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A STUDY ON WAVE BREAKING AND WAVE INTERACTION

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This paper aims to apply two methods namely, Vortex Method and Higher Order Spectral Method to study the wave breaking and wave interaction. The breaking of shallow water gravity wave is performed using Vortex method, while the interaction between two gravity waves is carried out using Higher Order Spectral (HOS) method. In Vortex method an interface defined by a density or velocity discontinuity is modeled as a vortex sheet. It comprises of a finite number of point vortices and the motion of these point vortices dictates the evolution of the interface. The simulation of wave breaking involves interaction between two interfaces, one is the free surface and the other is the bottom surface which is fixed. The interaction between the point vortices on both the interfaces causes the wave to break. The latter part of this paper is about the application of Higher Order Spectral method to study the interaction between a surface wave and an interfacial wave in a two layered density stratified fluid. This is a highly computationally efficient pseudospectral method to study the wave evolution and wave interactions.

***Index Terms*—Vortex method, Vortex sheet, Singularity, Point vortex, Higher order spectral method**

I. INTRODUCTION

Water waves such as gravity waves propagate with or without dispersion depending on whether the wavelength is comparable to the depth of the sea as long as their amplitude is small. On reaching the shallower waters if the amplitude of the wave is comparable to the depth then the wave breaks due to its interaction with the bottom. In this paper we model the wave breaking phenomena by considering an interaction between a free and fixed interface. We consider fluids to be incompressible, irrotational and inviscid. The free interface has a density and perturbation velocity discontinuity which results in existence of vorticity at the free interface and we can model it as a vortex sheet comprised of a finite number of point vortices. The major question in this analysis is how the bottom topography which is fixed can be modeled as a vortex sheet, since above the bottom interface the fluid has a velocity generated by the point vortices on the free interface and below it the velocity has to be zero it has a velocity discontinuity and hence can be modeled as a vortex sheet. The prime difference in considering one of the interfaces fixed is that the velocity normal to the fixed interface has to be zero and there is no separate evolution equation for the vortex strength of the fixed interface like we have for the free interfaces which will be discussed in the subsequent section. The vortex sheet strength of the fixed interface depends on the vortex sheet strength of the free interface and itself and hence we must solve iteratively for it. The methodology used in this paper is applicable to both linear and non linear waves and with the proper initial conditions we can model a shallow water gravity wave traveling with a constant phase speed of $\sqrt{gH}$ (where g is acceleration due to gravity and H is the water depth) or the breaking of a non-linear surface gravity wave when they reach shallow waters. Vortex methods can be used to simulate a variety of phenomenon and the advantage of this method is that there is no need to solve for the whole domain, only the interfaces are studied. The breaking of a non-linear wave due to the effect of bottom topography was also modeled by [Baker et al. (1970)] but our analysis is mathematically simpler and intuitive. Similar work on non-linear waves has been done using more conventional methods by [Chan and Steet (1970)], [Longuet-Higgins and Cokelet (1976)] etc.

Further, we have also studied the interaction between two waves in a two layered density stratified fluid using Higher Order Spectral method in a similar way as described by [Alam et al. (2009)]. This method is fast, efficient and accurate which relies on perturbation expansions along with the Fast Fourier Transform. However, a limitation of the method is that it is Eulerian, thus, making the study of wave turning or wave breaking difficult. Nevertheless, the method is very powerful in the study of wave-wave interaction or wave-bottom interaction. The basic idea of spectral method to use a number of Fourier modes to study the evolution of a wave. This is done up to any order M for the perturbation expansion of the velocity potential. Using the appropriate Kinematic and Dynamic boundary conditions, the boundary value problem is solved at every time step along with marching forward in time using a suitable time integration scheme.

