SOFT RADIATION
BEYOND LEADING POWER

Lorenzo Magnea

University of Torino - INFN Torino

WHEPP XIV - IIT Kanpur - 07/12/2015
Outline

• Introduction
• Threshold resummations at leading power
• Gathering evidence beyond leading power
• Next-to-eikonal approximation
• Solving a collinear conundrum
• Outlook
GATEWAY
Multi-scale problems in renormalizable quantum field theories have perturbative corrections of the form $\alpha_s^n \log^k \left( \frac{Q_i^2}{Q_j^2} \right)$, which may spoil the reliability of the perturbative expansion. However, they carry important physical information.

- **Renormalization and factorization logs:** $\alpha_s^n \log^n \left( \frac{Q^2}{\mu^2} \right)$
- **High-energy logs:** $\alpha_s^n \log^{n-1} (s/t)$
- **Sudakov logs:** $\alpha_s^n \log^{2n-1} \left( 1 - z \right)$, \hspace{1em} $1 - z = \frac{W^2}{Q^2}, \frac{1 - M^2}{\hat{s}}, \frac{Q_\perp^2}{Q^2}, \ldots$

Logarithms encode process-independent features of perturbation theory. For Sudakov logs: the structure of infrared and collinear divergences.

\[
\underbrace{\frac{1}{\epsilon}}_{\text{virtual}} + (Q^2)^\epsilon \int_0^{m^2} \frac{dk^2}{(k^2)^{1+\epsilon} (s/t)^{1+\epsilon}} \implies \ln \left( \frac{m^2}{Q^2} \right)
\]

- **For inclusive observables:** analytic resummation to high logarithmic accuracy.
- **For exclusive final states:** parton shower event generators, \((N(N))\)LL accuracy.

Resummation probes the all-order structure of perturbation theory.

- **Non-perturbative** contributions to QCD cross sections can be estimated.
- **Links to the strong coupling** regime can be established for special gauge theories.
Predictions for the Higgs boson $q_T$ spectrum at LHC (M. Grazzini, 05)

Predictions for the $q_T$ spectrum of Higgs bosons produced via gluon fusion at the LHC, with and without resummation, and theoretical uncertainty band of the resummed prediction.
Predictions for the q_T spectrum of Higgs bosons produced via gluon fusion at the LHC, with and without resummation, and theoretical uncertainty band of the resummed prediction.
Higgs $p_T$ distribution somewhat shifted to higher transverse momentum but of course statistically limited.
CDF data on $Z$ production compared with QCD predictions at fixed order (dotted), with joint resummation (dashed), and with the inclusion of power corrections (solid).
CDF data on $Z$ production compared with QCD predictions at fixed order (dotted), with (joint) resummation (dashed), and with the inclusion of power corrections (solid).

Note shift in the distribution due to non-perturbative corrections extrapolated from all-order resummed result.
Jet veto efficiency in Higgs and Z production (Banfi et al. 12)

Comparison of NNLO fixed order results and matched resummed NNLL-NNLO results for Higgs production with a jet veto (left) and Z production with a jet veto (right). See also Becher, Neubert, Rothen, 13 for a SCET approach, and further improvements including treatment of quark masses (Banfi et al. 13).
Jet veto efficiency in Higgs and Z production  (Banfi et al. 12)

Comparison of NNLO fixed order results and matched resummed NNLL-NNLO results for Higgs production with a jet veto (left) and Z production with a jet veto (right). See also Becher, Neubert, Rothen, 13 for a SCET approach, and further improvements including treatment of quark masses (Banfi et al. 13).
Jet veto efficiency in Higgs and Z production (Banfi et al. 12)

Note the significant reduction of the theoretical uncertainty upon resummation ...

... which does not always take place!

Logarithms working harder

Comparison of NNLO fixed order results and matched resummed NNLL-NNLO results for Higgs production with a jet veto (left) and Z production with a jet veto (right). See also Becher, Neubert, Rothen, 13 for a SCET approach, and further improvements including treatment of quark masses (Banfi et al. 13).
More logarithms

- **Threshold logarithms** are associated with kinematic variables $\xi$ that vanish at Born level and get corrections that are enhanced because phase space for real radiation is restricted near partonic threshold: examples are $1 - T$, $1 - M^2/\hat{s}$, $1 - x_{BJ}$.

