	Indian Institute of Technology Kanpur Department of Mathematics and Statistics **WRITTEN TEST FOR PH.D. ADMISSIONS IN MATHEMATICS**		
Maximum Marks : 120	Date : May 15, 2018		Time : 120 Minutes
Name of the Candidate			
Roll Number		Category (Tick One)	GEN / OBC / SC/ST/PwD

INSTRUCTIONS

(1) There are three sections; the first section has fill in the blanks, the second section has multiple choice questions and the third section has subjective type questions.
- In the first section, every correct answer will be awarded 3 marks and a wrong answer will be awarded 0 marks.
- The second section has questions with four choices where one, two or three correct answers:
  - if a wrong answer is selected in a question then that entire question will be awarded 0 marks.
  - the candidate gets full credit of 3 marks, only if he/she selects all the correct answers and no wrong answers; 1 mark will be awarded for an answer to a question if it is partially correct and a wrong answer is not selected.
- Provide detailed answers for questions in the third section, but within the space provided for each of them. Each question in this section carries a maximum of 10 marks.

(2) These question-cum-answer-sheets must be returned to the invigilator before leaving the examination hall.

(3) **Please enter your answers on this page in the space given below.** If required, use alternate blank pages for rough calculations.

Fill In The Blanks Questions

Q. No.	Answer	Q. No.	Answer
1		5	
2		6	
3		7	
4		8	

Multiple Choice Questions

Q. No.	Correct Option(s)	Q. No.	Correct Option(s)	Q. No.	Correct Option(s)	Q. No.	Correct Option(s)	Q. No.	Correct Option(s)
1		7		13		19		25	
2		8		14		20		26	
3		9		15		21		27	
4		10		16		22		28	
5		11		17		23		29	
6		12		18		24		30	

Name		Roll No.	

Answer to Subjective Question 1

Name		Roll No.	

Answer to Subjective Question 2

Name		Roll No.	

Answer to Subjective Question 3

Notations

I. We denote by $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$, the set of natural numbers, integers, rational numbers, real numbers and complex numbers, respectively.

Fill in the blanks

(1) Let A be a 5×8 real matrix such that nullity of $A^T A$ is 3. Then rank of A^T is ____________.

(2) The number of two sided ideals that $M_2(\mathbb{Z}/7\mathbb{Z})$ has is ____________.

(3) What is the value of the limit?

$$\lim_{z \to 0} z \sin \frac{1}{z}, \ z = x + iy \in \mathbb{C}.$$

(4) The Lapalace transform $F(s)$ of the function

$$f(t) = \begin{cases} 0, & 0 \leq t < 2, \\ t^2 - 5t + 6, & t \geq 2, \end{cases}$$

is $F(s) =$ ____________.

(5) The lowest eigenvalue for the following boundary value problem:

$$y'' + \lambda y = 0; y'(0) = y'(2) = 0; \text{ where } \lambda > 0:$$

is ____________.

(6) The solution $y(x)$ of the initial value problem

$$y'' = y' e^y, \quad y(0) = 0, \ y'(0) = 1,$$

is $y(x) =$ ____________.

(7) For a non-negative integer n, if $P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n}(x^2 - 1)^n$ denotes the Legendre polynomial of degree n, then the value of the integral

$$\int_{-1}^{1} (1 - x)^4 P_4(x)\, dx$$

is ____________.

(8) Let $(T_k)_{k=0}^{\infty}$ be a sequence of polynomials satisfying

$$T_k(x) = 2xT_{k-1}(x) - T_{k-2}(x), k \geq 2,$$

with $T_0(x) = 1$ and $T_1(x) = x$. For $n > 0$, if p be the polynomial that interpolates $f(x) = x^n$ at the zeros of T'_{n+1}, then $f - p = \alpha_n T'_{n+1}$ for $\alpha_n =$ ____________.

Questions with one, two or three correct choices

(1) Let A be a 5×5 matrix such that all of its entries are 1. Then which of the following statement is correct?

a. A is not diagonalizable.
b. A is idempotent.
c. A is nilpotent.
d. The characteristic polynomial and the minimal polynomial of A are not equal.

(2) Let A be a $n \times n$ matrix with characteristic polynomial $x^{n-2}(x^2-1)$. Then which of the following is/are true ?

a. $A^n = A^{n-2}$
b. Rank of A is 2.
c. Rank of A is atmost 2.
d. There exists nonzero vectors v and w such that $A(v+w) = v - w$.

(3) Which of the following statements involving Euler's ϕ function is/are true ?

a. $\phi(n)$ is even as many times as it odd.
b. $\phi(n)$ is odd for only two values of n.
c. $\phi(n)$ is even when $n > 2$.
d. $\phi(n)$ is odd when $n = 2$ or n is odd.