II. VORTEX METHOD

A. *Methodology*

Interfaces considered in this study could have a density discontinuity or a velocity discontinuity or both due to which vorticity exists at these interfaces. Apart from the interfaces fluids are considered irrotational, incompressible and inviscid. Inviscid approximation is crucial in this study since existence of vortex sheets is not feasible in viscous fluids due to existence of infinite shear stresses at the interfaces. The system is governed by Euler equation away from the interfaces

$$\mathbf{a_i} = -\frac{\mathbf{1}}{\rho_\mathbf{i}}\nabla \mathbf{p_i} - \mathbf{g}\hat{\mathbf{j}} \tag{1}$$

Where $i = 1, 2$ denotes fluid above and below the interface respectively. The interfaces have been modeled as vortex sheets and have an associated vortex sheet strength defined as the difference in the tangential velocity across the interface.

$$\gamma = (\mathbf{u_1} - \mathbf{u_2}) \cdot \mathbf{s} \tag{2}$$

Here $\mathbf{u_1}, \mathbf{u_2}$ are the velocities of the fluid above and below the interface and $\mathbf{s}$ is the unit tangent vector to the interface. The evolution of interface is studied with the help of the Lagrangian velocities of the point vortices comprising the interface. The interface velocity is a weighted sum of the velocities above and below the interface given as:

$$\mathbf{q} = \mathbf{U} + \frac{\mathbf{1}}{\mathbf{2}}\gamma\alpha\mathbf{s} \tag{3}$$

$$\frac{\mathbf{dX}}{\mathbf{dt}} = \mathbf{q} \tag{4}$$

Where $\mathbf{q}$ is the lagrangian velocity of the interface, $\mathbf{U}$ is the average of the velocities above and below the interface $\mathbf{U} = \frac{\mathbf{1}}{\mathbf{2}}(\mathbf{u_1} + \mathbf{u_2})$ and α is the weighting parameter (when $\alpha = 1, -1$ the interface moves with the velocity of the upper or lower fluid respectively). $\mathbf{X}$ is the position vector representing the interface which has been parametrized using the arc length coordinate ($\mathbf{X} = \mathbf{x}(\mathbf{s}, \mathbf{t})\hat{\mathbf{i}} + \mathbf{y}(\mathbf{s}, \mathbf{t})\hat{\mathbf{j}}$).

The average velocity $\mathbf{U}$ can be evaluated for an array of point vortices using the Biot-Savart law:

$$\mathbf{U}(s,t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{\hat{\mathbf{k}} \times (\mathbf{X}(s,t) - \mathbf{X}(\tilde{s},t))}{|\mathbf{X}(s,t) - \mathbf{X}(\tilde{s},t)|^2}\gamma(\tilde{s},t)d\tilde{s} \tag{5}$$

The Biot-Savart law for a periodic boundary condition (as is present in the interfaces we consider) reduces to Birkhoff-Rott equations:

$$u_l - iv_l = \sum_{k=1}^{M} \frac{i}{2\lambda}\int_0^\lambda \tilde{\gamma_k}\cot(\frac{\pi(z_l - \tilde{z_k})}{\lambda})d\tilde{s_k} \tag{6}$$

Here u,v are the horizontal and vertical components of $\mathbf{U}$, z is the complex position of the interface $z = x(s,t) + iy(s,t)$, λ is the wavelength of the interface and the indices k and l represent the k^{th} and l^{th} interfaces respectively . This equation can be understood as the velocity of a point vortex on some interface is due to the velocity induced by all the vortices on that interface superposed with the velocity induced by the point vortices on all the other interfaces. As can be seen in equation (6), for evaluating the velocities at some time t we need the values of vortex strength at that time. This means that we also need an evolution equation for the vortex strength of the interface which can be derived using the Euler equations and the kinematic conditions for acceleration of the fluid above and below the interface respectively. From the Euler equations we can get the following relation for accelerations:

$$A(\mathbf{a}_1 + \mathbf{a}_2) \cdot \mathbf{s} + 2Ag\mathbf{j} \cdot \mathbf{s} = (\mathbf{a}_1 - \mathbf{a}_2) \cdot \mathbf{s} \tag{7}$$

Here A is the Atwood number for the interface given by $A = \frac{\rho_2 - \rho_1}{\rho_2 + \rho_1}$, $\mathbf{a_1}, \mathbf{a_2}$ are the accelerations of the fluid above and below the free interface. Now we also have the kinematic conditions for the accelerations given by:

$$\mathbf{a}_i = \frac{d\mathbf{u}_i}{dt} - \frac{1}{2}\gamma(\alpha \pm 1)\frac{\partial \mathbf{u}_i}{\partial s} \tag{8}$$

Where the +(-) sign is for $i = 2(1)$. Using equations (7) and (8) we can get the evolution equation for vortex strength of free interface interface.

