| | Indian Institute of Technology Kanpur<br>Department of Mathematics and Statistics<br>WRITTEN TEST FOR PH.D. ADMISSIONS IN MATHEMATICS | | | | |
|---|---|---|---|---|---|
| Maximum Marks : 90 | Date : 9th May, 2019 | | Time : 90 Minutes | | |
| Name of the Candidate | | | | | |
| Roll Number | | Category (Tick One) | GEN | OBC-NCL /EWS | SC/ST/PwD |

INSTRUCTIONS

(1) There are 3 sections. Each section has 10 questions, out of which, first 5 are 'Fill in the blank' type and the remaining 5 are 'MCQ' type.
(2) For each 'Fill in the blank' question, 3 marks will be awarded for a correct answer, and 0 marks for all other cases.
(3) For each 'MCQ', 3 marks will be awarded for fully correct answers, 1 mark for a partially correct answer with no wrong answer, and 0 marks for all other cases.
(4) This question-cum-answer booklet must be returned to the invigilator before leaving the examination hall.
(5) **Please enter your answers only on this page in the space given below.**

**ANSWERS**

| Q. NO. | SECTION A | Q. NO. | SECTION B | Q. NO. | SECTION C |
|---|---|---|---|---|---|
| 1 | | 1 | | 1 | |
| 2 | | 2 | | 2 | |
| 3 | | 3 | | 3 | |
| 4 | | 4 | | 4 | |
| 5 | | 5 | | 5 | |
| 6 | | 6 | | 6 | |
| 7 | | 7 | | 7 | |
| 8 | | 8 | | 8 | |
| 9 | | 9 | | 9 | |
| 10 | | 10 | | 10 | |

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**Notations**

I. We denote by $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{C}$, the set of natural numbers, real numbers and complex numbers, respectively.

II. $M_n(\mathbb{R})$ denotes the set of all $n \times n$ real matrices. $GL_n(\mathbb{R})$ denotes the set of all invertible $n \times n$ real matrices.

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## Section A

(1) Compute $\lim_{n\to\infty} (1 + 2^n + 3^n)^{\frac{1}{n}}$. Ans. __________ .

(2) Identify the following subset of $\mathbb{R}$,

$$\left\{ \ell \in \mathbb{R} : \text{there is a sequence } \{a_n\}_{n=1}^{\infty} \text{ in } [-1,1] \cap \mathbb{Q} \text{ s.t. } \ell = \sum_{n=1}^{\infty} a_n \right\}.$$

Ans. __________ .

(3) Compute the value of the following limit:

$$\lim_{\delta \to 0} \frac{1}{\delta} \int_{1-\delta}^{1+\delta} \left( \int_0^x y e^{y^2}\, dy \right) dx.$$

Ans. __________ .

(4) Identify $M_2(\mathbb{R})$ with $\mathbb{R}^4$ by the map $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto (a, b, c, d)$ and let

$$SL_2(\mathbb{R}) := \{ g \in M_2(\mathbb{R}) : det\,(g) = 1 \}.$$

Let $\gamma : \mathbb{R} \longrightarrow SL_2(\mathbb{R})$ be a differentiable map with $\gamma(0) = I_2$ (where $I_2$ denotes the $2 \times 2$ identity matrix). Then the trace of $\gamma'(0)$ is __________ .

(5) Consider a $n \times n$ real symmetric matrix $Q$. Let $\mathrm{f} : \mathbb{R}^n \longrightarrow \mathbb{R}$ be defined as $\mathrm{f}(\mathrm{x}) = \mathrm{x}^t Q \mathrm{x}$, $\forall \mathrm{x} \in \mathbb{R}^n$, where $\mathrm{x}$ is viewed as a column vector. Then the gradient $\nabla \mathrm{f}(\mathrm{x}) =$ __________ .

(6) We say that a continuous function $\varphi : \mathbb{R} \longrightarrow \mathbb{R}$ has *compact support* if there exists a closed and bounded interval $I$ such that $\varphi$ vanishes identically outside $I$. Let $C_C^1(\mathbb{R})$ denote the set of all continuously differentiable functions from $\mathbb{R}$ to $\mathbb{R}$ with compact support. Then, which of the following statements is/are true:

(a) Any power series $\sum_{n=0}^{\infty} a_n x^n$, where all $a_n \in \mathbb{R}$, which converges everywhere on $\mathbb{R}$ is in $C_C^1(\mathbb{R})$.

