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INSTABILITIES in LONG and NARROW TWO-LAYERED LAKES UNDERGOING UPPER-LAYER CIRCULATION

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Abstract

Stability of two layered long and narrow water bodies like lakes and reservoirs to 2D normal mode perturbations is analysed. The inviscid stability of these water bodies is studied using a non-Boussinesq framework. The base state velocity profile being analysed is obtained when only the lighter upper-layer is undergoing a circulation, such circulation takes place when wind forcing is insufficient to induce a circulation in the heavier bottom-layer. The interfaces are characterized by the respective bulk Richardson numbers. Our analysis reveals instability over some range of wavenumbers and bulk Richardson numbers.

Keywords: *Hydrodynamic Stability; Normal Modes; Non-Boussinesq Flows; Mixing; Density Stratified Flows*

I. INTRODUCTION

The effect of wind on lakes has been studied by researchers, particularly the effect on long and narrow lakes (Heaps &Ramsbottom [1], Bye [2]). Long and narrow lakes afford us many simplifying approximations. Due to being narrow we can ignore Coriolis effect (Hutter [3]). Coriolis effect becomes important when the width of the water body is comparable to the Rossby deformation radius. The lake geometry is assumed to be rectangular for simplicity. Wind flow is taken along the length of the lake, due to this transverse flow can be ignored. Presence of a sharp thermocline in the lake gives rise to two-layered density stratification. The time-scale over which density stratification changes is the seasonal time-scale, while the wind changes on very shorter time-scales of the order of hours. On shorter time-scales of wind changes the background density stratification is essentially treated as quasi-steady. Mixing can be ignored on these shorter time-scales. As demonstrated by Heaps & Ramsbottom [1], working in the small deflections regime leads to non-linear effects being ignored, this greatly simplifies the problem. Using other simplifying assumptions like uniform and constant properties like viscosity and density in each layer one can arrive at the velocity profiles in each layer in terms of the wind forcing.

Forcing by wind in a simplified sense can be assumed to work like a two-layered lid driven cavity flow. If this assumption is followed, then the rectangular cavity will consist of a stable density stratification of two layers, over which a lid is dragged along the top surface. Such type of arrangement can set up a single circulation in the upper layer in the system. In this paper we will analyse the stability of representative base state profiles that are step up in shallow lakes where the depth of each layer is around 10 metres and the lake is a few kilometres long. For simplicity we can assume that the wind flowing over the lake has a constant velocity (U_1). Study of stability of lakes has applications in mixing of lakes and other similarly structured natural or artificial water bodies. Mixing in such water bodies affects the distribution of salt, heat, momentum and other chemical tracers such as pollutants and dissolved gases like oxygen which plays an important role in the health of aquatic flora and fauna. Classically instabilities are thought of as one of the mechanisms through which a transition takes place from laminar flow to turbulent flow (Drazin [4]). The present study may lead to a better understanding of the enhanced mixing that is caused by the instabilities which are taking place in lakes. Enhanced mixing can generate higher intensities of turbulent motions. By this present study we aim at trying to answer some of these issues.

II. METHODOLOGY

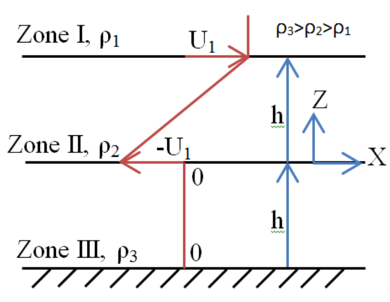

Fig. 1: Layout and base state profiles. Velocity undergoes a jump at the interface of zone II and III

Wind forcing is assumed invariant over the time-scales of the rest of the processes. Fluid is assumed to be incompressible, 2D and inviscid. To model the process we use Euler momentum equations along with continuity equation and material conservation of density. We are ignoring transverse flow in the narrow and long lake, implying that the flow is 2D. Considering a flow in the x-z plane we have

$$\frac{\partial u}{\partial x}+\frac{\partial w}{\partial z}=0 \tag{1}$$

$$\frac{D\rho}{Dt}=0\text{, i.e. }\frac{\partial \rho}{\partial t}+u\frac{\partial \rho}{\partial x}+w\frac{\partial \rho}{\partial z}=0 \tag{2}$$

$$\rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+w\frac{\partial u}{\partial z}\right)=-\frac{\partial p}{\partial x} \tag{3}$$

$$\rho\left(\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+w\frac{\partial w}{\partial z}\right)=-\frac{\partial p}{\partial z}-\rho g \tag{4}$$