- At **leading power** in the threshold variable $\xi$ logarithms are directly related to soft and collinear divergences: real radiation is proportional to factors of

$$\frac{1}{\xi^{1+\epsilon}} = -\frac{1}{p\epsilon} \delta(\xi) + \left(\frac{1}{\xi}\right) + - p\epsilon \left(\frac{\log \xi}{\xi}\right) + \ldots$$

  Cancels virtual IR poles

  Leading power threshold logs

- **Beyond the leading power**, $1/\xi$, the perturbative cross section takes the form

$$\frac{d\sigma}{d\xi} = \sum_{n=0}^{\infty} \left(\frac{\alpha_s}{\pi}\right)^n \sum_{m=0}^{2n-1} \left[ c_n^{(-1)} \left(\frac{\log^m \xi}{\xi}\right) + c_n^{(\delta)} \delta(\xi) + c_{nm}^{(0)} \log^m \xi + \ldots \right]$$

  Resummed to high accuracy

  All-order structure in some cases

  NLP threshold logs

- The structure of **NLP** threshold logarithms may be understood to all orders.
LEADING POWER
Electroweak annihilation

We will focus on processes involving parton annihilation into electroweak final states (Drell-Yan, Higgs, di-boson final states): very well understood at LP, simpler at NLP.

- **LP** threshold resummation is based on factorization: the Mellin-space partonic cross section reads

\[ \omega(N, \epsilon) = |H_{DY}|^2 \psi(N, \epsilon)^2 U(N) + O \left( \frac{1}{N} \right). \]

- **Collinear** poles can be subtracted with suitable parton distributions,

\[ \tilde{\omega}_{\text{MS}}(N) \equiv \frac{\omega(N, \epsilon)}{\phi_{\text{MS}}(N, \epsilon)^2} \]

- **Each factor in \( \omega \) obeys evolution equations near threshold, leading to exponentiation.

\[ \psi_R(N, \epsilon) = \exp \left\{ \int_0^1 dz \int_0^1 dy \frac{z^{N-1}}{1-z} \frac{1-y}{1-y} \kappa_{\psi} (\bar{\alpha} ((1-y)^2 Q^2), \epsilon) \right\}. \]

- **Real** and **virtual** contributions can be treated separately.
A well-established formalism exists for distributions in processes that are electroweak at tree level (Gardi, Grunberg 07). For an observable $r$ vanishing in the two-jet limit

$$\frac{d\sigma}{dr} = \delta(r)[1 + \mathcal{O}(\alpha_s)] + C_R\frac{\alpha_s}{\pi}\left\{\left[-\frac{\log r}{r} + \frac{b_1 - d_1}{r}\right]^+ + \mathcal{O}(r^0)\right\} + \mathcal{O}(\alpha_s^2)$$

The Mellin (Laplace) transform, $\sigma(N) = \int_0^1 dr \ (1 - r)^{N-1} \frac{d\sigma}{dr}$, exhibits $\log N$ singularities that can be organized in exponential form

$$\sigma(\alpha_s, N, Q^2) = H(\alpha_s) S(\alpha_s, N, Q^2) + \mathcal{O}(1/N)$$

where the exponent of the `Sudakov factor’ is in turn a Mellin transform

$$S(\alpha_s, N, Q^2) = \exp\left\{\int_0^1 \frac{dr}{r} \left[(1 - r)^{N-1} - 1\right] E(\alpha_s, r, Q^2)\right\}$$

and the general form of the kernel is

$$E(\alpha_s, r, Q^2) = \int_{r^2 Q^2}^{r Q^2} d\xi^2 \frac{A(\alpha_s(\xi^2)) + B(\alpha_s(r Q^2)) + D(\alpha_s(r^2 Q^2))}{\xi^2}$$

where $A$ is the cusp anomalous dimension, and $B$ and $D$ have distinct physical characters.
The perturbative exponent

A classic way to organize Sudakov logarithms in terms of the Mellin (Laplace) transform of the momentum space cross section (Catani et al. 93) is to write

\[ d\sigma(\alpha_s, N) = \sum_{n=0}^{\infty} \left( \frac{\alpha_s}{\pi} \right)^n \sum_{k=0}^{2n} c_{nk} \log^k N + O(1/N) \]

\[ = H(\alpha_s) \exp \left[ \log N g_1(\alpha_s \log N) + g_2(\alpha_s \log N) + \alpha_s g_3(\alpha_s \log N) + \ldots \right] + O(1/N) \]

This displays the main features of Sudakov resummation

- **Predictive**: a \( k \)-loop calculation determines \( g_k \) and thus a whole tower of logarithms to all orders in perturbation theory.

- **Effective**: • the range of applicability of perturbation theory is extended
  (finite order: \( \alpha_s \log^2 N \) small. NLL resummed: \( \alpha_s \) small);
  • the renormalization scale dependence is naturally reduced.

- **Theoretically interesting**: resummation ambiguities related to the Landau pole give access to non-perturbative power-suppressed corrections.