(b) The function $\Psi : \mathbb{R} \longrightarrow \mathbb{R}$ defined by

$$\Psi(x) = \begin{cases} e^{\frac{1}{x^2-1}} & \text{for } x \in (-1,1) \\ 0 & \text{elsewhere} \end{cases}$$

is in $C_C^1(\mathbb{R})$.

(c) There exists a continuous function $f : \mathbb{R} \longrightarrow \mathbb{R}$ which does not vanish identically on $\mathbb{R}$ but $\int_{-\infty}^{\infty} f(x) g(x)\, dx = 0$, whenever $g \in C_C^1(\mathbb{R})$.

(d) If $f : \mathbb{R} \longrightarrow \mathbb{R}$ is continuously differentiable then for all $g \in C_C^1(\mathbb{R})$, one has

$$\int_{-\infty}^{\infty} f(x) g'(x)\, dx = -\int_{-\infty}^{\infty} f'(x) g(x)\, dx\,.$$

(7) Let $f_n : [0,1] \longrightarrow \mathbb{R}$ be a sequence of functions which converges uniformly to $f$ on $[0,1]$. Which of the following statements is/are NOT true?
(a) $\forall n \in \mathbb{N}, f_n$ is bounded $\Longrightarrow f$ is bounded.
(b) $\forall n \in \mathbb{N}, f_n$ is continuous $\Longrightarrow f$ is continuous.
(c) $\forall n \in \mathbb{N}, f_n$ is differentiable $\Longrightarrow f$ is differentiable.
(d) $\forall n \in \mathbb{N}, f_n$ is integrable $\Longrightarrow f$ is integrable.

(8) Let $r > 0$ and $f : (-r,r) \longrightarrow \mathbb{R}$ be an infinitely differentiable function. For each $n \in \mathbb{N}$, define the polynomial

$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^k .$$

Then, which of the following statements is/are true:
(a) The sequence $\{P_n\}_{n=1}^{\infty}$ converges to $f$ pointwise in some open interval containing 0.
(b) If $J$ an open interval containing 0 such that the sequence $\{P_n\}_{n=1}^{\infty}$ converges to $f$ pointwise on $J$, then $P_n$ converges to $f$ uniformly on $J$.
(c) For all $n \in \mathbb{N}$, the polynomial $P_n$ satisfies

$$\lim_{x \to 0} \frac{f(x) - P_n(x)}{x^n} = 0. \qquad \text{(Eq.1)}$$

(d) For all $n \in \mathbb{N}$, $P_n$ is the unique polynomial of degree $\leq n$ satisfying (Eq.1).

**Notations and terminology for (9) and (10):** A function $\mathrm{f} = (f_1, f_2, f_3) : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ is said to be *Lipschitz* on a given $S \subseteq \mathbb{R}^2$ if there exists a $K > 0$ such that, $||\mathrm{f}(\mathrm{x}) - \mathrm{f}(\mathrm{y})|| \leq K||\mathrm{x} - \mathrm{y}||$ holds for all $\mathrm{x}, \mathrm{y} \in S$. We say f is of class $C^1$, if all the partial derivatives of f are continuous everywhere.

Consider the set

$$\mathfrak{S} := \left\{ \mathrm{x} \in \mathbb{R}^2 : \text{ rank of } \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(\mathrm{x}) & \frac{\partial f_1}{\partial x_2}(\mathrm{x}) \\ \frac{\partial f_2}{\partial x_1}(\mathrm{x}) & \frac{\partial f_2}{\partial x_2}(\mathrm{x}) \\ \frac{\partial f_3}{\partial x_1}(\mathrm{x}) & \frac{\partial f_3}{\partial x_2}(\mathrm{x}) \end{pmatrix} = 2 \right\}.$$

(9) Let f be $C^1$. With the above notations, pick the correct statement(s):
(a) The set $\mathfrak{S}$ is always compact and connected.
(b) The set $\mathfrak{S}$ is always compact but not necessarily connected.
(c) The set $\mathfrak{S}$ is never compact but always connected.
(d) None of the above.

(10) Let f be $C^1$. With notations as above, choose the correct statement(s):
(a) f is Lipschitz on every subset of $\mathbb{R}^2$ which is compact and convex.
(b) f is uniformly continuous on $\mathbb{R}^2$.
(c) If $\mathrm{x} \in \mathfrak{S}$, then f is one-one in some neighborhood of x.
(d) Neither (10b) nor (10c) is true.