(4) Let G be a group of order $2n$ for some integer n. Consider the map $\phi : G \to G$ defined by $\phi(x) = x^2$. Then

a. ϕ is injective.
b. ϕ is surjective.
c. ϕ is an isomorphism.
d. None of the above.

(5) Let S_n denote the group of permutations on n letters. Consider the following statements:
(A)There exists an onto group homomorphism from S_4 to S_3.
(B) There exists an onto group homomorphism from S_5 to S_4.
Then which of the following statement holds ?

a. Only (A) is true but (B) is false.
b. Only (B) is true but (A) is false.
c. Both (A) and (B) are true.
d. Both (A) and (B) are false.

(6) Which of the following groups has a proper subgroup of finite index

a. $\mathbb{Q}$. **b.** $\mathbb{Q}/Z$. **c.** $S^1 = \{z \in \mathbb{C} : |z| = 1\}$. **d.** $\mathbb{Z} \times \mathbb{Z}$.

(7) Which of the following rings are an integral domains ?

a. $\mathbb{R}[x]$.
b. $C^1[0,1]$, the ring of continuously differentiable functions on $[0,1]$.
c. $M_n(\mathbb{R})$, the ring of $n \times n$ matrices with real entries.
d. $\mathbb{Z}_4[x,y]$.

(8) Which of the following rings are fields ?

a. $\mathbb{Z}_3[x]/(x^2+1)$ **b.** $\mathbb{Q}[x]/(x^3+3x+3)$ **c.** $\mathbb{Z}[x]/(x^3+3x+3)$ **d.** $\mathbb{Q}[x]/(x^3-1)$

(9) Consider a sequence $\{a_n\}_{n\geq 0}$ of integers with $a_n \neq a_{n+1}$ for every $n \geq 0$. Which of the following is true:

a. $\{a_n\}_{n\geq 0}$ is not a Cauchy sequence.
b. $\{a_n\}_{n\geq 0}$ is not a convergence sequence.
c. $\{a_n\}_{n\geq 0}$ can not have a bounded sub-sequence.
d. $\{a_n\}_{n\geq 0}$ can not have a convergent sub-sequence.

(10) Which of these functions are uniformly continuous on (0, 1)?

a. x^2
b. $1/x^2$
c. $f(x) = 1$ for $x \in (0,1), f(0) = f(1) = 0$
d. $\sin(x)/x$

(11) Which of the following is not necessarily true about a continuous function f on $(0, 4)$?

- **a.** The function achieves its maximum on (0, 4).
- **b.** The function is bounded.
- **c.** For all Cauchy Sequences s_n on the set $(0, 4)$, $f(s_n)$ is also Cauchy.
- **d.** If $f(1) = 2$ and $f(3) = 5$, then $f(c) = 3$, for some $c \in (0, 4)$.

(12) Let $A = \{(x, y) \in \mathbb{R}^2 | \ x^2 + y^2 < 1\}$ and $B = \{(x, y) \in \mathbb{R}^2 | \ 1 < x^2 + y^2 < 2\}$, both taken with the subspace topology of $\mathbb{R}^2$. Choose the correct statement(s) from below.

- **a.** Every continuous function from A to $\mathbb{R}$ has bounded image.
- **b.** There exists a non-constant continuous function from B to $\mathbb{N}$.
- **c.** For every surjective continuous function from $A \cup B$ to a topological space X, X has at most two connected components.
- **d.** B is homeomorphic to the unit circle.

(13) Let $f(z) = e^z/(1 - e^z)$. Then which of the following are true.

- **a.** f is entire.
- **b.** f is meromorphic.
- **c.** all poles of f are simple.
- **d.** f has no zeros.

(14) Let X be a compact metric space with the metric d and let $\{x_n\}_{n \geq 0}$ be a dense subset of X. Let a denote the diameter of X. Consider the space $[0, a]^{\mathbb{N}}$ with product topology. Define $f : X \to [0, a]^{\mathbb{N}}$ by $f(x) = (d(x, x_1), d(x, x_2), \ldots,)$. Then,

- **a.** f is always injective, but may not be surjective.
- **b.** f is always surjective, but not injective.
- **c.** f is always a continuous injection.
- **d.** f is always a homeomorphism.

(15) Consider the Hilbert space $l^2(\mathbb{N})$ of square-summable sequences and for a sequence (b_n) of complex numbers, define a linear map $f(a_n) = \sum_{n=0}^{\infty} a_n b_n$ for all $(a_n) \in l^2(\mathbb{N})$ whenever the series $\sum_{n=0}^{\infty} a_n b_n$ converges.

