method based on a linear acceleration assumption. Assuming that structural acceleration $\ddot{x}_{i-1}^c$ and $\ddot{x}_i^c$ at the (i-1)th and ith time step are known, the linear acceleration assumption gives

$$\ddot{x}_{i+1}^p = \ddot{x}_i^c + \frac{\Delta t + \tau}{\Delta t} \cdot (\ddot{x}_i^c - \ddot{x}_{i-1}^c) = (2 + \frac{\tau}{\Delta t}) \cdot \ddot{x}_i^c - \frac{\tau}{\Delta t} \cdot \ddot{x}_{i-1}^c \quad (3.1)$$

where $\ddot{x}_{i+1}^p$ is the predicted acceleration for the (i+1)th time step; and Δt and τ are the integration time step and the actuator delay, respectively. The predicted displacement x_{i+1}^p can be calculated using the predicted acceleration with the Newmark family of integration algorithms (Newmark 1959) as

$$x_{i+1}^p = x_i^c + (\Delta t + \tau) \cdot \dot{x}_i^c + (0.5 - \beta) \cdot (\Delta t + \tau)^2 \cdot \ddot{x}_i^c + \beta \cdot (\Delta t + \tau)^2 \cdot \ddot{x}_{i+1}^p \quad (3.2)$$

where γ and β is integration parameter (where γ appears below in Table 3.1); and x_i^c, $\dot{x}_i^c$ and $\ddot{x}_i^c$ are the command displacement, velocity and acceleration response, respectively, for the ith time step calculated by the integration algorithm. Applying the discrete z-transforms to Eqns. 3.1 and 3.2 leads to an equivalent discrete transfer function $G_c(z) = X_d^p(z) / X_d^c(z)$ for the linear acceleration extrapolation written in the general form of Eqn. 2.1, where $X_d^c(z)$ and $X_d^p(z)$ are discrete z-transforms of x_{i+1}^c and x_{i+1}^p, respectively. The coefficients of the equivalent discrete transfer function $G_c(z)$ are tabulated in Table 3.1.

Table 3.1 Coefficients of equivalent discrete transfer function $G_c(z)$ for the linear acceleration extrapolation method using the Newmark family of integration algorithms

Denominator		Numerator	
d_4	$2\beta\Delta t^3$	n_4	*0*
d_3	$(1+2\gamma-4\beta)\Delta t^3$	n_3	$(1+2\gamma+4\beta)\Delta t^3 + (1+6\beta)\tau^2\Delta t + 2(1+\gamma+3\beta)\tau\Delta t^2 + 2\beta\tau^3$
d_2	$(1-2\gamma+2\beta)\Delta t^3$	n_2	$(1-2\gamma-10\beta)\Delta t^3 - 2(1+9\beta)\tau^2\Delta t - 2(1+2\gamma+9\beta)\tau\Delta t^2 - 6\beta\tau^3$
d_1	*0*	n_1	$8\beta\Delta t^3 + 2(\gamma+9\beta)\tau\Delta t^2 + (1+18\beta)\tau^2\Delta t + 6\beta\tau^3$
d_0	*0*	n_0	$-2\beta\Delta t^3 - 6\beta\tau\Delta t^2 - 6\beta\tau^2\Delta t - 2\beta\tau^3$

Applying the inverse discrete-z transform to the equivalent discrete transfer function for the linear acceleration extrapolation method leads to

$$x_{i+1}^{p} = \frac{n_4}{d_4} \cdot x_{i+1}^{c} + \frac{n_3}{d_4} \cdot x_{i}^{c} + \frac{n_2}{d_4} \cdot x_{i-1}^{c} + \frac{n_1}{d_4} \cdot x_{i-2}^{c} - \frac{d_3}{d_4} \cdot x_{i}^{p} - \frac{d_2}{d_4} \cdot x_{i-1}^{p} - \frac{d_1}{d_4} \cdot x_{i-2}^{p} \tag{3.3}$$

Eqn. 3.3 is the difference equation for the linear acceleration extrapolation compensation method, and indicates that the method can be interpreted as an extrapolation in the time domain using previous command displacements and previous predicted displacements.