# Section B

(1) Let $A$ be the matrix $\begin{bmatrix} 3 & 2 & -14 & 1 \\ -2 & -1 & 15 & 1 \\ -6 & -4 & 28 & -2 \end{bmatrix}$. Then the dimension of the null space of $A$ is __________ .

(2) Consider the following system of equations:

$$\begin{aligned} x + ky &= 1 \\ kx + y &= 1. \end{aligned}$$

Then the system has no solution when $k =$ __________ .

(3) Let $A = \begin{bmatrix} \frac{1}{4} & \frac{5}{4} \\ \frac{3}{4} & -\frac{1}{4} \end{bmatrix}$. Then $A^{2019} =$ __________ .

(4) Let

$$A = \begin{bmatrix} 0 & 0 & a_1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & a_2 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}.$$

If the matrix $A$ satisfies the polynomial $x^3 + 1$, then $(a_1, a_2) =$ __________ .

(5) Let $\mathbb{C}^{11}$ be the set of 11−tuples with entries in $\mathbb{C}$. Let $T : \mathbb{C}^{11} \to \mathbb{C}^{11}$ be a $\mathbb{C}$-linear transformation such that the dimension of the kernel of $T = 4$, dimension of the kernel of $T^3 = 9$ and dimension of the kernel of $T^4 = 11$. Then the dimension of the kernel of $T^2$ is __________ .

(6) Let $A \in \mathbb{M}_n(\mathbb{R})$ and its minimal polynomial be $t^2+t+1$. Then, which of the following statements is/are true:

**a.** When $n = 3$, the characteristic polynomial of $A$ will be $t^3 - 1$.
**b.** The value of $n$ cannot be 5.
**c.** The inverse of $-A$ is $A + I$.
**d.** There is a 1-dimensional subspace $W$ of $\mathbb{R}^n$ such that $\{Aw \mid w \in W\} \subset W$.

(7) Let $A$ be a $n \times n$ matrix with entries in $\mathbb{C}$. Let $\chi_A(x)$ denote its characteristic polynomial and $\mu_A(x)$ denote its minimal polynomial. Suppose $\chi_A(x) = (\mu_A(x))^2(x + (1 + i))$ and $(\mu_A(x))^3 = \chi_A(x)(x - i)(x + 1 + i)$. Further, let the dimension of the eigen space of $-1-i$ be 4. Then, which of the following statements is/are true:

**a.** $n = 7$ and $A$ cannot be $\begin{bmatrix} -1-i & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1-i & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1-i & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1-i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1-i & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & i & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & i \end{bmatrix}$.

b. $n = 7$ and $A$ cannot be $\begin{bmatrix} -1-i & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1-i & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1-i & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1-i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1-i & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & i & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & i \end{bmatrix}$.

c. $n = 7$ and $A$ cannot be $\begin{bmatrix} -1-i & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1-i & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1-i & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1-i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1-i & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & i & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & i \end{bmatrix}$.

d. $n$ is at least 7 and $-i$ as well as $-1+i$ are also eigenvalues of $A$.

(8) Let $V$ be the set of polynomials of degree at most 3 with real coefficients. For $S = \{f_i \in V : i = 1, 2, 3, 4\}$, which of the following statements is/are true:

a. If $\sum_{i=1}^{4} f_i(1) = 0$, then $S$ is necessarily a linearly dependent set over $\mathbb{R}$.

b. If $\sum_{i=1}^{4} f_i(0) = 0$, then $S$ is necessarily a linearly dependent set over $\mathbb{R}$.

c. If $f_i(1) = 0$ for each $i$, $1 \le i \le 4$, then $S$ is necessarily a linearly dependent set over $\mathbb{R}$.

d. If $f_i(0) = 1$ for each $i$, $1 \le i \le 4$, then $S$ is necessarily a linearly dependent set over $\mathbb{R}$.

(9) Let $A$ be a $n \times m$ matrix with real entries. Let $x_0 \in \mathbb{R}^n$ such that the system of equation $Ax = x_0$ has more than one solution. Then, which of the following statements is/are true:

a. $Ax = b$ has a solution for every $b \in \mathbb{R}^n$.

b. If the system $Ax = b$ has a solution for $b \in \mathbb{R}^n$, then it has infinitely many solutions.

c. The system $Ax = 0$ has a non-zero solution.

d. The rank of $A$ is strictly less than $n$.