- **a.** If $f(a_n) \in \mathbb{C}$, then $(b_n) \in l^2(\mathbb{N})$.
- **b.** If $f(a_n) \in \mathbb{C}$, then $(b_n) \notin l^2(\mathbb{N})$.
- **c.** If $(b_n) \in l^2(\mathbb{N})$, then f is bounded linear.
- **d.** If f is bounded linear, then $(b_n) \in l^2(\mathbb{N})$.

(16) Which of the following are true:

- **a.** There exists a sequence $\{p_n(x^2)\}$ of polynomials converging uniformly to x on $[-1, 1]$.
- **b.** There exists a sequence $\{p_n(|x|)\}$ of polynomials converging uniformly to x on $[-1, 1]$.
- **c.** There is no sequence $\{p_n(x^2)\}$ of polynomials converging uniformly to x on $[-1, 1]$.
- **d.** There is no sequence $\{p_n(|x|)\}$ of polynomials converging uniformly to x on $[-1, 1]$.

(17) Let $\{s_n\}$ be a sequence of real numbers on a bounded set S such that $\liminf s_n \neq \limsup s_n$. Which of the following statements are true?

- **a.** $\lim s_n$ does not exist.
- **b.** $\{s_n\}$ is Cauchy.
- **c.** $\liminf s_n < \limsup s_n$.
- **d.** There exists a convergent subsequence.

(18) Consider the ODE

$$y'' + p(x)y' + q(x)y = 0,$$

where p, q are continuous functions defined on $(-1, 1)$. If $y_1(x) = \cos x$ and $y_2(x)$ are solutions of this ODE, then which of the following can be chosen for $y_2(x)$?

a. $y_2(x) = \tan x$ **b.** $y_2(x) = \sin(x^2)$ **c.** $y_2(x) = x$ **d.** $y_2(x) = \cos(x^2)$

(19) Consider the ordinary differential equation

$$x^4 y'' - x^2 \sin x y' + 2\alpha(1 - \cos x)y = 0,$$

where α is a constant. Which among the following values of α leads to a single Frobenius series solution around $x = 0$?

a. $\alpha = 1$ **b.** $\alpha = -1$ **c.** $\alpha = -2$ **d.** $\alpha = -4$

(20) If $u(x, y)$ is the solution of the initial value problem

$$xu_x + yu_y = u, \quad u(x, 1) = x^2,$$

then $u(4, 2)$ is equal to

a. 1 **b.** 2 **c.** 4 **d.** 8

(21) Consider the initial value problem

$$\begin{aligned} &u_t - u_{xx} = 0, && 0 < x < 1, t > 0, \\ &u(x, 0) = 4x(1 - x), && 0 \le x \le 1, \\ &u(0, t) = u(1, t) = 0, && t > 0. \end{aligned}$$

Then, for $0 \le x \le 1, t \ge 0$, which of the following is/are true ?

a. $u(x, t) \le 1$ **b.** $u(x, t) \ge 0$ **c.** $u(x, t) \to 0$ as $t \to \infty$ **d.** $u(x, t) \to 1$ as $t \to \infty$

(22) Let the quadrature $Q : C([0, 1]) \to \mathbb{R}$ be given by

$$Q(f) = \frac{1}{4}\left(\frac{1}{2}f(0) + f(1/4) + f(1/2) + f(3/4) + \frac{1}{2}f(1)\right).$$

Then, for which of the following functions f, we have $Q(f) = \int_0^1 f(x)\,dx$?

a. $\cos(6\pi x)$ **b.** $\sin(6\pi x)$ **c.** $\cos(8\pi x)$ **d.** $\sin(8\pi x)$

Subjective questions

(1) Let A be a $n \times n$ matrix such that it is not a scalar multiple of the identity matrix. Show that A is similar to a matrix $B = (b_{ij})$, such that $b_{11} = 0$.

(2) Let $f : (a, b) \to \mathbb{R}$ be a function. Prove or disprove: The function f is continuous if f^2 and f^3 are continuous. Here $f^k(x) = f(x)^k$.

(3) (a) Find the second order non-homogeneous linear ordinary differential equation for which

$$y_1(x) = x^2, \; y_2(x) = x^2 + \exp(2x) \quad \text{and} \quad y_3(x) = 1 + x^2 + \exp(2x)$$

are solutions.

(b) Solve the following initial value problem by Laplace transform method

$$t\ddot{y} + 2\dot{y} + ty = 0, \; t > 0, \text{ with } y(0+) = 1, \; \dot{y}(0+) = 0.$$