- **Well understood**: • NLL Sudakov resummations exist for most inclusive observables at hadron colliders, NNLL and approximate \( N^3 \)LL in simple cases.
Non-logarithms

Delta-function terms arise from virtual corrections and phase space integration. They yield constants in Mellin space ("$\pi^2$") which can be controlled and "exponentiate" for simple processes (Parisi 80; Sterman 87; Eynck, Laenen, LM 03; Ahrens, Becher, Neubert, Yang 08),

- For EW annihilation, virtual terms reconstruct the full form factor.
- In dimensional regularization, each term exponentiates with no prefactor.
- Real and virtual factors are separately finite.
- An improved resummation formula can be written for DY, DIS and Higgs total rates: all constants are defined in the exponent.

$$\omega(N, \epsilon) = |H_{DY} \mathcal{R}(\epsilon)\sqrt{U_V(\epsilon)}|^2 \psi_R(N, \epsilon)^2 U_R(N) + \mathcal{O}\left(\frac{1}{N}\right)$$

$$= |\Gamma(Q^2, \epsilon)|^2 \psi_R(N, \epsilon)^2 U_R(N, \epsilon) + \mathcal{O}\left(\frac{1}{N}\right),$$

$$\hat{\omega}_{MS} (N) \equiv \frac{\omega(N, \epsilon)}{\phi_{MS} (N, \epsilon)^2} = \left( \frac{|\Gamma(Q^2, \epsilon)|^2}{\phi_V(\epsilon)^2} \right) \left( \frac{\psi_R(N, \epsilon)^2 U_R(N, \epsilon)}{\phi_R(N, \epsilon)^2} \right) + \mathcal{O}\left(\frac{1}{N}\right)$$

$$\hat{\omega}_{MS} (N) = \left[ \frac{\Gamma(Q^2, \epsilon)}{\Gamma(-Q^2, \epsilon)} \right]^2 \left( \frac{\Gamma(-Q^2, \epsilon)}{\phi_V(\epsilon)} \right)^2 \exp \left[ F_{MS} (\alpha_s) \right] \times$$

$$\exp \left[ \int_0^1 dz \frac{z^{N-1} - 1}{1 - z} \left\{ 2 \int_{Q^2}^{(1-z)^2 Q^2} \frac{d\mu^2}{\mu^2} A (\alpha_s(\mu^2)) + D (\alpha_s ((1-z)^2 Q^2)) \right\} \right] + \mathcal{O}\left(\frac{1}{N}\right).$$

Less predictive than conventional resummation: the exponent receives corrections order by order. Empirically, exponentiated lower-order constants provide much of the exact result.
A LONG HISTORY
The LBKD Theorem

The earliest evidence that infrared effects can be controlled at NLP is Low’s theorem (Low 58)

\[ M_\mu = e \left( \frac{\hat{p}_{1\mu}'}{\hat{p}_{1'} \cdot \hat{k}} - \frac{\hat{p}_{1\mu}}{\hat{p}_{1} \cdot \hat{k}} \right) T(\nu, \Delta) \]

\[ + e \left( \frac{\hat{p}_{2\mu}'}{\hat{p}_{2'} \cdot \hat{k}} - \frac{\hat{p}_{2\mu}}{\hat{p}_{1} \cdot \hat{k}} \right) \frac{\partial T(\nu, \Delta)}{\partial \nu} + O(k), \]

Low’s original expression for the radiative matrix element

A radiative matrix element
The LBKD Theorem

The earliest evidence that infrared effects can be controlled at NLP is Low's theorem (Low 58)

\[ M_\mu = e \left( \frac{p_{1\mu}'}{p_{1\mu}' \cdot k} - \frac{p_{1\mu}}{p_{1\mu} \cdot k} \right) T(\nu, \Delta) \]

\[ + e \left( \frac{p_{1\mu}'}{p_{1\mu}'} - \frac{p_{2\mu}'}{p_{2\mu}'} - \frac{p_{1\mu} p_{2\mu}}{p_{1\mu} \cdot k} \right) \frac{\partial T(\nu, \Delta)}{\partial \nu} + O(k), \]

A radiative matrix element

Eikonal approximation

Low's original expression for the radiative matrix element

Next-to-eikonal contribution
The LBKD Theorem

The earliest evidence that infrared effects can be controlled at NLP is Low's theorem (Low 58)

A radiative matrix element

Low's original expression for the radiative matrix element

The radiative matrix element for the emission of a (next-to-) soft photon is determined by the Born amplitude $T$ and its first derivative with respect to external momenta.