3.2 Derivative Feedforward Compensation

Jung and Shing (2006) used derivative feedforward compensation in real-time testing. The block diagram for real-time testing using the derivative feedforward compensation method is given in Figure 1, where the discrete transfer function $G_{ff}(z)$ calculates the derivative of the command displacements from the ramp generator and adds them to the displacement signal for the servo-hydraulic actuator. A ramp generator is used in Figure 1 to smoothly apply the command displacement to the experimental structure. If the discrete transfer function $G_{PID}(z)$ represents the digital PID servo-controller and $G_a(z)$ represents the servo-hydraulic dynamics, the discrete transfer function $G_{hc}(z)$ that relates the measured displacement to command displacement can be written as

$$G_{hc}(z) = \frac{[G_{PID}(z) + G_{ff}(z)] \cdot G_a(z)}{1 + G_{PID}(z) \cdot G_a(z)} = \frac{G_{PID}(z) + G_{ff}(z)}{G_{PID}(z)} \cdot \frac{G_{PID}(z) \cdot G_a(z)}{1 + G_{PID}(z) \cdot G_a(z)} \tag{3.4}$$

Similarly the transfer function for the servo-hydraulic control loop without derivative feedforward compensation can be written as $G_{hc}(z) = G_{PID}(z) \cdot G_a(z) / (1 + G_{PID}(z) \cdot G_a(z))$. The equivalent discrete transfer function for derivative feedforward compensation can therefore be written as

$$G_c(z) = \frac{G_{PID}(z) + G_{ff}(z)}{G_{PID}(z)} \tag{3.5}$$

If $G_{PID}(z)$ for the servo controller is written as $G_{PID}(z) = K_p(1 + \delta t / T_i / (1 - z^{-1}))$, where K_p and T_i are a proportional gain and integrative time constant, and $G_{ff}(z)$ is written as $G_{ff}(z) = K_{ff}(z-1)/\delta t / z$, where K_{ff} is the derivative feedforward gain, from Eqn. 3.5 the equivalent discrete transfer function $G_c(z)$ for derivative feedforward compensation can be written as

$$G_c(z) = \frac{(K_p \cdot \delta t + K_p \cdot \delta t \cdot \delta t / T_i + K_{ff}) \cdot z^2 - (2K_{ff} + K_p \cdot \delta t) \cdot z + K_{ff}}{K_p \cdot \delta t \cdot (1 + \delta t / T_i) \cdot z^2 - K_p \cdot \delta t \cdot z} \tag{3.6}$$

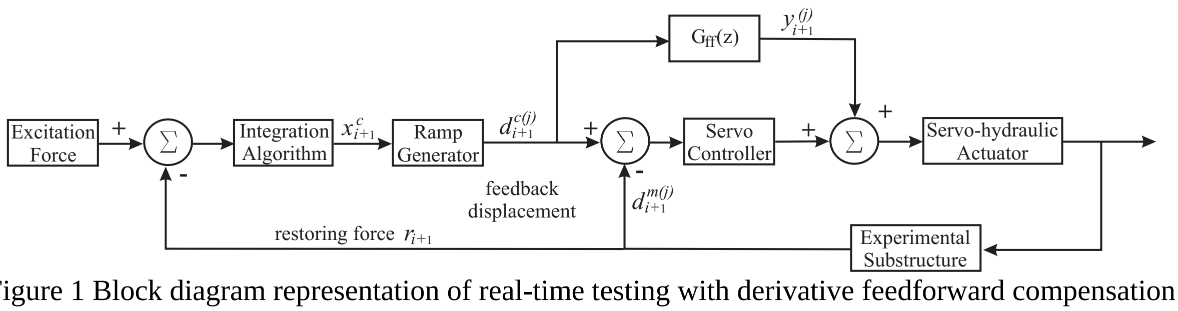

Figure 1 Block diagram representation of real-time testing with derivative feedforward compensation

Applying the inverse discrete z-transform to Eqn. 3.6 leads to

$$d_{i+1}^{p(j+1)} = \frac{(K_p\delta t + K_p\delta t\delta t/T_i + K_{ff})d_{i+1}^{c(j+1)} - (2K_{ff} + K_p\delta t)d_{i+1}^{c(j)} + K_{ff}d_{i+1}^{c(j-1)} + K_p\delta t d_{i+1}^{p(j)}}{K_p \cdot \delta t \cdot (1+\delta t/T_i)} \quad (3.7)$$

Eqn. 3.7 again indicates that the derivative feedforward compensation can be interpreted as an extrapolation in the time domain using the command displacements and the predicted displacement.