$$\frac{d\gamma}{dt} = 2A\frac{d\mathbf{U}}{dt} \cdot \mathbf{s} + \frac{\alpha + A}{4}(\frac{\partial \gamma^2}{\partial s}) - (1 + \alpha A)\gamma\frac{\partial \mathbf{U}}{\partial s} \cdot \mathbf{s} + 2Ag\frac{\partial y}{\partial s} \tag{9}$$

Numerical simulation of the interfaces require that we consider interfaces to be comprised of a finite number of point vortices. Thus we need to discretize equation (6) (Birkhoff-Rott). The circulation strength of point vortices is given by:

$$\Gamma_j = \gamma_j \Delta s_j \tag{10}$$

Equation (9) is the evolution equation for the vortex strength of the free interface. The vortex strength of the fixed bottom can be evaluated using the fact that the velocity below the interface is zero and that the velocity above the bottom is only present in the direction tangential to the bottom interface.

$$\gamma = (\mathbf{u_1} - \mathbf{0}) \cdot \mathbf{s} \tag{11}$$

Where $\mathbf{u_1}$ is the velocity above the fixed interface and the velocity below the interface is 0. Now since the Birkhoff-Rott equation evaluates the average velocity at the interface we have $\mathbf{U} = \mathbf{u}_1 + \mathbf{u}_2/2 = \mathbf{u}_1/2$. So, the vortex strength of the fixed interface becomes:

$$\gamma = 2(\mathbf{U}) \cdot \mathbf{s} \tag{12}$$

But the principal vortex velocity $\mathbf{U}$ is itself calculated from the vortex strengths of both free and fixed interfaces as given by equation (6). Since the equation for vortex strength of the bottom interface is implicit in nature we must solve for vortex strength of the bottom interface iteratively.

B. *Numerical technique*

The interaction between a free and fixed interface is described by the system of equations (3),(4),(6),(9) and (12) which represents an initial value problem hence the knowledge of the initial shape of the interfaces and their vortex strengths is imperative which will be discussed in the subsequent section. Suppose we know the initial conditions, we can calculate the velocities at the first time level using the vortex strengths and guess the velocity and vortex strengths of free and fixed interfaces respectively at the second time level. Now we integrate equation (9) using Euler scheme and express the acceleration term as a forward difference in time $\frac{d\mathbf{U}}{dt} = \frac{\mathbf{U^{(2)}} - \mathbf{U^{(1)}}}{\Delta t}$ and using the vortex strength at second time level we can update the free interface using trapezoidal scheme. Now in the Birkhoff-Rott equation we use updated values of vortex strength of free interface and guess values of vortex strengths of the bottom interface and calculate the velocities of the point vortices on the bottom interface and using equation (12) we can update the vortex strength of bottom interface. Now that we have updated values of vortex strengths of both free and fixed interfaces we can calculate the updated velocities of the free interface and apply an iterative procedure for convergence of velocity of free interface and vortex strength of the bottom.

The methodology at some general time level n is similar to the above described technique except that we don't need to guess velocities at n+1 level instead we can estimate the acceleration term in equation (9) at the previous time level n-1 as $\frac{d\mathbf{U}}{dt}^{n-1} = \frac{\mathbf{U^{(n)}} - \mathbf{U^{(n-1)}}}{\Delta t}$. Now after getting some value of γ_f at n+1 we can update the interface. Now we can guess vortex strength of bottom interface γ_b at n+1 and using this and updated vortex strength of free interface γ_f we can calculate velocity at the bottom interface and update γ_b using equation (12). From the updated values of interface position and vortex strengths (γ_f and γ_b) we can get updated velocities of the free interface at n+1 and now we can iterate until convergence of acceleration but we will use central differencing for acceleration at time level n $\frac{d\mathbf{U}}{dt}^{n} = \frac{\mathbf{U^{(n+1)}} - \mathbf{U^{(n-1)}}}{2\Delta t}$.