- Low's result established for a single charged scalar particle, follows from gauge invariance.
- It generalizes the well known properties of soft emissions in the eikonal approximation.
- The theorem was extended by (Burnett, Kroll 68) to particles with spin.
- The LBK theorem applies to massive particles and uses the mass as a collinear cutoff.
- It was extended to massless particles by (Del Duca 90), as discussed below.
An important source of known NLP logarithms is the DGLAP anomalous dimension. Non-trivial connections between LP and NLP logarithms in DGLAP were uncovered (Moch, Vermaseren, Vogt 08) and made systematic (Dokshitzer, Marchesini, Salam 08).

Conventional DGLAP for a quark distribution reads

\[
\mu^2 \frac{\partial}{\partial \mu^2} q(x, \mu^2) = \int_x^1 \frac{dz}{z} q \left( \frac{x}{z}, \mu^2 \right) P_{qq}(z, \alpha_s(\mu^2))
\]

\[
\mu^2 \frac{\partial}{\partial \mu^2} \tilde{q}(N, \mu^2) = \gamma_N(\alpha_s(\mu^2)) \tilde{q}(N, \mu^2).
\]

The large-$N$ behavior of the anomalous dimension is single-logarithmic in the MS scheme. NLP terms suppressed by $N$ are related to LP

\[
\gamma_N(\alpha_s) = -A(\alpha_s) \ln \tilde{N} + B_{\delta}(\alpha_s) - C_\gamma(\alpha_s) \frac{\ln \tilde{N}}{N} + D_\gamma(\alpha_s) \frac{1}{N} + \mathcal{O}\left(\frac{1}{N^2}\right),
\]

\[
C_1 = 0, \quad C_2 = 4C_FA_1, \quad C_3 = 8C_FA_2.
\]

Especially the relation for $C_3$ is very suggestive and seems to call for a structural explanation.

These relation extend to the function $D$, and recursively to all orders: modified splitting functions can be defined which vanish at large $x$ beyond one loop.
DMS, with refinements implied by (Basso, Korchemsky 06) propose to modify DGLAP as

\[ \mu^2 \frac{\partial}{\partial \mu^2} \psi(x, \mu^2) = \int_x^1 \frac{dz}{z} \psi \left( \frac{x}{z}, z^\sigma, \mu^2 \right) P(z, \alpha_s \left( \frac{\mu^2}{z} \right)) . \]

applying to both PDF’s and fragmentation, with \( \sigma = \pm 1 \) respectively, and the same kernel (Gribov-Lipatov reciprocity). The resulting kernel \( P \) and is claimed to vanish as \( z \to 1 \) beyond one loop in the “physical” MC scheme where \( \alpha_s = \gamma_{\text{cusp}} \). Therefore

\[ P(z, \alpha_s) = \frac{A(\alpha_s)}{(1-z)_+} + B_\delta(\alpha_s) \delta(1-z) + \mathcal{O}(1-z). \]

The modified equation cannot be diagonalized by Mellin transform: it must be solved by iteration, using a formal translation operator

\[ \mu^2 \frac{\partial}{\partial \mu^2} \psi(x, \mu^2) = \int_x^1 \frac{dz}{z} e^{-\ln z (\beta(\alpha_s) \frac{\partial}{\partial \alpha_s} - \sigma \frac{\partial}{\partial \ln \mu^2})} \times \psi \left( \frac{x}{z}, \mu^2 \right) P(z, \alpha_s \left( \frac{\mu^2}{z} \right)) , \]

In practice, this procedure constructs a modified kernel where high-order terms are generated by shifts of lower-orders.
An educated guess

Available NLP information can be combined in an ansatz for generalized threshold resummation applicable to EW annihilation processes and DIS (Laenen, LM, Stavenga 08).

\[
\ln[\hat{\omega}(N)] = \mathcal{F}_{\text{DY}}(\alpha_s(Q^2)) + \int_0^1 \, dz \, z^{N-1} \left\{ \frac{1}{1-z} \, D \left[ \alpha_s \left( \frac{(1-z)^2 Q^2}{z} \right) \alpha_s \right] \right. \\
+ 2 \int_{Q^2} \, dq^2 \frac{q^2}{Q^2} P_s[z, \alpha_s(q^2)] \left. \right\},
\]

- Exponentiation of constants
- Refinement of phase space
- DMS kernel

This expression, and similar ones for DIS and Higgs production via gluon fusion, incorporate

- The exponentiation of N-independent terms.
- A treatment of phase space consistent up to O(1-z), including running coupling effects.
- The DMS modification of the DGLAP kernel, including the NLP term in the LO kernel.
- Note: DMS brings to the exponent a $C_F^2$ contribution crucial to fit two-loop NLP logs.
**An educated guess: Drell-Yan**

\[
\hat{\omega}(N) = \sum_{i=0}^{\infty} \left( \frac{Q_s}{\pi} \right)^n \left[ \sum_{m=0}^{2n} a_{nm} \ln^m \tilde{N} + \sum_{m=0}^{2n-1} b_{nm} \frac{\ln^m \tilde{N}}{N} \right] + \mathcal{O} \left( \frac{\ln^p N}{N^2} \right),
\]

**Table 1**

Comparison of exact and resummed 2-loop coefficients for the Drell-Yan cross section. For each color structure, the left column contains the exact results, the right column contains the prediction from resummation.