3.3 Inverse Compensation Method

Under the interpolated command displacement $d_{i+1}^{c(j+1)}$, the actual response of the actuator can be idealized as a linear response shown in Figure 2. The duration for the actuator to achieve the command displacement is assumed to be t_d and designated as $\alpha\,\delta t$. The delayed actuator displacement of $d_{i+1}^{m(j+1)}$ can be expressed as

$$d_{i+1}^{m(j+1)} = d_{i+1}^{m(j)} + \frac{1}{\alpha}\cdot(d_{i+1}^{c(j+1)} - d_{i+1}^{m(j)}) \quad (3.8)$$

Applying the discrete z-transform to Eqn. 3.8 leads to

$$G_d(z) = \frac{X_d^m(z)}{X_d^c(z)} = \frac{z}{\alpha\cdot z - (\alpha - 1)} \quad (3.9)$$

where $X_d^m(z)$ and $X_d^c(z)$ are discrete z-transforms of $d_{i+1}^{m(j+1)}$ and $d_{i+1}^{c(j+1)}$, respectively. Chen (2007) proposed to use the inverse of Eqn. 3.9 for actuator delay compensation in real-time testing, whereby the equivalent discrete transfer function for the resulting inverse compensation method can be written as

$$G_c(z) = \frac{X_d^p(z)}{X_d^c(z)} = \frac{\alpha\cdot z - (\alpha-1)}{z} \quad (3.10)$$

Applying inverse discrete z-transform to Eqn. 3.10, the extrapolation in the time domain can be expressed as

$$d_{i+1}^{p(j+1)} = \alpha\cdot d_{i+1}^{c(j+1)} - (\alpha-1)\cdot d_{i+1}^{c(j)} \quad (3.11)$$

Eqn. 3.11 indicates that the inverse compensation can be interpreted as an extrapolation in the time domain using the previous command displacements.

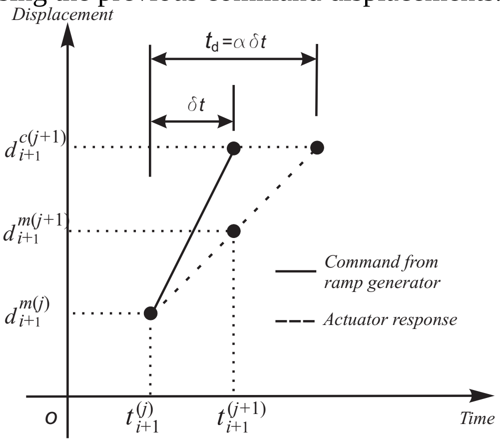

Figure 2 Conceptual actuator response

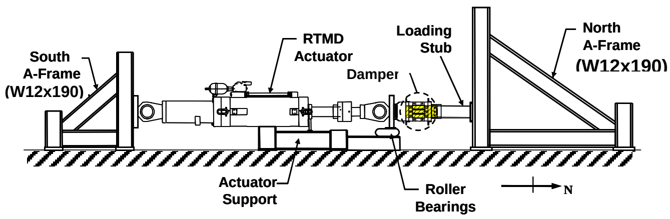

Figure 3 Frequency response of servo-hydraulic system

4. FREQUENCY RESPONSE ANALYSIS OF SELECTED COMPENSATION METHODS

A real-time hybrid testing bed at the NEES Lehigh RTMD Equipment Site (Chen et al. 2008) is used in this paper to experimentally validate the proposed approach. Figure 3 shows the experimental setup. Experiments with predefined sinusoidal displacements were conducted to identify the dynamics of the actuator attached to the elastomeric damper. The experimental results are converted to the frequency response properties in magnitude (dB) and phase (degrees), and are presented in Figure 4. Phase error can be observed in Figure 4(b), where for the frequency range of 0 to 5 Hz the phase lag varies almost linearly with respect to frequency. The amplitude error is also observed in Figure 4(a) to be frequency dependent, which is observed to be 0 dB for small frequencies and -3.9 dB for 5 Hz.

For the purpose of illustration, a constant actuator delay of τ=36/1024 sec. is selected to determine the parameters for the three compensation methods. When the integration parameters of γ=1/2 and β=0 are used and the time step Δt is selected to be 10/1024 sec. for the linear acceleration extrapolation method, the resulting equivalent discrete transfer functions can be written as $G_c(z) = (25.76z^2 - 40.32z + 16.56)/2z^2$. Considering the PID controller used in the experiments, the equivalent discrete transfer function for derivative feedforward compensation can be determined using where the controller parameters of $K_p=20$, $T_i=5$ and K_{ff}=0.71. The equivalent discrete transfer function for the inverse compensators can be written as $G_c(z) = (36z - 35)/z$.