C. *Initial conditions*

The interaction between free and fixed interface is governed by system of equations described by equations (3),(4),(6),(9) and (12) which represents an initial value problem thus for solving this system we require the knowledge of the initial shape of the interfaces and the vortex strength associated with each of the interfaces.The initial conditions for a non-linear wave have been used from the work of [Baker et al. (1970)].The initial shape of the free and bottom interface are as below:

$$y_f = a\cos(x)$$
$$y_b = -d + b\cos(x)$$

The corresponding initial vortex strengths of the free and bottom interface are:

$$\gamma_f = a(\tanh(d))^{\frac{-1}{2}}(1+\tanh(d))\cos(x)$$
$$\gamma_b = 0$$

Here a and b are the amplitudes of the free and bottom interface respectively and d is the depth of the water. For simulating wave breaking the amplitude of the free interface disturbance has been kept comparable to the depth of water. The values of the different parameters are $a = 0.5$, $b = 0$ and $d = 1$. The vortex strength of the bottom fixed interface can be assumed to be zero initially and it will itself take the appropriate values as we iterate for it in the next time step.

III. HIGHER ORDER SPECTRAL METHOD

A. *Schematic of the Domain*

In Fig 1., the surface wave and the interfacial waves are at $z = 0$ and $z = -h_1$ respectively along with the bottom at $z = -h_1 - h_2$. The wave displacement from the mean position is given by $\eta_1(x,t)$ and $\eta_2(x,t)$ respectively. Further, we have velocity potentials $\eta_1(x,z,t)$ and $\eta_2(x,z,t)$ defined in the two regions assuming irrotational flow. The elevation of bottom from its mean position is $\eta_b(x)$. Finally, the density of the top and the bottom fluids are ρ_1 and ρ_2 respectively.

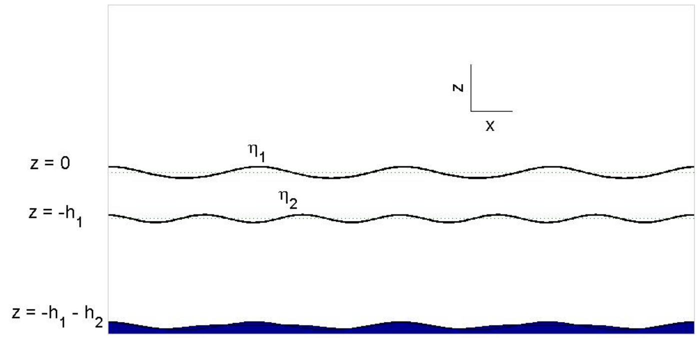

Fig. 1. **Domain**

B. *Methodology and Numerical Technique*

Higher Order Spectral methods are a class of computationally efficient methods for studying waves and wave interaction developed by [Dommermuth and Yue (1987)]. This method employs solving the Initial Boundary Value Problem using Fourier Basis Functions along with the use of Fast Fourier Transform (FFT) and Perturbation Expansion. This method can as well be employed for two layered fluid having different densities [Alam et al. (2009)]. For an irrotational and an incompressible flow, we have the continuity equations in terms of velocity potential ϕ as,

$$\nabla^2\phi_1 = 0 \quad \text{for } -h_1 + \eta_2 < z < \eta_1 \tag{13}$$

$$\nabla^2\phi_2 = 0 \quad \text{for } -h_1 - h_2 + \eta_b < z < -h_1 + \eta_2 \tag{14}$$

Kinematic Boundary Conditions-

$$\eta_{1,t} + \eta_{1,x}\phi_{1,x} = \phi_{1,z} \quad \text{at } z = \eta_1 \tag{15}$$

$$\eta_{2,t} + \eta_{2,x}\phi_{1,x} = \phi_{1,z} \quad \text{at } z = -h_1 + \eta_2 \tag{16}$$

$$\eta_{2,t} + \eta_{2,x}\phi_{2,x} = \phi_{2,z} \quad \text{at } z = -h_1 + \eta_2 \tag{17}$$

$$\eta_{b,x}\phi_{2,x} = \phi_{2,z} \quad \text{at } z = -h_1 - h_2 + \eta_b \tag{18}$$

The Dynamic Boundary Conditions respectively at the surface ant the interface are

$$\phi_{1,t} + \frac{1}{2}(\phi_{1,x}^2 + \phi_{1,z}^2) + g\eta_1 = 0 \tag{19}$$

$$\rho_1[\phi_{1,t} + \frac{1}{2}(\phi_{1,x}^2 + \phi_{1,z}^2) + g\eta_2] = \rho_2[\phi_{2,t} + \frac{1}{2}(\phi_{2,x}^2 + \phi_{2,z}^2) + g\eta_2] \tag{20}$$

The Kinematic boundary conditions provide the evolution equation for surface and interface elevations whereas, corresponding potentials evolve using the dynamic boundary conditions.