<table>
<thead>
<tr>
<th>( b_{23} )</th>
<th>( C_f^2 )</th>
<th>( C_A C_F )</th>
<th>( n_f C_F )</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>4</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( \frac{7}{2} )</td>
<td>( \frac{7}{2} )</td>
<td>( \frac{11}{6} )</td>
<td>( -\frac{1}{3} )</td>
</tr>
<tr>
<td>( 8\zeta_2 - \frac{43}{4} )</td>
<td>( 8\zeta_2 - 11 )</td>
<td>( -\zeta_2 + \frac{239}{18} )</td>
<td>( -\zeta_2 + \frac{133}{18} )</td>
</tr>
<tr>
<td>( -\frac{1}{2} \zeta_2 - \frac{3}{4} )</td>
<td>( 4\zeta_2 )</td>
<td>( -\frac{7}{4} \zeta_3 + \frac{275}{216} )</td>
<td>( \frac{7}{4} \zeta_3 + \frac{11}{3} \zeta_2 - \frac{101}{54} )</td>
</tr>
</tbody>
</table>

**Table 2**

Comparison of exact and resummed 2-loop coefficients for the DIS structure function. For each color structure, the left column contains the exact results, the right column contains the prediction from resummation.

<table>
<thead>
<tr>
<th>( d_{23} )</th>
<th>( C_f^2 )</th>
<th>( C_A C_F )</th>
<th>( n_f C_F )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{1}{4} )</td>
<td>( \frac{1}{4} )</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( \frac{7}{10} )</td>
<td>( \frac{7}{10} )</td>
<td>( \frac{11}{48} )</td>
<td>( -\frac{1}{27} )</td>
</tr>
<tr>
<td>( \frac{7}{4} \zeta_2 - \frac{49}{12} )</td>
<td>( -\frac{1}{4} \zeta_2 - \frac{105}{32} )</td>
<td>( -\frac{1}{2} \zeta_2 + \frac{1333}{336} )</td>
<td>( -\frac{1}{2} \zeta_2 + \frac{555}{336} )</td>
</tr>
<tr>
<td>( \frac{15}{16} \zeta_3 - \frac{47}{16} \zeta_2 - \frac{431}{64} )</td>
<td>( -\frac{11}{12} \zeta_3 + \frac{13}{48} \zeta_2 - \frac{17570}{3276} )</td>
<td>( \frac{5}{3} \zeta_3 + \frac{7}{16} \zeta_2 - \frac{9510}{3276} )</td>
<td>( \frac{1}{24} \zeta_2 - \frac{1699}{864} )</td>
</tr>
</tbody>
</table>

- Only one-loop **NLP** and **DMS** input has been used in the resummation formula.
- **Leading NLP** logarithms are reproduced exactly for all color structures at two loops.
- **NLL** and **NNLL** NLP logarithms are well approximated but not exact.
- Similar results hold for three-loop **DIS**, using two-loop information in the exponent.
Towards systematics

The problem of NLP threshold logarithms has been of interest for a long time, and several different approaches have been proposed. Recent years have seen a resurgence of interest, both from a theoretical point of view and for phenomenology.

Early attempts include a study of the impact of NLP logs on the Higgs cross section by Kraemer, Laenen, Spira (98); work on \( F_L \) by Akhoury and Sterman (99) (logs without plus distributions are however leading) and work by Grunberg et al. (07-09) on DIS.

Important results can be obtained by using physical kernels (Vogt et al. 09-14) which are conjectured to be single-logarithmic at large \( z \), which poses constraints on their factorized expression. Note in particular a recent application to Higgs production by De Florian, Mazzitelli, Moch, Vogt (14).

Useful approximations can be obtained by combining constraints from large \( N \) with high-energy constraints for \( N \sim 1 \) and analyticity (Ball, Bonvini, Forte, Marzani, Ridolfi, 13), together with phase space refinements.

SCET techniques can be applied and indeed may be well-suited to the problem: a thorough one-loop analysis was given in (Larkoski, Neill, Stewart,15).

A lot of recent formal work on the behavior of gauge and gravity scattering amplitudes beyond the eikonal limit was triggered by a link to asymptotic symmetries of the \( S \) matrix (many authors from A(ndy Strominger) to Z(vi Bern), 14-15).
NEXT-TO-SOFT
On the eikonal approximation

- Taking the soft approximation at leading power on emissions from an energetic (or very massive) particle yields a set of simplified Feynman rules.