Figures 5(a) and 5(c) present the frequency response of $G_c(z)$ for the three selected compensation methods between 0 and 5 Hz. For the linear acceleration method, the magnitude increases and reaches a magnitude of 3.0 dB (141%) at 5 Hz. The phase varies almost linearly with respect to frequency where the maximum phase angle reaches 75 degree at 5 Hz. The derivative feedforward compensation introduces both amplification and phase lead. The maximum magnitude and phase occur at 5 Hz and are 3.5 dB (150%) and 48 degrees, respectively. The frequency response for the inverse compensator is shown to have the maximum amplification and phase lead of 3.5 dB (135%) and 45 degree at 5 Hz.

The expected frequency response of the servo-hydraulic actuator under the three selected compensation methods can be derived by combining the frequency response of the actuator dynamics with each of the compensation methods, and is presented in Figures 5(b) and 5(d). For the linear acceleration method, the magnitude has a minimum value of around -1 dB (90%) at 5 Hz. An over-compensation for phase is observed to occur, where a maximum phase lead value of 22 degrees occurs at 5 Hz. For derivative feedforward compensation, the phase will remain around 0 degree and have a minimum value of -5.8 degree at 5 Hz. The magnitude has a maximum value of 0.8 dB (108%). Thus the derivative feedforward compensation with a feedforward gain of 0.71 is observed to reduce the phase error due to the actuator dynamics but have a slight amplitude error when applied to real-time testing. The expected frequency response for the servo hydraulic actuator with the inverse compensator has a maximum magnitude of 0.7 dB at 4 Hz and a slight under-compensation for phase can be observed to be -6.2 degrees around 5 Hz.

5. EXPERIMENTAL VALIDATION

Experiments with predefined sine sweep displacement are conducted to validate the above frequency response analysis of the selected compensation methods using the equivalent discrete transfer function. The sine sweep signal has a magnitude of 25 mm and a frequency content of 0 to 5 Hz. The three selected compensation methods are implemented by programming their corresponding equivalent discrete transfer functions onto the Lehigh RTMD real-time integrated control system.

Figure 6 shows the experimental results when different methods are used for actuator delay compensation. The occurrence of a decrease in amplitude and a phase lead can be observed between the command and predicted displacements. For the linear acceleration extrapolation, Figure 6(a) ~ 6(c) show a slight decrease in the amplitude occurs at 35 sec. (corresponding to 3.5 Hz) and 45 sec. (corresponding to 4.5 Hz), which is expected in accordance with the frequency response analysis results, while a slight phase lead between the measured and command displacement can be observed. This indicates a slight phase over-compensation. This phase lead is also expected in accordance with the frequency response analysis results. The experimental results for derivative feedforward compensation (Figure 6(d) ~ 6(f) and inverse compensation (Figure 6(g) ~ 6(i)) are almost the same as they have also almost the same anticipated actuator response. Consistent with the frequency response analysis presented, the amplitude of the measured displacement is a little larger than that of the command displacement for frequencies around 3.5 Hz and 4.5 Hz. A slight phase lag also occurs around 4.5 Hz.