All the above four equations are applied to both the layers of the water body. Density difference between air and top layer water is quite high therefore one cannot make use of the Boussinesq approximation. Boussinesq approximation can be used when the density differences are small, it neglects the variation of density in inertial terms and its effect is only considered in the terms, where the density variation is amplified by gravity. This leads to the filtering out of the sound waves in the medium (Turner [5]). Boussinesq approximation is applied at interface between the lighter top water layer and the heavier bottom water layer. We linearize the equations (1-4) about the base state horizontal velocity and density given by (5) and (6) respectively.

$$U=\begin{cases}U_1 & Z\geq h\\ \dfrac{2U_1}{h}Z-U_1 & h\geq Z\geq 0\\ 0 & 0\geq Z\geq -h\end{cases} \tag{5}$$

$$\rho=\begin{cases}\rho_1 & Z>h\\ \rho_2 & h>Z>0\\ \rho_3 & 0>Z>-h\end{cases} \tag{6}$$

Following the seminal approaches outlined by Taylor [6] and Goldstein [7] we aim to obtain a single equation in terms of eigenfunctions of the vertical velocity. On linearizing and assuming normal mode perturbations of the form $\hat{\chi}(z)\exp(i\alpha(x-ct))$ we obtain the following non-Boussinesq Taylor-Goldstein equation. In the normal mode form, the $\hat{\chi}$ is the eigenfunction of quantities like density, velocity and pressure. From the form of normal modes we can see that exponential growth will occur if Im(c) is non-zero, and the corresponding growth rate being αIm(c).

$$\begin{aligned}&\frac{d\bar{\rho}}{dz}[(U-c)\frac{d\hat{w}}{dz}-\frac{dU}{dz}\hat{w}-\frac{g}{U-c}\hat{w}]\\&+\bar{\rho}[(U-c)(\frac{d^2\hat{w}}{dz^2}-\alpha^2\hat{w})-\frac{d^2U}{dz^2}\hat{w}]=0\end{aligned} \tag{7}$$

Here U, $\bar{\rho}$ are base state velocity and density respectively; $\hat{w}$ is the vertical velocity eigenfunction; c, α are the wave speed and wavenumber respectively. Equation (7) obtained is similar to the one obtained by Barros and Choi [8] ,[9]. Solving equation (7) in zones I,II, and III (see fig. 1) and applying pressure continuity and kinematic condition across the interfaces we obtain a quartic (in wave speed, c) dispersion relation. The interface between zone I and II is characterised by bulk Richardson number J_1, and that between zone II and III is characterised by bulk Richardson number J_2. Bulk Richardson number is the ratio of buoyancy term and velocity gradient term (Turner [5]). The dispersion relation is solved numerically to generate growth rate plots [10]. Some contour plots for the growth rate are shown in the results section.J_1 and $J_2$are defined as given below in equation (8) and (9), respectively.

$$J_1=(\frac{\rho_2-\rho_1}{\rho_2})\frac{gh}{U_1^2} \tag{8}$$

$$J_2=(\frac{\rho_3-\rho_2}{\rho_2})\frac{gh}{U_1^2} \tag{9}$$

For simplicity we can assume that ρ_1 to be negligible as compared to ρ_2 , this stands true because density of air is orders of magnitude small as compared to the density of water. Such simplification is also done in case of water waves. Bulk Richardson numbers can be combined to give a stratification parameter, $R=\frac{J_1}{J_2}=\frac{\rho_2}{\rho_3-\rho_2}$ (10)

R	**ρ2 (kg/m^3)**	**ρ3 (kg/m^3)**
95	971.82	982.03
	983.20	993.54
285	992.22	995.70
	995.65	999.14

Table 1: Some representative values of density and stratification parameter, R

Max. Growth Rate	R	Wavelength (m)	U_1 (m/s)	U_1 on Beaufort Scale
0.0439	95	15.52	10.26	5
0.0019	285	8.22	5.92	4

Table 2: Maximum growth rate, wavelength of the perturbations, and the wind forcing required for the less dominant mode

III. RESULTS

Results for some representative values of stratification parameter R are presented below. Density stratification can be present due to any stratifying agent that has long time scales of diffusion and advection.