- These rules correspond to emissions from a Wilson line oriented along the trajectory of the energetic particle, in the same color representation.

\[ \Phi_n(\lambda_2, \lambda_1) = P \exp \left[ ig \int_{\lambda_1}^{\lambda_2} d\lambda \, n \cdot A(\lambda n) \right]. \]

- The results do not depend on the energy and spin of the emitter, only on its direction and color charge.

- Physically, we are neglecting the recoil of the emitter: the only effect of interaction with soft radiation is that the emitter acquires a phase.

- The soft limit of a multi-particle amplitude is a correlator of Wilson lines.
Infrared exponentiation

All correlators of Wilson lines, regardless of shape, resum in exponential form.

\[ S_n \equiv \langle 0 | \Phi_1 \otimes \ldots \otimes \Phi_n | 0 \rangle = \exp(\omega_n) \]

Diagrammatic rules exist to compute directly the logarithm of the correlators.

\[ \omega_{2,\text{QED}} = \]

Only connected photon subdiagrams contribute to the logarithm.

\[ \omega_{2,\text{QCD}} = \]

Only gluon subdiagrams which are two-eikonal irreducible contribute to the logarithm. They have modified color factors.

For eikonal form factors, these diagrams are called webs (Gatheral; Frenkel, Taylor; Sterman).
Multiparticle webs

The concept of web generalizes non-trivially to the case of multiple Wilson lines. (Gardi, Smillie, White, et al).

A web is a set of diagrams which differ only by the order of the gluon attachments on each Wilson line. They are weighted by modified color factors.

Writing each diagram as the product of its natural color factor and a kinematic factor

$$ D = C(D) \mathcal{F}(D) $$

a web $W$ can be expressed as a sum of diagrams in terms of a web mixing matrix $R$

$$ W = \sum_D \tilde{C}(D) \mathcal{F}(D) = \sum_{D,D'} C(D') R(D',D) \mathcal{F}(D) $$

The non-abelian exponentiation theorem holds: each web has the color factor of a fully connected gluon subdiagram (Gardi, Smillie, White).
Beyond the eikonal

The soft expansion can be organized beyond leading power using either path integral techniques (Laenen, Stavenga, White 08) or diagrammatic techniques (Laenen, LM, Stavenga, White 10). The basic idea is simple, but the combinatorics cumbersome. For spinors

\[ \frac{p + k}{2p \cdot k + k^2} \gamma^\mu u(p) = \left[ \frac{p^\mu}{p \cdot k} + \frac{k \gamma^\mu}{2p \cdot k} - k^2 \frac{p^\mu}{2(p \cdot k)^2} \right] u(p) + \mathcal{O}(k) \]

- A class of factorizable contributions exponentiate via NE webs

\[ \mathcal{M} = \mathcal{M}_0 \exp \left[ \sum_{D_{\text{eik}}} \tilde{C}(D_{\text{eik}}) \mathcal{F}(D_{\text{eik}}) + \sum_{D_{\text{NE}}} \tilde{C}(D_{\text{NE}}) \mathcal{F}(D_{\text{NE}}) \right] . \]

- Feynman rules exist for the NE exponent, including “seagull” vertices.

\[ \mathcal{M} = \mathcal{M}_0 \exp [\mathcal{M}_{\text{eik}} + \mathcal{M}_{\text{NE}}] (1 + \mathcal{M}_r) + \mathcal{O}(\text{NNE}) . \]

- Non-factorizable contributions involve single gluon emission from inside the hard function, and must be studied using LBDK’s theorem.
Double real two-loop Drell-Yan

Multiple real emission contributions to EW annihilation processes involve only factorizable contributions. NE Feynman rules can be tested this level.

Defining the Drell-Yan K-factor as

\[ K^{(n)}(z) = \frac{1}{\sigma^{(0)}} \frac{d\sigma^{(n)}(z)}{dz}, \]

As a test, we (re)computed the \( C_F^2 \) part of \( K \) at NNLO from ordinary Feynman diagrams, and then using NE Feynman rules, finding complete agreement. As expected, plus distributions arise from the eikonal approximation.

Next-to-eikonal terms arise from single-gluon corrections: seagull-type contributions vanish for the inclusive cross section.

The abelian part of the NNLO K-factor from real emission, omitting constants
COLLINEAR EMISSION
COLLINEAR EMISSION
Non-factorizable contributions start at NNLO. For massive particles they can be traced to the original LBK theorem. For massless particles a new contribution to NLP logs emerges.