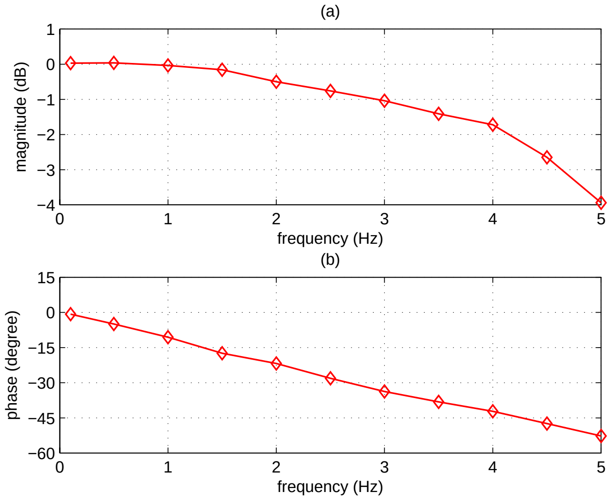

Figure 4 Frequency response of actuator dynamics

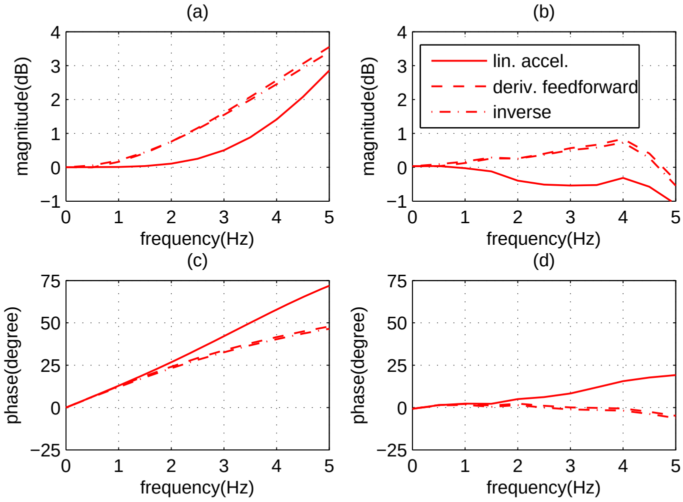

Figure 5 Frequency response of different compensation methods

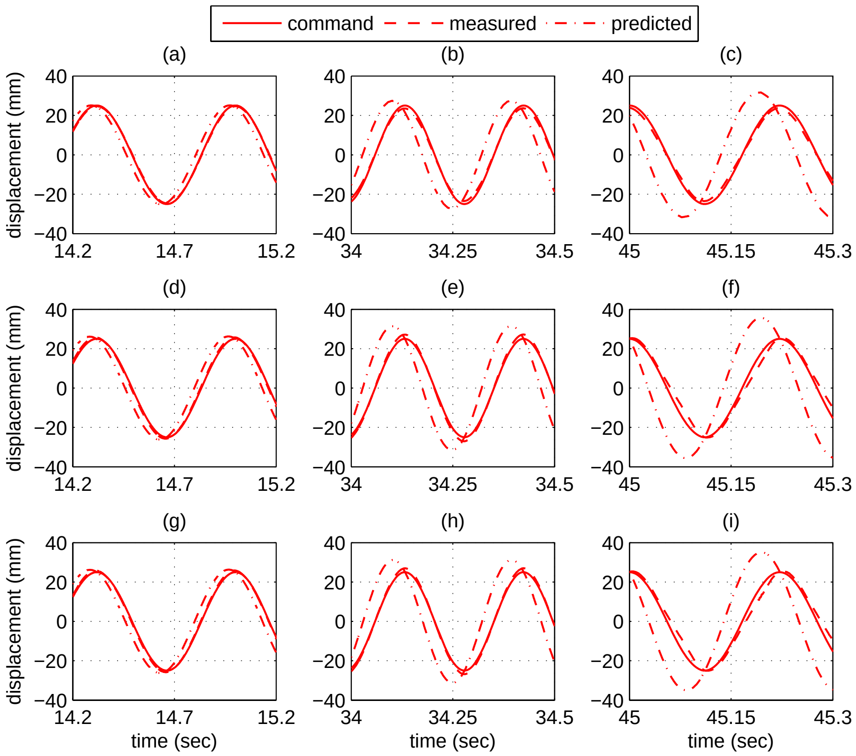

Figure 6 Experimental results using different compensation methods

Comparing the experiment results using the three compensation methods, the derivative feedforward compensation and the inverse compensation methods have a small phase under-compensation and slight amplitude increase, while the linear acceleration extrapolation method has a small phase over-compensation and slight amplitude decrease. In practice, a researcher must tune the compensator over the frequency range of interest, and attempt to minimize phase and magnitude error in the measured displacement response.

6. SUMMARY AND CONCLUSION

A general approach that is based on using a discrete transfer function and frequency response analysis of the function is proposed for analyzing actuator delay compensation methods for real-time testing. Three compensation methods are selected to illustrate the proposed approach. The selected compensation methods are shown to have different characteristics of amplitude and phase in their frequency response. Using the same value of actuator delay, the three compensation methods are experimentally applied in predefined sine sweep tests. The frequency analysis using the equivalent discrete transfer functions is validated by the experimental results, which illustrates the effectiveness of the proposed general approach. The proposed approach is shown to present a useful tool to analyze compensation methods for real-time testing. It is also worth noting that the proposed approach can also be extended to other compensation methods used for real-time testing.

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