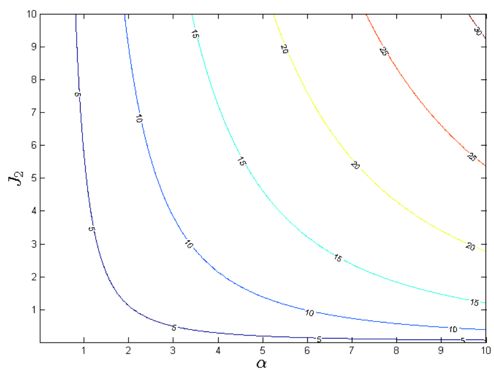

Fig. 2: Contour plot for growth rate vs. J_2 for stratification parameter, R =95 for the dominant KH mode

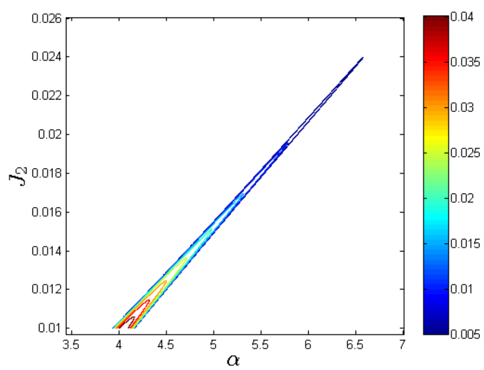

Fig. 3: Contour plot for growth rate vs. J_2 for stratification parameter, R =95 for the less dominant mode

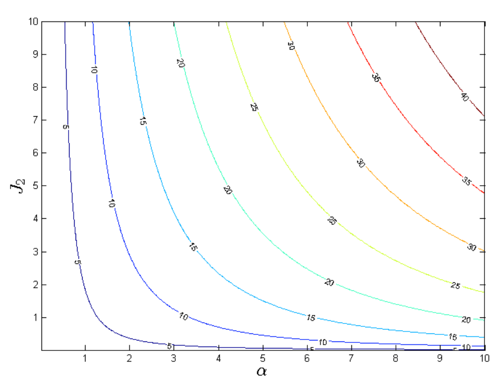

Fig.4 : Contour plot for growth rate vs. J_2 for stratification parameter, R =285 for the dominant KH mode

IV. DISCUSSION AND CONCLUSIONS

As can be seen from Fig.2 and Fig.4, the normal mode perturbations undergo exponential growth. Strong stable stratification requires large wind forcing and inertia to overcome the stabilizing effect of gravity, which tries to keep the denser fluid at the bottom (see Taylor [6] and Goldstein [7]). For Kelvin-Helmholtz instability (KH) to occur, a jump in the velocity profile is needed (Drazin [4]). In our system there is a presence of jump in the velocity, so KH will occur. In fact the most energetic instability in our system is the KH. In KH instabilities the growth rate has an approximate direct correlation with wavenumber (α), this behaviour is evident in our results (see Fig.2 and Fig.3). Classical KH instabilities are stationary instabilities, that do not have any wave propagation speed (Drazin [4]). In our system the KH type instability is undergoing propagation since the mean speed of the interface is not zero. The wind speeds required for instability are feasible and are shown in Table 2, the wind speeds are also given in terms of Beaufort scale which is an empirical scale used to describe and report wind speeds.

Along with KH, there is a presence of less dominant instability with growth rate that is orders of magnitude smaller than the KH mode growth rate, contour plot for which is shown in Fig.3.

REFERENCES

1. Heaps, N.S. and Ramsbottom, A.E., 1966. Wind effects on the water in a narrow two-layered lake. *Philosophical Transactions of the Royal Society of London A: Mathematical, Physical and Engineering Sciences*, *259*(1102), pp.391-430.

2. Bye, J.A.T.: Wind-driven circulation in unstratified lakes.Limnol. Oceanogr.10, 451–458 (1965)

3.Hutter, K., Wang, Y. and Chubarenko, I.P., 2010. *Physics of Lakes: Volume 1: Foundation of the Mathematical and Physical Background*. Springer Science & Business Media.

4. Drazin, P.G., 2002. *Introduction to Hydrodynamic Stability* (Vol. 2).Cambridge University Press.

5. Turner, J.S., 1973. Bouyancy Effects in Fluids: Cambridge Monographs on Mechanics and Applied Mathematics.

6. Taylor, G.I., 1931. Effect of variation in density on the stability of superposed streams of fluid. *Proceedings of the Royal Society of London.Series A, Containing Papers of a Mathematical and Physical Character*, *132*(820), pp.499-523

7. Goldstein, S. "On the stability of superposed streams of fluids of different densities." *Proceedings of the Royal Society of London. Series A, Containing Papers of a Mathematical and Physical Character* 132, no. 820 (1931): 524-548.

8. Barros, R. and Choi, W., 2014. Elementary stratified flows with stability at low Richardson number. *Physics of Fluids (1994-present)*, *26*(12), p.124107.

9. Barros, R. and Choi, W., 2011. Holmboe instability in non-Boussinesq fluids.*Physics of Fluids (1994-present)*, *23*(12), p.124103.

10. MATLAB inbuilt routines have been used to solve the non-linear equation.

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