Introducing a surface potential and an interface potential,

$$\phi_1^S(x,t) = \phi_1(x, \eta_1(x,t), t)$$

$$\phi_{u/l}^I(x,t) = \phi_{u/l}(x, -h_1 + \eta_2(x,t), t)$$

Further, we have Density ratio,

$$R = \frac{\rho_1}{\rho_2}$$

Finally, we obtain the set of Evolution equations, by using the variable transformation as described by [Zakharov and Cokelet (1968)].

$$\eta_{1,t} = -\eta_{1,x}\phi_{1,x}^S + (1+\eta_{1,x}^2)\phi_{1,z} \tag{21}$$

$$\eta_{2,t} = -\eta_{2,x}\phi_{2,x}^I + (1+\eta_{2,x}^2)\phi_{2,z} \tag{22}$$

$$\phi_{1,t}^S = -g\eta_1 - \frac{1}{2}(\phi_{1,x}^S)^2 + \frac{1}{2}(1+\eta_{1,x}^2)\phi_{,z}^2 \tag{23}$$

$$\psi_{,t}^I = \frac{1}{2}(R(\phi_{1,x})^2 - (\phi_2, x)^2) + \frac{1}{2}(1+\eta_{2,x}^2)(\phi_{2,z}^2 - R\phi_{1,z}^2) - g\eta_2(1-R) \tag{24}$$

At every time step, ϕ and η is specified for each surface. Therefore, their horizontal derivatives are easily evaluated. However, the vertical velocity i.e. $\phi_{1,z}$ and $\phi_{2,z}$ can't be obtained without solving the boundary value problem because the values of velocity potential is not known below the surface/interface making it difficult to calculate the vertical derivatives of the velocity potential.

In order to solve the Boundary Value Problem, we do a perturbation expansion of ϕ_1 and ϕ_2 as,

$$\phi_1(x,z,t) = \sum_{m=1}^{M} \phi_1^{(m)}(x,z,t)$$

$$\phi_2(x,z,t) = \sum_{m=1}^{M} \phi_2^{(m)}(x,z,t)$$

Here, the superscript (m) denotes the order of steepness (ϵ). Using the normal mode form of Velocity Potentials, we have, for each m,

$$\phi_1^{(m)} = \sum_{n=-N}^{N-1} \{A_n^{(m)}(t)\frac{\cosh k_n(z+h_1)}{\cosh(k_n h_1)} + B_n^{(m)}(t)\frac{\sinh k_n(z+h_1)}{\cosh(k_n h_1)}\}e^{ik_n x} \tag{25}$$

$$\phi_2^{(m)} = \sum_{n=-N}^{N-1} \{C_n^{(m)}(t)\frac{\cosh k_n(z+h_1+h_2)}{\cosh(k_n h_2)} + D_n^{(m)}(t)\frac{\sinh k_n(z+h_1+h_2)}{\cosh(k_n h_2)}\}e^{ik_n x} \tag{26}$$

Using the boundary conditions along with the expansion of ϕ viz. equation (25) and (26), we obtain the values of modal coefficients A, B, C and D at every time step as described in the paper by [Alam et al. (2009)] After obtaining the coefficients, we have the value of Velocity potentials at all the points. Therefore, the surface and interface velocities along with the other spatial derivatives of velocity potential can be easily calculated. Using these at every time step, we march forward in time using 4th order Runge-Kutta time integration scheme.

C. *Initial Conditions*

Initially, we have taken two Stokes wave of steepness $\epsilon = 0.03$, one in the interface of wave number 4 on the surface and other of wave number 6 in the interface. The density ratio taken is equal to 0.9. The distance between the two waves is kept as such so that the waves can feel the effect of each other, which implies that the eigenfunctions of either waves do not die down to zero at the location of the other wave. However, the bottom is substantially far away, making the fluid to behave like deep water ($kH >> 1$) so that the bottom has minimal effect on the waves.