(10) Let $A$ be a $m \times n$ matrix of rank $m$ with real entries. Then which of the following statements is/are true:

a. There exists a $B \in M_m(\mathbb{R})$ such that $BA = [I_m | \mathbf{0}_{\mathbf{n-m}}]$.

b. There exists a $C \in M_n(\mathbb{R})$ such that $AC = [I_m | \mathbf{0}_{\mathbf{n-m}}]$.

c. There exists a $B \in GL_m(\mathbb{R})$ and $C \in GL_n(\mathbb{R})$ such that $BAC = [I_m | \mathbf{0}_{\mathbf{n-m}}]$.

d. There exist unique $B \in M_m(\mathbb{R})$ and $C \in M_n(\mathbb{R})$ such that $BAC = [I_m | \mathbf{0}_{\mathbf{n-m}}]$.

# Section C

(1) If the $x$-axis is tangent to the graph of a solution $y(x)$ of the ordinary differential equation

$$y'' + \cos(x)y = 0$$

at the point $(3, 0)$, then $y(2)$ is ________ .

(2) Let $y = \sin(x) + xe^x$ be a solution of the fourth order ordinary differential equation $y'''' + ay''' + by'' + cy' + dy = 0$, where $a, b, c$ and $d$ are real constants. Then the value of $b - a$ is equal to ________ .

(3) Let $y(t)$, for $t \geq 0$, be a continuous function which satisfies $y(t) + \int_0^t (t - \tau)y(\tau)\, d\tau = t^2$. Then, the value of $y\left(\frac{\pi}{2}\right)$ is ________ .

(4) A rain drop falls on the surface $z = y^2 - x^2$ at the point $(1, 2, 3)$. Assume that the path along which the rain drop goes down is given by the parametric curve

$$t \longmapsto \left(t,\ b(t),\ \frac{4}{t^2} - t^2\right).$$

Then $b(t)$ is ________ .

(5) Let $f(x)$ be a continuously differentiable function on $\mathbb{R}$. If the ordinary differential equation

$$(3y^2 - x)f(x + y^2)\, dx + 2y(y^2 - 3x)f(x + y^2)\, dy = 0$$

is exact, then $xf'(x) + 3f(x)$ is ________ .

(6) Consider the following sequence of functions defined by the iterative formula,

$$y_n(t) = 1 + \int_0^t (s + y_{n-1}(s))\, ds,\ \ n \geq 1, n \in \mathbb{N},$$

$$y_0(t) = 1, \quad t \in \mathbb{R}.$$

Then, which of the following statements is/are true:

a. $\{y_n\}$ converges pointwise to the function $2e^t - t - 1$ in some neighbourhood of 0.
b. $\{y_n\}$ does not have a pointwise limit in any neighbourhood of 0.
c. $\{y_n\}$ converges uniformly to some function in some neighbourhood of 0.
d. $\{y_n\}$ converges pointwise to the function $3e^t - t - 2$ on $\mathbb{R}$.

(7) Consider the ordinary differential equation (ODE)

$$y'' - y = -1.$$

Let $y(x)$ be the solution of this ODE which passes through the origin and remains bounded as $x \to \infty$. Then, which of the following statements is/are true:

a. $y(-1) = 1 - e$.
b. $y(-1) = 0$.
c. $\lim_{x \to 0} \frac{y(x)}{x} = 0$.
d. $\lim_{x \to 0} \frac{y(x)}{x} = 1$.

(8) Consider the initial value problem:

$$y'(t) = (y-1)^{\frac{1}{2}}, \quad y(0) = y_0. \qquad (A)$$

Then, which of the following statements is/are true:

**a.** if $y_0 = 1$, $(A)$ has a unique solution in some neighbourhood of $t = 0$.
**b.** if $y_0 = 1$, $(A)$ has more than one solution in some neighbourhood of $t = 0$.
**c.** if $y_0 = 2$, $(A)$ has a unique solution in some neighbourhood of $t = 0$.
**d.** if $y_0 = 2$, $(A)$ has more than one solution on the interval $(-1, \infty)$.

(9) Consider the ordinary differential equation

$$y'' + p(x)y' + q(x)y = 0,$$

where $p, q$ are continuous functions defined on $(-1, 1)$. If $y_1(x) = \cos x$ and $y_2(x)$ are solutions of this ordinary differential equation, then which of the following can be chosen for $y_2(x)$?

**a.** $1 + x^2$.
**b.** $x$.
**c.** $\sin(x^2)$.
**d.** $x^2$.

(10) Consider the problem

$$y'(t) = y^{\frac{2}{3}}, \quad y(0) = 0.$$

Then the above problem has

**a.** unique solution.
**b.** infinitely many solutions.
**c.** at most countably many solutions.
**d.** exactly two solutions.