- Gluon $k_2$ is always (next-to) soft for EW annihilation near threshold.
- When $k_1$ is (next-to) soft all logs are captured by NE rules.
- Contributions with $k_1$ hard and collinear are missed by the soft expansion.
- The collinear pole interferes with soft emission and generates NLP logs.
- The problem first arises at NNLO

These contributions are missed by the LBK theorem: it applies to an expansion in $E_k/m$.
They can be analyzed using the method of regions: the relevant factor is $(p \cdot k_2)^{-\epsilon/\epsilon}$.
They cause the breakdown of next-to-soft theorems for amplitudes beyond tree level.
⇒ the soft expansion and the limit $\epsilon \to 0$ do not commute.
They require an extension of LBK to $m^2/Q < E_k < m$. It was provided by Del Duca (90).
LP factorization: pictorial

A pictorial representation of soft-collinear factorization for fixed-angle scattering amplitudes
The precise functional form of this graphical factorization is

\[
M_L \left( p_i/\mu, \alpha_s(\mu^2), \epsilon \right) = S_{LK} \left( \beta_i \cdot \beta_j, \alpha_s(\mu^2), \epsilon \right) H_K \left( \frac{p_i \cdot p_j}{\mu^2}, \frac{(p_i \cdot n_i)^2}{n_i^2 \mu^2}, \alpha_s(\mu^2) \right) \\
\times \prod_{i=1}^{n} \left[ J_i \left( \frac{(p_i \cdot n_i)^2}{n_i^2 \mu^2}, \alpha_s(\mu^2), \epsilon \right) / J_i \left( \frac{(\beta_i \cdot n_i)^2}{n_i^2 \mu^2}, \alpha_s(\mu^2), \epsilon \right) \right],
\]

Here we introduced dimensionless four-velocities \( \beta_i^\mu = Q p_i^\mu, \beta_i^2 = 0 \), and factorization vectors \( n_i^\mu, n_i^2 \neq 0 \) to define the jets,

\[
J \left( \frac{(p \cdot n)^2}{n^2 \mu^2}, \alpha_s(\mu^2), \epsilon \right) u(p) = \langle 0 | \Phi_n (\infty, 0) \psi(0) | p \rangle.
\]

where \( \Phi_n \) is the Wilson line operator along the direction \( n^\mu \).

The soft function \( S \) is a matrix, mixing the available color tensors. It is defined by a correlator of Wilson lines.

\[(c_L)_{\{a_k\}} S_{LK} (\beta_i \cdot \beta_j, \epsilon) = \langle 0 | \prod_{k=1}^{n} [\Phi_{\beta_k} (\infty, 0)]_{a_k}^{b_k} | 0 \rangle (c_K)_{\{b_k\}} ,\]

where the \( c_L \) are a basis of color tensors for the process at hand.
NLP factorization: a new jet

Soft radiation can arise either from the jets or from the hard function

\[ A_\mu e^\mu(k) = A^J_\mu e^\mu(k) + A^H_\mu e^\mu(k) , \]

The amplitude for emission from the jets can be precisely defined in terms of a new jet function

\[ A^J_\mu = \sum_{i=1}^{2} H(p_i - k; p_j, n_j) J_\mu(p_i, k, n_i) \prod_{j \neq i} J(p_j, n_j) \equiv \sum_{i=1}^{2} A^J_{\mu i} . \]

defines the radiative jet.

\[ J_\mu(p, n, k, \alpha_s(\mu^2), \epsilon) u(p) = \int d^d y \ e^{i(p-k) \cdot y} \langle 0 | \Phi_n(y, \infty) \psi(y) j_\mu(0) | p \rangle , \]

Factorized contributions to the radiative amplitude

- At tree level the radiative jet displays the expected dependence on spin.
- Dependence on the gauge vector \( n^\mu \) starts at loop level: simplifications arise for \( n^2 = 0 \).
Beyond Low’s theorem

A slightly modified version of Del Duca’s result gives the radiative amplitude in terms of the non-radiative one, its derivatives, and the two “jet” functions.

\[ A^\mu(p_j, k) = \sum_{i=1}^{2} \left\{ q_i \left( \frac{(2p_i - k)^\mu}{2p_i \cdot k - k^2} + G_i^{\nu\mu} \frac{\partial}{\partial p_i^\nu} \right) + G_i^{\nu\mu} \left[ \frac{J_\nu(p_i, k, n_i)}{J(p_i, n_i)} - q_i \frac{\partial}{\partial p_i^\nu} \left( \ln J(p_i, n_i) \right) \right] \right\} A(p_i; p_j). \]

The tensors \( G^{\mu\nu} \) project out the eikonal contribution present in the first term.

\[ \eta^{\mu\nu} = G^{\mu\nu} + K^{\mu\nu}, \quad K^{\mu\nu}(p; k) = \frac{(2p - k)^\nu}{2p \cdot k - k^2} k^\mu, \]

The factorized expression for the radiative amplitude can be simplified.