IV. RESULTS AND DISCUSSION

A. *Vortex method*

The numerical computations for the interaction between a free and fixed interface have been performed for an Atwood number of 1.0 which could represent an air water interface, the amplitude has been kept comparable to the depth in order to simulate wave breaking. If the computations are done for a small amplitude wave with the initial conditions derived from the linear theory it does move with a constant phase speed of $\sqrt{gH}$ although that is not the focal point of this paper. The results for wave breaking for Atwood number equal to 1.0 are shown in figure 1, it can be clearly seen that at t = 4.9 s the wave has a plunging breaker which is as expected.

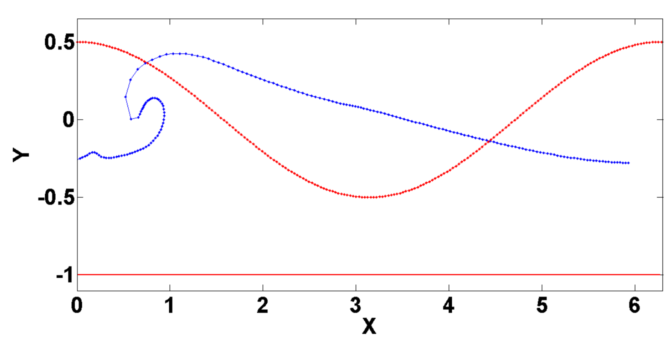

Fig. 2. **Initial shape of the interface and configuration at t = 4.9 s, At = 1.0**

B. *HOS method for wave interaction*

For the case when the density ratio of the fluids are similar, here R=0.9, we see that the interface of wave number 4 assumes a shape which is similar to that of the surface wave meaning that although the interfacial wave was initially of wave number 4, later the most dominant wave number is 4.

Although the surface wave starts to beat due to the effect of the interfacial wave, there is not much appreciable change in its shape. The most dominant wave number of the wave shape remains the same as it was initially.

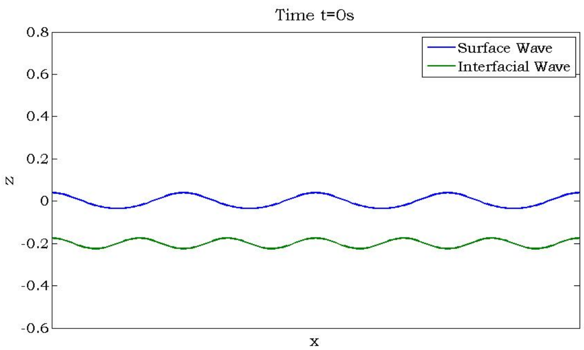

Fig. 3. **Initial shapes of the interfaces (vertical scales are exaggerated)**

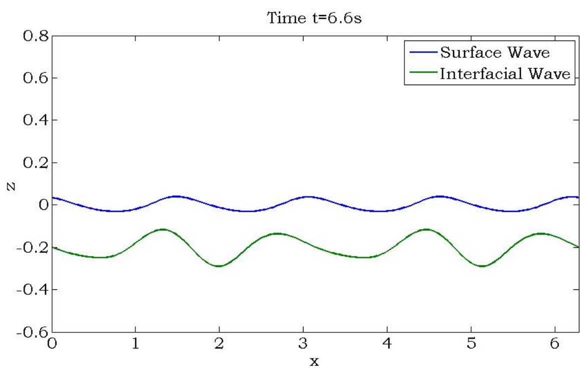

Fig. 4. **Wave shapes at t=6.6s (vertical scales are exaggerated)**

V. CONCLUSIONS

The present paper aims to study wave breaking and wave interactions using vortex and higher order spectral methods respectively. The results presented for the breaking of a non-linear wave due to the effect of bottom topography using vortex methods correctly represents the phenomenon of breaking of a wave that is seen in nature. When the amplitude of the free surface disturbance is small compared to the depth then we observe a propagating wave with a constant phase speed of $\sqrt{gH}$ but the initial conditions in that case are not the same as what has been presented here, the initial conditions then are derived using linear theory but that is not the focal point of this study. As for the HOS method, the interaction between two gravity waves in a two layered stratification is studied. The density ratio closer to 1 results in the interfacial wave being dominated over by the surface wave.

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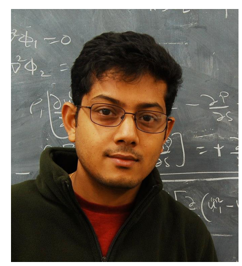
VI. BIOGRAPHICAL SKETCH OF A. GUHA



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