- The jet factor is RG invariant: it can be computed in bare perturbation theory.
- With this choice one can use that \( J(p, n) = 1 \) for \( n^2 = 0 \): it is a pure counterterm.
- The choice of reference vectors is then physically motivated (and confirmed by a complete analysis using the method of regions): we take \( n_1 = p_2 \) and \( n_2 = p_1 \).

\[ A^\mu(p_j, k) = \sum_{i=1}^{2} \left( q_i \frac{(2p_i - k)^\mu}{2p_i \cdot k - k^2} + q_i G_i^{\nu\mu} \frac{\partial}{\partial p_i^\nu} + G_i^{\nu\mu} J_\nu(p_i, k) \right) A(p_i; p_j). \]

For general amplitudes, a full subtraction of the residual \( n \) dependence should be aimed at.
To achieve NNLO accuracy at NLP, we need the radiative jet function at one loop. As a test, we compute the $C_F^2$ contributions. They simplify considerably for $n^2 = 0$.

Abelian-like Feynman diagrams for the bare one-loop radiative jet

The result has a simple structure, with characteristic scale and spin dependence

\[
J^{(1)}_{\nu}(p, n, k; \epsilon) = (2p \cdot k)^{-\epsilon} \left[ \left( \frac{2}{\epsilon} + 4 + 8\epsilon \right) \left( \frac{n \cdot k}{p \cdot k} \frac{p^\nu}{p \cdot n} - \frac{n^\nu}{p \cdot n} \right) - (1 + 2\epsilon) \frac{i k_\alpha \Sigma^\alpha\nu}{p \cdot k} \right.
\]
\[
+ \left( \frac{1}{\epsilon} - \frac{1}{2} - 3\epsilon \right) \frac{k^\nu}{p \cdot k} + (1 + 3\epsilon) \left( \frac{\gamma^\nu \not{n}}{p \cdot n} - \frac{p^\nu}{p \cdot k n} \right) \bigg] + \mathcal{O}(\epsilon^2, k),
\]

As a test, it obeys the Ward identity

\[
k^\mu J_\mu(p, n, k, \alpha_s(\mu^2), \epsilon) = q J(p, n, \alpha_s(\mu^2), \epsilon),
\]
Real-virtual two-loop Drell-Yan

Real-virtual corrections to EW annihilation processes involve non-factorizable contributions. NE rules cannot reproduce the perturbative result at NLP, due to collinear interference.

As a test of the LBDK factorization, we computed the $C_F^2$ part of the real-virtual K-factor at NNLO from ordinary Feynman diagrams, and then using the radiative amplitude integrated over phase space. As expected, plus distributions arise from the eikonal approximation, fully determined by the dressed non-radiative amplitude. Derivative terms and the projected radiative jet contribute at NLP.

The abelian part of the NNLO K-factor from real-virtual diagrams, omitting constants

\[
K_{RV}^{(2)}(z) = \left( \frac{\alpha_s}{4\pi} C_F \right)^2 \left\{ \frac{32}{e^2} \left[ D_0(z) - 1 \right] + \frac{16}{e^2} \left[ -4 D_1(z) + 3 D_0(z) + 4 L(z) - 6 \right] + \frac{4}{e} \left[ 16 D_2(z) - 24 D_1(z) + 32 D_0(z) - 16 L^2(z) + 52 L(z) - 49 \right] - \frac{128}{3} D_3(z) + 96 D_2(z) - 256 D_1(z) + 256 D_0(z) + \frac{128}{3} L^3(z) - 232 L^2(z) + 412 L(z) - 408 \right\},
\]

The abelian part of the NNLO K-factor from real-virtual diagrams, omitting constants.
OUTLOOK
A Perspective

- Perturbation theory continues to display **new and unexplored** structures.
- **Leading power** threshold resummation is highly developed and provides some of the **most precise** predictions in perturbative QCD.
- Mellin-space **constants** naturally reside in the exponent for simple processes.
- Low’s theorem is the first of many hints that **NLP logs** can be understood and organized.
- **Different approaches** catch a number of **towers of NLP logs** in simple processes.
- The **next-to-soft** approximation is well understood, using both diagrammatic and path integral approaches, even for multi-parton processes.
- **Hard collinear** emissions spoil Low’s theorem: a **new radiative jet function** emerges.
- A complete treatment of **NLP threshold logs** is at hand.
- **Much work** to do to organize a true resummation formula, even for EW annihilation: we have a more intricate “factorization”, we must make sure to control **double countings**.
- In order to achieve **complete generality**, we will need to include **final state jets**.
धन्यवाद